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STABILITY BY THE LINEAR APPROXIMATION OF DISCRETE DYNAMICAL SYSTEMS IN SPACE conv $(\mathbb{R}^n)$

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ABSTRACT: The stability of the fixed points of discrete dynamical systems (DDS) in the semilinear space conv $(\mathbb{R}^n)$ (the space of nonempty convex compact sets of $\mathbb{R}^n$) is investigated. Equation in variations in the neighborhood of the fixed point of DDS is derived. Based on the results of the spectral theory of linear operators in Banach space, the conditions of localization of the spectrum are obtained. Examples of DDS in conv (R^n) for which the stability conditions of the fixed points are reduced to the Schur-Cohn criterion for polynomials are considered.

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1. INTRODUCTION

One of the important problems of the control theory for dynamical systems is the construction or approximation the attainability domains. The study of attainability domains for controlled systems is of great importance especially it is essential for the estimation the limiting possibilities of the controlled system, for selection of the optimal or suboptimal control, for estimation the spread of the trajectories under the influence of disturbances, etc. A fundamental contribution to the theory of estimation of attainability sets was made by F.L. Chernous'ko [1]. Some problems of qualitative analysis for attainability domains and their ellipsoidal approximations have been considered in [2, 3]. The dynamical systems (semiflows or cascades) are naturally occurred in the space $\operatorname{conv}(\mathbb{R}^n)$ (the metric space of convex compacts with the Hausdorff metric) while studying the attainability domains of linear controlled systems. The qualitative study of discrete dynamical systems in the space $\operatorname{conv}(\mathbb{R}^n)$ have been considered in monograph [4] and papers [5], [6] where the direct Lyapunov method and comparison method have been generalized and the general conditions of Lyapunov stability of the fixed points have beenobtained. In paper [7] based on the method of comparison and the methods of convex geometry the stability in terms of two measures of DDS in $\operatorname{conv}(\mathbb{R}^n)$, as well as Lyapunov stability of fixed points are studied.

Consider a linear controlled system with discrete time

$$x_{p+1} = \mathbf{A}x_p + u_p, \tag{1.1}$$

where $x_p \in \mathbb{R}^n$ is a vector of phase state, $\mathbf{A} \in L(\mathbb{R}^n)$, $u_p \in \mathbb{R}^n$ is a control vector. Here and further, if $(X, \|.\|_X)$ is a Banach space, then $L(X)$ is a Banach algebra of bounded linear operators in X. Assume that the initial values belong to the initial set i.e. $x_0 \in X_0 \in \operatorname{conv}(\mathbb{R}^n)$ and control $u_p \in U \in \operatorname{conv}(\mathbb{R}^n)$, U is a set of admissible controls. The set of attainability X_p is defined as the set of all possible trajectories of the controlled system at a time $p \in \mathbb{Z}_+$. This set satisfies the equation

$$X_{p+1} = \mathbf{A}X_p + U, \tag{1.2}$$

where $X_p \in \operatorname{conv}(\mathbb{R}^n)$, $\mathbf{A}$ is an extension of the action of operator $\mathbf{A} \in L(\mathbb{R}^n)$ to the space $\operatorname{conv}(\mathbb{R}^n)$.

We now consider the generalization of the evolution problem for attainability domain of the controlled system. Assume that at each step of system functioning the information about the volume of attainability domain is known and this information is used for correcting the law of motion and the domain of admissible controls for considered discrete dynamical system, i.e.,

$$x_{p+1} = \varphi(V[X_p])\mathbf{A}x_p + \psi(V[X_p])u_p, \tag{1.3}$$

where $x_p \in \mathbb{R}^n$, $u_p \in U \in \operatorname{conv}(\mathbb{R}^n)$, $x_0 \in X_0 \in \operatorname{conv}(\mathbb{R}^n)$, and φ, $\psi \in C(\mathbb{R}_+; \mathbb{R}_+)$ are the feedback functions. Thus, during evolution of the system its parameters can be corrected depending on the volume of attainability domain, and this problem can be interpreted as a problem of control the attainability domain. In this case, the attainability domain satisfies the equation

$$X_{p+1} = \varphi(V[X_p])\mathbf{A}X_p + \psi(V[X_p])U, \tag{1.4}$$

where $X_p \in \operatorname{conv}(\mathbb{R}^n)$, $U \in \operatorname{conv}(\mathbb{R}^n)$. This equation generates a (semi) dynamical system in the space $\operatorname{conv}(\mathbb{R}^n)$. If this system has the asymptotically stable fixed point $X^* \in \operatorname{conv}(\mathbb{R}^n)$, then the attainability domain of the controlled system is stabilized to the set X^*. Therefore it is important to find out under what conditions on the functions φ and ψ a dynamical system in the space $\operatorname{conv}(\mathbb{R}^n)$ has the asymptotically stable fixed point.

In this paper we consider a class of nonlinear dynamical systems in space $\operatorname{conv}(\mathbb{R}^n)$, including a system of the form (1.4). On the basis of the classical results of convex geometry of H. Minkowski and A.D. Alexandrov, stability theory of differential equations in a Banach space and spectral theory of linear operators, the sufficient conditions for the asymptotic stability of the fixed points are obtained.

2. MATHEMATICAL PRELIMINARIES

Let $\operatorname{conv}(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$d_H(X, Y) = \inf\{\lambda \geq 0 \mid X \subset Y + \lambda K, \quad Y \subset X + \lambda K\}, \ \ X, Y \in \mathbb{R}^n,$$

where $K = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, $\|\cdot\|$ is the Euclidean norm.

Define the operations of addition and nonnegative scalar multiplication on $\operatorname{conv}(\mathbb{R}^n)$ as below: For $X \in \operatorname{conv}(\mathbb{R}^n)$, $Y \in \operatorname{conv}(\mathbb{R}^n)$, let

$$X + Y = \{x + y \mid x \in X, \ y \in Y\}, \quad \lambda X = \{\lambda x \mid x \in X\}, \quad \lambda \geq 0.$$

Then $X+Y \in \operatorname{conv}(\mathbb{R}^n)$, $\lambda X \in \operatorname{conv}(\mathbb{R}^n)$.

To ensure that the difference between any two elements of the space $\operatorname{conv}(\mathbb{R}^n)$ is always defined, the space $\operatorname{conv}(\mathbb{R}^n)$ is isometrically and isomorphically embedded as a wedge in the Banach space $C(S^{n-1}))$, the space of continuous functions on the unit sphere S^{n-1} in $\mathbb{R}^n$. Such embedding was realized in [8].

We recall that to each element of the space $\operatorname{conv}(\mathbb{R}^n)$ we can bijectively match its support function $h_X(p)$, defined in space $\mathbb{R}^n$.

Considering the restriction of this function on the unit sphere S^{n-1} of space $\mathbb{R}^n$, we can set the correspondence

$$\operatorname{conv}(\mathbb{R}^n) \ni X \leftrightarrow h_X(p) \in C(S^{n-1}),$$

which is an isometric isomorphism, i.e., if $X \leftrightarrow h_X$, $Y \leftrightarrow h_Y$, $\lambda \geq 0$, then

$$X+Y \leftrightarrow h_X+h_Y, \quad \lambda X \leftrightarrow \lambda h_X, \quad d_H(X,Y)=\|h_X-h_Y\|_{C(S^{n-1})}.$$

Here $\|h_X\|_{C(S^{n-1})} = \max\limits_{p\in S^{n-1}} |h_X(p)|$.

In further discussion, the elements of space $\operatorname{conv}(\mathbb{R}^n)$ will be identified with their support functions.

Let us introduce a binary (equivalence) relation ϱ on the product space $\operatorname{conv}(\mathbb{R}^n) \times \operatorname{conv}(\mathbb{R}^n)$ as follows:

$$\varrho : (X,Y)\varrho(X_1,Y_1) \Leftrightarrow X+Y_1 = Y+X_1.$$

One can shown that on the quotient space $\Omega = (\operatorname{conv}(\mathbb{R}^n))^2/\varrho$ the operations of addition and multiplication by a scalar are well defined and if $[(X,Y)] \in \Omega$, $[(X_1,Y_1)] \in \Omega$, then by definition we get

$$[(X_1,Y_1)]+[(X,Y)] = [(X+X_1, Y+Y_1)],$$

$$\lambda[(X,Y)] = \begin{cases} [(\lambda X, \lambda Y)], & \lambda \geq 0, \\ [(|\lambda| Y, |\lambda| X)], & \lambda \leq 0, \end{cases}$$

and the norm

$$\|[(X,Y)]\|_\Omega = \|h_X-h_Y\|_{C(S^{n-1})} = d_H(X,Y).$$

The space $(\Omega, \|.\|_\Omega)$ is a normed linear space. Note also that Ω is not a complete space, but its completion coincides with $C(S^{n-1})$ (see [8]). The space $\operatorname{conv}(\mathbb{R}^n)$ isomorphically and isometrically embedded in the space Ω by the rule $X \to [(X,0)]$. In the space Ω the difference of two elements from $\operatorname{conv}(\mathbb{R}^n)$ is defined:

$$X - Y = [(X,Y)].$$

We now recall a few results from the theory of mixed volumes [8], that we need for our study in this paper.

Let $X_k \in \operatorname{conv}(\mathbb{R}^n)$, $\lambda_k \geq 0$, $k = \overline{1,m}$, $X = \sum\limits_{k=1}^{m} \lambda_k X_k \in \operatorname{conv}(\mathbb{R}^n)$. It is known [9] that a volume $V[X]$ of a convex body X is a homogeneous polynomial of degree n relative to the variables λ_k:

$$V[X] = \sum_{k_1,\dots,k_n} V_{k_1,\dots,k_n} \lambda_{k_1} \dots \lambda_{k_n}, \tag{2.1}$$

where the sum is taken over all indices $k_1, \cdots, k_n$ which vary independently over all values from 1 to m. At the same time the coefficients of $V_{k_1,\dots,k_n}$ are determined so that they do not depend on the order of the indices. Since $V_{k_1,\dots,k_n}$ depend only on the bodies $X_{k_1}, \cdots, X_{k_n}$ it is natural to write it in the form $V[x_{k_1}, ..., x_{k_n}]$. These coefficients are called the *mixed volumes.*

The functional $V[X_1, ..., X_n]$ has the following properties:

1. $V[X_1, ..., X_n]$ is additive and positively homogeneous with respect to each variable, i.e. for all λ', $\lambda'' \in \mathbb{R}_+$, $X_k \in \operatorname{conv}(\mathbb{R}^n)$, $Y_k \in \operatorname{conv}(\mathbb{R}^n)$, $k = \overline{1,n}$, $i = 1,2$

$$\begin{aligned} &V[X_1, ..., \lambda' Y_1 + \lambda'' Y_2, ..., X_n] \\ &\qquad = \lambda' V[X_1, ..., Y_1, ..., X_n] + \lambda'' V[X_1, ..., Y_2, ..., X_n]; \end{aligned}$$

2. $V[X_1, ..., X_n]$ is translation invariant and invariant with respect to permutation of arguments, as well as a continuous respect to the totality of variables [9].

From these properties the following Steiner formula is derived:

$$V[X_1 + \varrho X_2] = \sum_{k=0}^{n} C_n^k \varrho^k V_k[X_1, X_2], \quad \varrho \in \mathbb{R}_+, \tag{2.2}$$

where $V_k[X_1, X_2] = V\underbrace{[X_1, ..., X_1, \overbrace{X_2, ..., X_2}^{k}]}_{n}$ — k–th mixed volume, $C_n^k = \frac{n!}{k!(n-k)!}$.

Consider a functional $V[X_1, ..., X_{n-1}, z]$ for a fixed $X_i \in \mathrm{conv}\,(\mathbb{R}^n)$. $V[X_1, ..., X_{n-1}, z]$ is a linear and continuous functional in a Banach space $C(S^{n-1})$, so it can be represented as a Stieltjes–Radon integral of a continuous function $z \in C(S^{n-1})$ by a uniquely defined additive set function on the unit sphere S^{n-1}. The functional $V[X_1, ..., X_{n-1}, z]$ is completely determined by specifying of convex bodies $X_i \in \mathrm{conv}\,(\mathbb{R}^n)$, so we can state that

$$V[X_1, ..., X_{n-1}, z] = \frac{1}{n} \int\limits_{S^{n-1}} z(p) F[X_1, ..., X_{n-1}; d\omega], \quad z \in C(S^{n-1}), \qquad (2.3)$$

where $F[X_1, ..., X_{n-1}; d\omega]$ is a function of set ω on the unit sphere S^{n-1}, which is uniquely determined by the convex compacts $X_i \in \mathrm{conv}\,(\mathbb{R}^n)$. This function is called a mixed superficial function of convex compacts $u_i \in \mathrm{conv}\,(\mathbb{R}^n)$. One can show that

$$F[X_1, ..., X_{n-1}; d\omega] \geq 0.$$

If $X_1 = X_2 = ... = X_{n-1} = X$, then we set $F[X_1, ..., X_{n-1}; d\omega] = F[X; d\omega]$.

3. PROBLEM STATEMENT

Consider in the space $\mathrm{conv}\,(\mathbb{R}^n)$ a continuous mapping $\mathfrak{G} : \mathrm{conv}\,(\mathbb{R}^n) \to \mathrm{conv}\,(\mathbb{R}^n)$.

Definition 3.1. The family of mappings $\{\mathfrak{G}^p\}$ defined inductively

$$\mathfrak{G}^0 X = X, \quad \mathfrak{G}^p X = \mathfrak{G}(\mathfrak{G}^{p-1} X),$$

forms a semigroup, which will be called a discrete dynamical system and denote $\mathfrak{G}$.

The main goal of this paper is to study the stability of the fixed points of the dynamical system $\mathfrak{G}$ for some specific dynamical systems in the space $\mathrm{conv}\,(\mathbb{R}^n)$.

Definition 3.2. A point $X^* \in \operatorname{conv}(\mathbb{R}^n)$ is called a fixed point of the discrete dynamical system $\mathfrak{G}$, if $\mathfrak{G}X^* = X^*$.

Definition 3.3. A fixed point $X^* \in \operatorname{conv}(\mathbb{R}^n)$ of discrete dynamical system $\mathfrak{G}$ is

(1) stable by Lyapunov if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that the inequality $d_H(X, X^*) < \delta$ implies the estimate $\sup\limits_{p \in \mathbb{Z}_+} d_H(\mathfrak{G}^p(X), X^*) < \varepsilon$;

(2) asymptotically stable by Lyapunov if it is stable and there exists a positive number ρ such that for all $X \in \operatorname{conv}(\mathbb{R}^n)$, from inequality $d_H(X, X^*) < \rho$ follows that $d_H(\mathfrak{G}^p(X), X^*) \to 0$ for $p \to \infty$.

Next, we consider a more restricted class of discrete dynamical systems, namely, assume that $\mathfrak{G}(X) = G(X, V[X])$, where $G \in C(\operatorname{conv}(\mathbb{R}^n) \times \mathbb{R}_+; \operatorname{conv}(\mathbb{R}^n))$ is a mapping satisfying the following condition: for any $(X_0, v_0) \in \operatorname{conv}(\mathbb{R}^n) \times \mathbb{R}_+$ there exist $G'_X(X_0, v_0) \in L(C(S^{n-1}))$ and $G'_V(X_0, v_0) \in C(S^{n-1})$ and a neighborhood $\mathcal{U} \subset \operatorname{conv}(\mathbb{R}^n) \times \mathbb{R}_+$ such that, for all $(X, v) \in \mathcal{U}$ the following equality holds

$$\begin{aligned} h_{G(X,v)} = h_{G(X_0,v_0)} + G'_X(X_0, v_0)(h_X - h_{X_0}) + G'_v(X_0, v_0)(v - v_0) \\ +o(\|h_X - h_{X_0}\|_{C(S^{n-1})} + |v - v_0|). \end{aligned} \tag{3.1}$$

In the following sections we formulate the general stability conditions for the fixed points of the DDS in $\operatorname{conv}(\mathbb{R}^n)$, and also consider the particular cases and examples.

4. STABILITY CONDITIONS

We represent a dynamical system in the form

$$\overline{X} = G(X, V[X]), \tag{4.1}$$

where $X \in \operatorname{conv}(\mathbb{R}^n)$, $G : \operatorname{conv}(\mathbb{R}^n) \times \mathbb{R}_+ \to \operatorname{conv}(\mathbb{R}^n)$ is a mapping satisfying the condition (3.1).

Let $\delta\overline{X} = h_{\overline{X}} - h_{X^*}$, $\delta X = h_X - h_{X^*}$, then, using the formula of Steiner (2.2), the properties of mixed volumes and the assumptions made above regarding the mapping G, we obtain for $\|\delta X\|_{C(S^{n-1})} \to 0$

$$V[X^*+\delta X] = V[X^*] + nV_1[X^*,\delta X] + o(\|\delta X\|_{C(S^{n-1})}),$$
$$h_{G(X^*+\delta X, V[X^*+\delta X])} = h_{G(X^*,V[X^*])} + G'_X(X^*,V[X^*])\delta X$$
$$+G'_V(X^*,V[X^*])(V[X^*+\delta X]-V[X^*])$$
$$+o(\|\delta X\|_{C(S^{n-1})} + |V[X^*+\delta X]-V[X^*]|).$$

From the integral expression (2.3), it follows that for some positive constant c_0 the inequality $|V_1[X^*,\delta X]| \le c_0\|\delta X\|_{C(S^{n-1})}$ is fulfilled, and therefore from (3.1) we get

$$h_{G(X^*+\delta X, V[X^*+\delta X])} = h_{G(X^*,V[X^*])} + G'_X(X^*,V[X^*])\delta X$$
$$+nG'_V(X^*,V[X^*])V_1[X^*,\delta X] + o(\|\delta X\|_{C(S^{n-1})}).$$

Equation (4.1) we represent in the form

$$\delta\overline{X} = G'_X(X^*,V[X^*])\delta X + nG'_V(X^*,V[X^*])V_1[X^*,\delta X] + o(\|\delta X\|_{C(S^{n-1})}).$$

Using the integral expression (2.3), the variational equation can be represented in abstract form

$$\delta\overline{X} = \mathcal{Z}\delta X.$$

Here $\mathcal{Z} \in L(C(S^{n-1}))$ is a linear operator

$$\mathcal{Z}f = G'_X(X^*,V[X^*])f + G'_V(X^*,V[X^*]) \int\limits_{S^{n-1}} fF[X^*,d\omega], \quad f \in C(S^{n-1}).$$

The stability problem for a fixed point X^* is solved on the basis of a general theorem on stability with respect to linear approximation for a fixed point of the mappings of Banach spaces [10].

Theorem 4.1. *Let $r(\mathcal{Z})$ be the spectral radius of linear operator $\mathcal{Z}$. If the inequality*

$$r(\mathcal{Z}) < 1$$

is fulfilled, then the fixed point $X^* \in \operatorname{conv}(\mathbb{R}^n)$ of DDS $\mathfrak{G}$ is asymptotically stable.

Next we give a simple method for localization the spectrum of $\mathcal{Z}$.

Let $R(z,.)$ denote the resolvent and $\rho(.)$ – the resolvent set for an appropriate operator.

Lemma 4.1. *Let $z \in \rho(G'_X(X^*, V[X^*]))$ and define the function*

$$D(z) = 1 - \int\limits_{S^{n-1}} R(z, G'_X(X^*, V[X^*]))G'_V(X^*, V[X^*])F[X^*; d\omega], \quad z \in \mathbb{C}$$

and the set

$$M = \{z \in \rho(G'_X(X^*, V[X^*])) \mid D(z) = 0\} \subset \mathbb{C}.$$

Then

$$\sigma(\mathcal{Z}) \subset \sigma(G'_X(X^*, V[X^*])) \cup M.$$

Proof. Let $\lambda \in \rho(G'_X(X^*, V[X^*])) \cap CM$. Then there exists a bounded linear operator $R(\lambda, G'_X(X^*, V[X^*])) \in L(C(S^{n-1}))$. Let $f \in C(S^{n-1})$, consider a linear equation

$$\lambda h - (G'_X(X^*, V[X^*])h + G'_V(X^*, V[X^*]) \int\limits_{S^{n-1}} hF[X^*, d\omega]) = f. \tag{4.2}$$

This equation is equivalent to equation

$$h - R(\lambda, G'_X(X^*, V[X^*]))G'_V(X^*, V[X^*]) \int\limits_{S^{n-1}} hF[X^*, d\omega]$$
$$= R(\lambda, G'_X(X^*, V[X^*]))f. \tag{4.3}$$

Multiplying (4.3) by the function $F[X^*, d\omega]$ and integrating over the surface of the sphere S^{n-1}, we obtain

$$\int\limits_{S^{n-1}} hF[X^*, d\omega]$$
$$- \int\limits_{S^{n-1}} R(\lambda, G'_X(X^*, V[X^*]))G'_V(X^*, V[X^*])F[X^*, d\omega] \int\limits_{S^{n-1}} hF[X^*, d\omega]$$
$$= \int\limits_{S^{n-1}} R(\lambda, G'_X(X^*, V[X^*]))fF[X^*, d\omega].$$

From (4.3) we get

$$h = \frac{1}{D(\lambda)} R(\lambda, G'_X(X^*, V[X^*]))G'_V(X^*, V[X^*])$$
$$\int\limits_{S^{n-1}} R(\lambda, G'_X(X^*, V[X^*]))fF[X^*, d\omega]$$
$$+ R(\lambda, G'_X(X^*, V[X^*]))f.$$

Since the right-hand side of the last equality continuously depends on the function f, the equation (4.2) for any $f \in C(S^{n-1})$ has a unique solution $h \in C(S^{n-1})$ which is continuously depends on function $f \in C(S^{n-1})$ and, therefore, $\lambda \in \rho(\mathcal{Z})$. The lemma is proved.

5. DYNAMICAL SYSTEMS IN SPACE conv $(\mathbb{R}^2)$

We concretize the conditions of the asymptotic stability formulated in the previous section, for some special cases, when the mapping $G(X, s)$ is of the form

$$G(X, s) = \varphi(s)\mathbf{A}X + \psi(s)B, \tag{5.1}$$

where $B \in \operatorname{conv}(\mathbb{R}^n)$, φ, $\psi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbf{A} \in L(\mathbb{R}^2)$ is a linear operator the action of which is naturally extended to space conv $(\mathbb{R}^2)$

$$\mathbf{A}X = \{\mathbf{A}x \mid x \in X\} \in \operatorname{conv}(\mathbb{R}^2).$$

Next, this operator extends to the space Ω and, by continuity, on the space $C(S^{n-1})$. This dilatation of operator we denote as $\mathcal{A}$, then

$$(\mathcal{A}h)(p) = \|\mathbf{A}^*p\| h\Big(\frac{\mathbf{A}^*p}{\|\mathbf{A}^*p\|}\Big), \quad h \in C(S^{n-1}).$$

Here $\mathbf{A}^*$ is an adjoint operator.

The operator $\mathcal{Z} : C(S^{n-1}) \to C(S^{n-1})$ is of the form

$$(\mathcal{Z}h)(p) = \varphi(S[X^*])(\mathcal{A}h)(p) + (\varphi'(S[X^*])(\mathcal{A}h_{X^*})(p)$$
$$+\psi'(S[X^*])h_B(p)) \int\limits_{S^{n-1}} h(x)F[X^*; d\omega(x)].$$

If $n = 2$ then it is possible to introduce the angular variable function $\theta \in [0, 2\pi]$, $H_X(\theta) = h_X(\cos\theta, \sin\theta)$ instead the support function $h_X(p)$, $X \in \operatorname{conv}(\mathbb{R}^2)$. Assume that $H''_{X^*}(\theta)$ exists and $H''_{X^*}(\theta) \in L^1[0, 2\pi]$, then the conditions for asymptotic stability of a fixed point X^* are determined from the inequalities

$$r(\mathcal{Z}) < 1,$$

where $\mathcal{Z} \in L(C[0, 2\pi])$ is a linear continuous operator acting according to the formula

$$\mathcal{Z}H(\theta) = \varphi(S[X^*])a(\theta)H(\gamma(\theta)) + \Gamma(\theta)\int_0^{2\pi} N(\tau)H(\tau)\, d\tau,$$

where

$$\Gamma(\theta) = \varphi'(S[X^*])a(\theta)H_{X^*}(\gamma(\theta)) + \psi'(S[X^*])H_B(\theta),$$
$$N(\theta) = H_{X^*}(\theta) + H''_{X^*}(\theta).$$

Here, we use the following notations

$$a(\theta) = \sqrt{(a_{11}\cos\theta + a_{21}\sin\theta)^2 + (a_{12}\cos\theta + a_{22}\sin\theta)^2},$$
$$\cos\gamma(\theta) = \frac{a_{11}\cos\theta + a_{21}\sin\theta}{a(\theta)}, \quad \sin\gamma(\theta) = \frac{a_{12}\cos\theta + a_{22}\sin\theta}{a(\theta)}.$$

$A = [a_{ij}]_{i,j=1,2}$ is a matrix of the linear operator $\mathbf{A}$ relative to a fixed orthonormal basis of the space $\mathbb{R}^2$.

Consider the problem of the localization of the spectrum for the operator $\mathcal{Z}$ in various special cases.

Case 1. Assume that $\operatorname{tr}^2\mathbf{A} - 4\det\mathbf{A} < 0$, then, without loss of generality, we can assume that matrix A of a linear operator is of the form

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It's obvious that $a(\theta) = \sqrt{\det\mathbf{A}}$, $\gamma(\theta) = \theta + \alpha$, where α determined from the relations

$$\cos\alpha = \frac{\operatorname{tr}\,\mathbf{A}}{2\sqrt{\det\mathbf{A}}}, \quad \sin\alpha = \frac{\sqrt{4\det\mathbf{A} - \operatorname{tr}^2\mathbf{A}}}{2\sqrt{\det\mathbf{A}}}, \quad \omega = \sqrt{\det\mathbf{A}}\varphi(S[X^*]).$$

Denote $\mathcal{U}_\alpha \in L(C[0, 2\pi])$, $\mathcal{U}_\alpha H(\theta) = H(\theta + \alpha)$.

Let $\alpha q \in 2\pi\mathbb{Z}$, $q > 0$, then

$$\mathcal{Z}^i H(\theta) = \omega^i \mathcal{U}_{\alpha i} H(\theta) + \int_0^{2\pi} \sum_{k=1}^{i} \mathrm{A}_{ki}(\theta)\mathrm{B}_{ki}(\tau)H(\tau)\, d\tau,$$

where the functions $\mathrm{A}_{ki}(\theta)$, $\mathrm{B}_{ki}(\tau)$, $k = 1, 2, ..., i$ are determined from the recurrence equations

$$\mathrm{A}_{k,i+1}(\theta) = \mathrm{A}_{ki}(\theta), \quad k = \overline{1, i}, \quad \mathrm{A}_{i+1,i+1}(\theta) = \omega^i \Gamma(\theta + i\alpha),$$

$$\mathrm{B}_{k,i+1}(\tau) = \omega \mathrm{B}_{ki}(\tau - \alpha) + N(\tau) \int_0^{2\pi} \mathrm{B}_{ki}(\xi)\Gamma(\xi)\, d\xi, \quad k = \overline{1,i}, \quad \mathrm{B}_{i+1,i+1}(\tau) = N(\tau) \quad (5.2)$$

with initial values $\mathrm{A}_{11}(\theta) = \Gamma(\theta)$, $\mathrm{B}_{11}(\tau) = N(\tau)$.

Indeed, by definition,

$$\begin{aligned}
\mathcal{Z}^{i+1}H(\theta) &= \omega^i \mathcal{Z}H(\theta + i\alpha) + \int_0^{2\pi} \sum_{k=1}^{i} \mathrm{A}_{ki}(\theta)\mathrm{B}_{ik}(\tau)\mathcal{Z}H(\tau)\, d\tau \\
&= \omega^i(\omega H(\theta + \alpha(i+1)) + \Gamma(\theta + i\alpha) \int_0^{2\pi} N(\tau)H(\tau)\, d\tau) \\
&+ \int_0^{2\pi} \sum_{k=1}^{i} \mathrm{A}_{ki}(\theta)\mathrm{B}_{ki}(\tau)(\omega H(\tau + \alpha) + \Gamma(\tau) \int_0^{2\pi} N(\xi)H(\xi)\, d\xi)\, d\tau \\
&= \omega^{i+1}H(\theta + \alpha(i+1)) + \int_0^{2\pi} (\sum_{k=1}^{i} \mathrm{A}_{ki}(\theta)(\omega \mathrm{B}_{ki}(\tau - \alpha) \\
&+ \int_0^{2\pi} \mathrm{B}_{ki}(\xi)\Gamma(\xi)\, d\xi N(\tau)) + \omega^i \Gamma(\theta + i\alpha)N(\tau))H(\tau)\, d\tau \\
&= \omega^{i+1}H(\theta + \alpha(i+1)) + \int_0^{2\pi} \sum_{k=1}^{i+1} \mathrm{A}_{k,i+1}(\theta)\mathrm{B}_{k,i+1}(\tau)H(\tau)\, d\tau.
\end{aligned}$$

Therefore,

$$\mathcal{Z}^q H(\theta) = \omega^q H(\theta) + \int_0^{2\pi} \sum_{k=1}^{q} \mathrm{A}_{kq}(\theta)\mathrm{B}_{kq}(\tau)H(\tau)\, d\tau \equiv \omega^q H(\theta) + \mathcal{T}H.$$

Let us define a polynomial

$$D_1(\lambda) = \det \Big[\lambda \delta_{mn} - \int_0^{2\pi} \mathrm{A}_{mq}(\tau)\mathrm{B}_{nq}(\tau)\, d\tau\Big].$$

Then, by spectral theorem for compact linear operator, we get

$$\sigma(\mathcal{Z}^q) = \omega^q + \sigma(\mathcal{T}), \quad \sigma(\mathcal{T}) = \Big\{\lambda \in \mathbb{C} \mid D_1(\lambda) = 0\Big\} \cup \{0\}.$$

If we define the characteristic polynomial $D(\lambda) = D_1(\lambda - \omega^q)$, then

$$\sigma(\mathcal{Z}^q) = \{\lambda \in \mathbb{C} \mid D(\lambda) = 0\} \cup \{0\}.$$

Let $\alpha \notin 2\pi\mathbb{Q}$. We introduce a characteristic function

$$D(z) = \begin{cases} 1 + \sum\limits_{k=0}^{\infty} z^k \omega^{-k-1} \int\limits_0^{2\pi} N(\theta)\Gamma(\theta - (k+1)\alpha)\, d\theta, & |z| < \omega, \\ 1 - \sum\limits_{k=0}^{\infty} z^{-k-1} \omega^{k} \int\limits_0^{2\pi} N(\theta)\Gamma(\theta + k\alpha)\, d\theta, & |z| > \omega. \end{cases}$$

The following assertion allows us to establish the estimate of the spectrum $\sigma(\mathcal{Z})$.

Lemma 5.1. *Let* $B_\omega = \{z \in \mathbb{C} \mid |z| = \omega\}$, $M = \{z \in \mathbb{C} \mid D(z) = 0\}$. *Then the following inclusion holds*

$$\sigma(\mathcal{Z}) \subset M \cup B_\omega.$$

Proof. The proof is similar to that of Lemma 4.1.

Theorem 5.1. *[11] Suppose that* $\operatorname{tr}^2 \mathbf{A} - 4 \det \mathbf{A} < 0$, *and the following inequalities hold*

$$\varphi(S[X^*])\|\mathbf{A}\| < 1, \quad \sup\{|z| \mid z \in \mathbb{C},\ D(z) = 0\} < 1. \tag{5.3}$$

Then the fixed point $X = X^* \in \operatorname{conv}(\mathbb{R}^2)$ *of the DDS* (5.1) *is Lyapunov asymptotically stable.*

Case 2. Assume that $\operatorname{tr}^2 \mathbf{A} - 4 \det \mathbf{A} \geq 0$, $\det \mathbf{A} \neq 0$. The operator $\mathcal{Z}$ can be represented in the form

$$\mathcal{Z}H(\theta) = \varphi(S[u^*])\mathcal{V}H(\theta) + \Gamma(\theta) \int\limits_0^{2\pi} N(\tau)H(\tau)\, d\tau,$$

where $\mathcal{V} \in L(C[0, 2\pi])$ is a bounded linear operator such that $\mathcal{V}H(\theta) = a(\theta)H(\gamma(\theta))$.

For $z \in \mathbb{C}$, $|z| > \varphi(S[X^*])\|\mathbf{A}\|$ we consider a function

$$D_1(z) = 1 - \sum_{k=0}^{\infty} z^{-k-1} \int\limits_0^{2\pi} \varphi^k(S[X^*]) a^k(\theta) \Gamma(\gamma^{[k]}(\theta)) N(\theta)\, d\theta,$$

where $\gamma^{[k]}$ is a k-th iteration of mapping $\gamma : S^1 \to S^1$.

The following theorem is a consequence of the above considerations and Lemma 4.1.

Theorem 5.2. *[11] Suppose that* $\operatorname{tr}^2 \mathbf{A} - 4 \det \mathbf{A} \geq 0$, $\varphi(S[X^*])\|\mathbf{A}\| < 1$, *and the equation* $D_1(z) = 0$ *has no solutions for which* $|z| > \varphi(S[X^*])\|\mathbf{A}\|$.

Then the fixed point $X^* \in \operatorname{conv}(\mathbb{R}^2)$ *of the DDS* (5.1) *is Lyapunov asymptotically stable.*

6. EXAMPLES

In this section we consider some applications of the obtained results that complement the results presented in Section 5. In particular, we consider the degenerate cases.

Example 6.1. Consider the DDS (5.1) when $\det \mathbf{A} = 0$ and $\mathbf{A}$ is a diagonalizable operator. Assume that there exists a fixed point $X^* \in \operatorname{conv}(\mathbb{R}^2)$ of this system and denote $s^* = S[X^*]$. Without loss of generality, we may assume that the matrix of $\mathbf{A}$ is of the form

$$A = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\mu > 0$, then the operator $\mathcal{Z}$ is of the form

$$\mathcal{Z}H(\theta) = \begin{cases} \omega|\cos\theta|H(0) + \Gamma(\theta)\int\limits_0^{2\pi} N(\tau)H(\tau)\,d\tau, & \theta \in [-\pi/2, \pi/2), \\ \omega|\cos\theta|H(\pi) + \Gamma(\theta)\int\limits_0^{2\pi} N(\tau)H(\tau)\,d\tau & \theta \in [\pi/2, 3\pi/2). \end{cases}$$

Here $N(\theta) = H_{X^*}(\theta) + H''_{X^*}(\theta)$ and

$$\Gamma(\theta) = \begin{cases} \varphi'(s^*)\|\mathbf{A}\||\cos\theta|H_{X^*}(0) + \psi'(s^*)H_B(\theta), & \theta \in [-\pi/2, \pi/2), \\ \varphi'(s^*)\|\mathbf{A}\||\cos\theta|H_{X^*}(\pi) + \psi'(s^*)H_B(\theta), & \theta \in [\pi/2, 3\pi/2). \end{cases}$$

It is obvious that $\mathcal{Z} \in L(C[0, 2\pi])$.

Let $f \in C[0, 2\pi]$ and consider a linear equation

$$\lambda H(\theta) - \mathcal{Z}H(\theta) = f(\theta).$$

This equation is equivalent to the relations

$$\lambda H(\theta) - \omega \cos\theta H(0) - \Gamma(\theta) \int\limits_0^{2\pi} N(\tau) H(\tau)\, d\tau = f(\theta), \quad \theta \in [-\pi/2, \pi/2),$$
$$\lambda H(\theta) + \omega \cos\theta H(\pi) - \Gamma(\theta) \int\limits_0^{2\pi} N(\tau) H(\tau)\, d\tau = f(\theta) \quad \theta \in [\pi/2, 3\pi/2). \tag{6.1}$$

Let $\theta = 0$ then from the first relation, we get

$$(\lambda - \omega) H(0) - \Gamma(0) \int\limits_0^{2\pi} N(\tau) H(\tau)\, d\tau = f(0). \tag{6.2}$$

Let $\theta = \pi$, then from the second relation, we get

$$(\lambda - \omega) H(\pi) - \Gamma(\pi) \int\limits_0^{2\pi} N(\tau) H(\tau)\, d\tau = f(\pi). \tag{6.3}$$

Multiplying the first and second relation (6.1) on $N(\theta)$, and integrating them in the range from $-\pi/2$ to $\pi/2$ and from $\pi/2$ to $3\pi/2$ respectively, we obtain

$$\lambda \int\limits_{-\pi/2}^{\pi/2} N(\theta) H(\theta)\, d\theta - \omega H(0) \int\limits_{-\pi/2}^{\pi/2} N(\theta) \cos\theta\, d\theta$$
$$- \int\limits_{-\pi/2}^{\pi/2} N(\theta)\Gamma(\theta) d\theta \int\limits_0^{2\pi} N(\tau) H(\tau)\, d\tau = \int\limits_{-\pi/2}^{\pi/2} N(\theta) f(\theta)\, d\theta,$$

$$\lambda \int\limits_{\pi/2}^{3\pi/2} N(\theta) H(\theta)\, d\theta + \omega H(\pi) \int\limits_{\pi/2}^{3\pi/2} N(\theta) \cos\theta\, d\theta$$
$$- \int\limits_{\pi/2}^{3\pi/2} N(\theta)\Gamma(\theta) d\theta \int\limits_0^{2\pi} N(\tau) H(\tau)\, d\tau = \int\limits_{\pi/2}^{3\pi/2} N(\theta) f(\theta)\, d\theta.$$

Adding the last two equations, we obtain

$$
\begin{gathered}
-\omega \int_{-\pi/2}^{\pi/2} N(\theta)\cos\theta\, d\theta H(0) + \omega \int_{\pi/2}^{3\pi/2} N(\theta)\cos\theta\, d\theta H(\pi) \\
+(\lambda - \int_0^{2\pi} N(\theta)\Gamma(\theta) d\theta) \int_0^{2\pi} N(\tau)H(\tau)\, d\tau = \int_0^{2\pi} N(\theta) f(\theta)\, d\theta.
\end{gathered} \tag{6.4}
$$

Considering the equation (6.2)–(6.4) as a system of linear nonhomogeneous equations, we conclude that under the condition $D(\lambda) \neq 0$, where

$$
D(\lambda) =
$$

$$
= \begin{bmatrix}
\lambda - \omega & 0 & -\Gamma(0) \\
0 & \lambda - \omega & -\Gamma(\pi) \\
-\omega \int\limits_{-\pi/2}^{\pi/2} N(\theta)\cos\theta\, d\theta & \omega \int\limits_{\pi/2}^{3\pi/2} N(\theta)\cos\theta\, d\theta & \lambda - \int\limits_0^{2\pi} N(\theta)\Gamma(\theta) d\theta
\end{bmatrix}
$$

this system has a unique solution which continuously depends on function $f(\theta)$. From this system we can determine the values $H(0)$, $H(\pi)$ and $\int\limits_0^{2\pi} N(\tau)H(\tau)\, d\tau$. If $\lambda \neq 0$, then by substituting these values in the relations (6.1) we obtain the function $H(\theta)$. This function is continuous and continuously depends on the function $f(\theta)$. Therefore $\sigma(\mathcal{Z}) \subset \{\lambda_1, \lambda_2, \lambda_3, 0\}$, where λ_i, $i = 1, 2, 3$ are the roots of the equation $D(\lambda) = 0$. It is easy to see that the numbers λ_i, $i = 1, 2, 3$ and $\lambda = 0$ are the eigenvalues of the operator $\mathcal{Z}$, i.e. $\sigma(\mathcal{Z}) = \{\lambda_1, \lambda_2, \lambda_3, 0\}$.

The conditions $|\lambda_i| < 1$ are necessary and sufficient for the asymptotic stability of the fixed point X^* of the DDS (5.1). These conditions are equivalent to the inequalities

$$
\begin{gathered}
\|\mathbf{A}\|\varphi(s^*) < 1, \quad \mu > 0, \\
-1 + \Big| \|\mathbf{A}\|\varphi(s^*) + \int_0^{2\pi} N(\theta)\Gamma(\theta)\, d\theta \Big| < \|\mathbf{A}\|\varphi(s^*) \Big[\int_0^{2\pi} N(\theta)\Gamma(\theta) d\theta \\
+\Gamma(\pi) \int_{\pi/2}^{3\pi/2} N(\theta)\cos\theta\, d\theta - \Gamma(0) \int_{-\pi/2}^{\pi/2} N(\theta)\cos\theta\, d\theta \Big] < 1.
\end{gathered}
$$

The case $\mu \le 0$ is considered similarly. The operator $\mathcal{Z}$ is of the form

$$\mathcal{Z}H(\theta) = \begin{cases} -\omega|\cos\theta|H(0) + \Gamma(\theta)\int\limits_0^{2\pi} N(\tau)H(\tau)\,d\tau, & \theta \in [\pi/2, 3\pi/2), \\ \omega|\cos\theta|H(\pi) + \Gamma(\theta)\int\limits_0^{2\pi} N(\tau)H(\tau)\,d\tau, & \theta \in [-\pi/2, \pi/2), \end{cases}$$

and the function $\Gamma(\theta)$ has the form

$$\Gamma(\theta) = \begin{cases} -\varphi'(s^*)\|\mathbf{A}\||\cos\theta|H_{X^*}(0) + \psi'(s^*)H_B(\theta), & \theta \in [\pi/2, 3\pi/2), \\ \varphi'(s^*)\|\mathbf{A}\||\cos\theta|H_{X^*}(\pi) + \psi'(s^*)H_B(\theta), & \theta \in [-\pi/2, \pi/2). \end{cases}$$

The conditions of asymptotic stability of the fixed point $X^* \in \operatorname{conv}(\mathbb{R}^2)$ consist in the fact that the roots of the characteristic equation

$$D(\lambda) = \begin{bmatrix} \lambda & -\omega & -\Gamma(0) \\ -\omega & \lambda & -\Gamma(\pi) \\ \omega \int\limits_{\pi/2}^{3\pi/2} N(\theta)\cos\theta\,d\theta & -\omega \int\limits_{-\pi/2}^{\pi/2} N(\theta)\cos\theta\,d\theta & \lambda - \int\limits_0^{2\pi} N(\theta)\Gamma(\theta)d\theta \end{bmatrix} = 0$$

satisfy the inequalities $|\lambda_i| < 1$, $i = 1, 2, 3$ and could be represented as a system of inequalities for the coefficients of the characteristic polynomial, based on the known criteria of localization of the roots of polynomials [12].

The characteristic polynomial is of the form

$$\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0 = 0,$$

$$d_2 = -\int_0^{2\pi} N(\tau)\Gamma(\tau)\,d\tau,$$

$$d_1 = \omega\Gamma(0)\int_{\pi/2}^{3\pi/2} N(\tau)\cos\tau\,d\tau - \omega\Gamma(\pi)\int_{-\pi/2}^{\pi/2} N(\tau)\cos\tau\,d\tau - \omega^2,$$

$$d_0 = \omega^2\Gamma(\pi)\int_{\pi/2}^{3\pi/2} N(\tau)\cos\tau\,d\tau - \omega^2\Gamma(0)\int_{-\pi/2}^{\pi/2} N(\tau)\cos\tau\,d\tau + \omega^2\int_0^{2\pi} N(\tau)\Gamma(\tau)\,d\tau.$$

In this case the conditions of the asymptotic stability of the fixed point X^* follow from the Schur-Cohn criteria and have the form of inequalities (6.5).

$$|d_2 + d_0| < 1 + d_1, \quad |d_1 - d_0 d_2| < 1 - d_0^2. \tag{6.5}$$

Example 6.2. Consider the DDS (5.1) when $\mathbf{A}^2 = 0$, $\mathbf{A} \neq O$. The fixed points of this system defined by the formulas

$$s^* = \varphi(s^*)\psi^2(s^*)S[\mathbf{A}B, B] + \psi^2(s^*)S[B],$$
$$X^* = \psi(s^*)(\varphi(s^*)\mathbf{A}B + B).$$

Without loss of generality, we may assume that the matrix of $\mathbf{A}$ is of the form

$$A = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}.$$

Let $\mu > 0$, then the operator $\mathcal{Z}$ is of the form

$$\mathcal{Z}H(\theta) = \begin{cases} \omega|\cos\theta|H(\pi/2) + \Gamma(\theta)\int\limits_0^{2\pi} N(\tau)H(\tau)\,d\tau, & \theta \in (-\pi/2, \pi/2], \\ \omega|\cos\theta|H(3\pi/2) + \Gamma(\theta)\int\limits_0^{2\pi} N(\tau)H(\tau)\,d\tau & \theta \in (\pi/2, 3\pi/2]. \end{cases}$$

Here $N(\theta) = H_{X^*}(\theta) + H''_{X^*}(\theta)$ and

$$\Gamma(\theta) = \begin{cases} \varphi'(s^*)\omega|\cos\theta|H_{X^*}(\pi/2) + \psi'(s^*)H_B(\theta), & \theta \in (-\pi/2, \pi/2], \\ \varphi'(s^*)\omega|\cos\theta|H_{X^*}(3\pi/2) + \psi'(s^*)H_B(\theta), & \theta \in (\pi/2, 3\pi/2]. \end{cases}$$

Let $\mu > 0$, then similar to Example 6.1, we obtain the sufficient conditions for asymptotic stability of the fixed point $X^* \in \operatorname{conv}(\mathbb{R}^2)$ of the DDS in the form

$$-1 + \Big|\int\limits_0^{2\pi} \Gamma(\theta)N(\theta)\,d\theta\Big| < \psi'(s^*)\omega\Big(H_B(3\pi/2)\int\limits_{\pi/2}^{3\pi/2} N(\theta)\cos\theta\,d\theta$$
$$-H_B(\pi/2)\int\limits_{-\pi/2}^{\pi/2} N(\theta)\cos\theta\,d\theta\Big) < 1.$$

If $\mu < 0$ then the conditions for asymptotic stability of the fixed point $X^* \in \operatorname{conv}(\mathbb{R}^2)$ have the form

$$-1 + \Big|\int\limits_0^{2\pi} \Gamma(\theta)N(\theta)\,d\theta\Big| < \omega\psi'(s^*)H_B(\pi/2)\Big(\int\limits_{\pi/2}^{3\pi/2} N(\tau)\cos\tau\,d\tau$$
$$-H_B(3\pi/2)\int\limits_{-\pi/2}^{\pi/2} N(\tau)\cos\tau\,d\tau\Big) < 1.$$

Example 6.3. Consider the DDS (5.1) when $\mathbf{A} = \mathbf{J}$, $\mathbf{J}$ is a symplectic identity operator, i.e. $\mathbf{J}^2 = -\mathbf{I}$, $\mathbf{J}^* = -\mathbf{J}$.

Assume that there exists a fixed point $X^* \in \operatorname{conv}(\mathbb{R}^2)$ for dynamical system (5.1).

Operator $\mathcal{Z}$ has the form

$$\mathcal{Z}H(\theta) = \varphi(s^*)H(\theta + \frac{\pi}{2}) + \Gamma(\theta)\int\limits_0^{2\pi} N(\tau)H(\tau)\,d\tau.$$

Here $\Gamma(\theta) = \varphi'(s^*)H_{X^*}(\theta + \frac{\pi}{2}) + \psi'(s^*)H_B(\theta)$, $N(\theta) = H''_{X^*}(\theta) + H_{X^*}(\theta)$.

Let us define the constants

$$\omega = \varphi(s^*), \quad n_k = \int\limits_0^{2\pi} N(\tau - \frac{k\pi}{2})\Gamma(\tau)\,d\tau, \quad k = 0, 1, 2, 3.$$

Using the recurrence formulas (5.2) we get

$$\mathrm{A}_{14}(\theta) = \Gamma(\theta), \quad \mathrm{A}_{24}(\theta) = \omega\Gamma(\theta + \frac{\pi}{2}),$$

$$\mathrm{A}_{34} = \omega^2\Gamma(\theta + \pi), \quad \mathrm{A}_{44}(\theta) = \omega^3\Gamma(\theta + \frac{3\pi}{2}).$$

$$\mathrm{B}_{14}(\tau) = \omega^3 N(\tau - \frac{3\pi}{2}) + n_0\omega^2 N(\tau - \pi) + \omega(\omega n_1 + n_0^2)N(\tau - \frac{\pi}{2}) + (\omega^2 n_2 + 2\omega n_0 n_1 + n_0^3)N(\tau),$$

$$\mathrm{B}_{24}(\tau) = \omega^2 N(\tau - \pi) + \omega n_0 N(\tau - \frac{\pi}{2}) + (\omega n_1 + n_0^2)N(\tau),$$

$$\mathrm{B}_{34}(\tau) = \omega N(\tau - \frac{\pi}{2}) + n_0 N(\tau), \quad \mathrm{B}_{44}(\tau) = N(\tau).$$

Let define the matrix with elements

$$d_{mn} = \int\limits_0^{2\pi} \mathrm{A}_{m4}(\tau)\mathrm{B}_{m4}(\tau)\,d\tau,$$

then using the simple calculations we obtain

$$d_{11} = \omega^3 n_3 + \omega^2(2n_0 n_2 + n_1^2) + 3n_0^2 n_1\omega + n_0^4,$$

$$d_{12} = \omega^2 n_2 + 2\omega n_0 n_1 + n_0^3, \quad d_{13} = \omega n_1 + n_0^2, \quad d_{14} = n_0,$$

$$d_{21} = \omega[\omega^3 n_0 + \omega^2(n_0 n_3 + 2n_2 n_1) + \omega(n_0^2 n_2 + 2n_0 n_1^2) + n_0^3 n_1],$$
$$d_{22} = \omega[\omega^2 n_3 + \omega(n_0 n_2 + n_1^2) + n_1 n_0^2], \quad d_{23} = \omega[\omega n_2 + n_0 n_1], \quad d_{24} = \omega n_1,$$
$$d_{31} = \omega^2[\omega^3 n_1 + (n_0^2 + n_2^2 + n_1 n_3)\omega^2 + (n_0^2 n_3 + 2n_0 n_1 n_2)\omega + n_0^3 n_2],$$
$$d_{32} = \omega^2[\omega^2 n_0 + (n_0 n_3 + n_1 n_2)\omega + n_0^2 n_2], \quad d_{33} = \omega^2[\omega n_3 + n_0 n_2], \quad d_{34} = \omega^2 n_2,$$
$$d_{41} = \omega^3[\omega^3 n_2 + \omega^2(2n_0 n_1 + n_2 n_3) + (n_0^3 + 2n_0 n_1 n_3)\omega + n_0^3 n_3],$$
$$d_{42} = \omega^3[\omega^2 n_1 + (n_0^2 + n_1 n_3)\omega + n_0^2 n_3], \quad d_{43} = \omega^3[\omega n_0 + n_0 n_3], \quad d_{44} = \omega^3 n_3.$$

Define the characteristic polynomial

$$D(\lambda) = \det[(\lambda - \omega^4)I - D].$$

The conditions of asymptotic stability of a fixed point X^* for considered DDS have a form of the system of inequalities, which is obtained from a well-known criteria for localization of the spectrum of the matrix in the unit circle [12] to which it is necessary to add a condition $\omega < 1$.

Consider the particular case $B = \overline{B}_1(0)$.

Then $X^* = \frac{\psi(s^*)}{1-\varphi(s^*)}\overline{B}_1(0)$ is the fixed point for DDS (5.1). Here s^* is defined from the conditions

$$s^* = \pi\Big(\frac{\psi(s^*)}{1-\varphi(s^*)}\Big)^2, \quad \varphi(s^*) < 1.$$

It is easy to verify that for all $k = 1, 2, 3$

$$n_0 = n_k = 2\pi\Big(\frac{\varphi'(s^*)\psi(s^*)}{1-\varphi(s^*)} + \psi'(s^*)\Big)\frac{\psi(s^*)}{1-\varphi(s^*)} = 2[s^*\varphi'(s^*) + \sqrt{s^*\pi}\psi'(s^*)].$$

Therefore

$$D(\lambda) = (\lambda - \omega^4)^4 - \operatorname{tr} D(\lambda - \omega^4)^3,$$
$$\operatorname{tr} D = n_0[(n_0 + \omega)^3 + \omega(n_0 + \omega)^2 + \omega^2(n_0 + \omega) + \omega^3] = (n_0 + \omega)^4 - \omega^4.$$

Hence, the conditions of the asymptotic stability of the fixed point of X^* are reduced to the inequalities

$$|2(s^*\varphi'(s^*) + \sqrt{s^*\pi}\psi'(s^*)) + \varphi(s^*)| < 1, \quad \varphi(s^*) < 1.$$

Consider a numerical example. Let $B = \overline{B}_1(0) \subset \mathbb{R}^2$, $\varphi(s) = \lambda s$, $\psi(s) = \frac{1}{1+\lambda s}$, then $S[B, \mathbf{J}^i B] = 1$, $i = 0, 1, 2, 3$. The equilibrium value $s^* \geq 0$ is defined from the equation

$$\frac{s^*}{\pi} = \frac{1}{(1 - \lambda^2 s^{*2})^2}, \tag{6.6}$$

and the fixed point $X^* \in \operatorname{conv}(\mathbb{R}^2)$ of the DDS (5.1) has the form

$$X^* = \frac{1}{1-\lambda^2 s^{*2}} \overline{B}_1(0).$$

Since in $\operatorname{conv}(\mathbb{R}^2)$ the operation of multiplication by nonnegative scalar is defined, it follows from the last equality that $\lambda s^* < 1$. Therefore from two branches of the equilibrium curve, which is obtained from the equation (6.6), we have to select a branch, which lies in the region $\lambda s^* < 1$, i.e., the equilibrium curve of dependence $s^* = s^*(\lambda)$ is defined by the equality

$$\lambda^2 = \frac{\sqrt{s^*} - \sqrt{\pi}}{s^{*2}\sqrt{s^*}}.$$

Since $\frac{d\lambda^2}{ds^*} = -2s^{*-3} + 5/2\sqrt{\pi} s^{*-7/2} = 0$ for $\overline{s}^* = (5/4)^2\pi$, the maximum value $\lambda_{\max} = \frac{16}{25\pi\sqrt{5}}$. For any $\lambda \in [0, \lambda_{\max})$ there are two fixed points of the considered dynamical system.

Consider the stability problem of the fixed point X^*. We get

$$n_0 = 2\omega\Big(1 - \frac{1-\omega}{1+\omega}\Big) = \frac{4\omega^2}{1+\omega}, \quad \omega = \lambda s^*.$$

So, the asymptotic stability conditions are reduced to the inequalities

$$\omega < 1, \quad \omega + \frac{4\omega^2}{1+\omega} < 1.$$

The solution of this inequality is $\omega \in [0, \omega^*)$, where $\omega^* = \frac{1}{\sqrt{5}}$. Then all fixed points, the area of which is less than a certain critical value $s^{**} = \frac{25\pi}{16}$ will be asymptotically stable.

7. CONCLUSION

In some cases, the investigation of the stability of the fixed points of the DDS in the semilinear metric space $\operatorname{conv}(\mathbb{R}^n)$ can be carried based on the localization of the roots of a polynomial, exactly the same as in the case of the DDS in the finite-dimensional space. For further investigation is of interest the study of stability of the fixed points and cycles of the DDS in the space $\operatorname{conv}(\mathbb{R}^n)$, and also the questions of universality the behavior of this class of dynamical systems, in the context of the results of M. Feigenbaum [13] and Yakov G.

Sinai [14]. We note also that the obtained results can be applied for the study of the dynamics of discrete controlled systems and systems functioning in the conditions of incomplete information.

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