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A CONNECTION BETWEEN INFINITE MATRIX AND SEMINORM TO ORIGINATE ORLICZ VECTOR VALUED SEQUENCE SPACES AND THEIR STATISTICAL CONVERGENCE

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ABSTRACT: In the present paper we introduce and study some vector valued sequence spaces by using infinite matrix, seminorm and a sequence of Orlicz functions with real n-normed space as base space. We make an effort to study some topological and algebraic properties of these spaces. We also show that these spaces are complete paranormed spaces when the base space is n-Banach space and investigate some inclusion relations between the spaces. Finally, we study statistical convergence of these spaces.

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1. INTRODUCTION AND PRELIMINARIES

The studies on vector valued sequence spaces are done by Rath and Srivastava [25], Das and Choudhary [3], Tripathy and Sen [28], Et et al. [5] and many others. The scope for the studies on sequence spaces was extended on introducing the notion of associated multiplier sequences. Goes and Goes [10] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E with the help of multiplier sequences (k^{-1}) and (k), respectively. The studies on the multiplier sequence spaces are done by Çolak [1] and many others. Recently, Kórus [14] studied some recent results concerning Λ^2-strong convergence numerical sequences. He gave a new appropriate definition for the Λ^2-strong convergence. Moreover, Kórus [15] generalized the results on the L^1-convergence of Fourier series. In [16], he also studied the uniform convergence of mearurable functions by extended results of Móricz [19] and gave examples for appropriate functions.

Let w be the set of all sequences of real or complex numbers and l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $||x||_\infty = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers.

Let q_1 and q_2 be seminorms on a vector space X. Then q_1 is said to be stronger than q_2 if whenever (x_k) is a sequence such that $q_1(x_k) \to 0$, then $q_2(x_k) \to 0$ also. If each is stronger than the others q_1 and q_2 are said to be equivalent (see [29]). Throughout the paper $w(X)$, $c(X)$, $c_0(X)$ and $l_\infty(X)$ will represent the spaces of all, convergent, null and bounded X valued sequence spaces. For $X = \mathbb{C}$, the field of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\theta = (0, 0, \cdots, 0)$, where θ is the zero element of X.

The notion of difference sequence spaces was introduced by Kızmaz [13], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [6] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Recently, Dutta [4] introduced and studied the following difference sequence spaces:

Let n, m be non-negative integers, then for $Z = l_\infty$, c and c_0, we have sequence spaces,

$$Z(\Delta^n_{(m)}) = \{x = (x_k) \in w : (\Delta^n_{(m)} x_k) \in Z\},$$

where $\Delta^n_{(m)} x = (\Delta^n_{(m)} x_k) = (\Delta^{n-1}_{(m)} x_k - \Delta^{n-1}_{(m)} x_{k-m})$ and $\Delta^0_{(m)} x_k = x_k$ for all

$k \in \mathbb{N}$ which is equivalent to the following binomial representation

$$\Delta^n_{(m)} x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k-mv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [13].

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function. A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-*condition* if there exist constants a, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l^1_+$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq K M_k(u) + c_k$$

hold for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$. For more details about sequence spaces (see [20], [22], [23], [24]) and references therein.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{A_n(x)\}$ and the A-transform of x is in Y, where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathrm{N}$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$.

For a sequence space X , the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has been employed by several authors [26].

The concept of 2-normed spaces was initially developed by Gähler [9] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [18]. Since then many others have studied this concept and obtained various results (see [11]). Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ called a n-normed space over the field $\mathbb{R}$.

Example 1.1. We may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$;
2. $p(-x) = p(x)$, for all $x \in X$;
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;
4. if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma \; as \; n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [29], Theorem 10.4.2, P-183).

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars such that $(\alpha_k) \leq 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be symmetric if $(x_n) \in E$ implies $(x_{\pi(n)}) \in E$, where $\pi(n)$ is a permutation of elements of $\mathbb{N}$.

A sequence space E is said to be sequence algebra if $xy \in E$ whenever $x, y \in E$.

Lemma 1.2. *[12] A sequence space E is solid implies E is monotone.*

Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive reals such that $u_k \neq 0$ for all k, X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q_k for each $k \in \mathbb{N}$, $A = (a_{nk})$ be an infinite matrix and $(X, ||\cdot, \cdots, \cdot||)$ is an $n-$normed space. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \ \lambda_1 = 1, \ \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. Then for every $z_1, \cdots, z_{n-1} \in X$, we define the following sequence spaces in the present paper:

$$w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) =$$
$$\Bigg\{x = (x_k) \in w(X) : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \Bigg[q_k\Bigg(\Bigg\|\frac{u_k \Delta^n_{(m)} x_k}{\rho}, z_1, ..., z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k} \to 0$$
$$\text{as } n \to \infty, \text{ for some } \rho > 0\Bigg\},$$

$$w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) =$$
$$\Bigg\{x = (x_k) \in w(X) : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \Bigg[q_k\Bigg(\Bigg\|\frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, ..., z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k} \to 0$$
$$\text{as } n \to \infty, \text{ for some } \rho > 0 \text{ and } L \in X\Bigg\},$$

$$w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) =$$
$$\Bigg\{x = (x_k) \in w(X) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \Bigg[q_k\Bigg(\Bigg\|\frac{u_k \Delta^n_{(m)} x_k}{\rho}, z_1, ..., z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k}$$
$$< \infty, \text{ for some } \rho > 0\Bigg\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$w_0^\lambda(A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) =$$
$$\Bigg\{x = (x_k) \in w(X) : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Bigg[q_k\Bigg(\Bigg\|\frac{u_k \Delta^n_{(m)} x_k}{\rho}, z_1, ..., z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k} \to 0$$

$$\text{as } n \to \infty, \text{ for some } \rho > 0 \Bigg\},$$

$$w^{\lambda}(A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) =$$

$$\Bigg\{ x = (x_k) \in w(X) : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \to 0$$

$$\text{as } n \to \infty, \text{ for some } \rho > 0 \text{ and } L \in X \Bigg\},$$

$$w^{\lambda}_{\infty}(A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) =$$

$$\Bigg\{ x = (x_k) \in w(X) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} < \infty,$$

$$\text{for some } \rho > 0 \Bigg\}.$$

If we take $p = (p_k) = 1$ for all $k \in \mathbb{N}$, we have

$$w^{\lambda}_0(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|)$$

$$\Bigg\{ x = (x_k) \in w(X) : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right] \to 0$$

$$\text{as } n \to \infty, \text{ for some } \rho > 0 \Bigg\},$$

$$w^{\lambda}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|) =$$

$$\Bigg\{ x = (x_k) \in w(X) : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right] \to 0$$

$$\text{as } n \to \infty, \text{ for some } \rho > 0 \text{ and } L \in X \Bigg\},$$

$$w^{\lambda}_{\infty}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|) =$$

$$\Big\{x=(x_k)\in w(X):\sup_n \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\Big[q_k\Big(\Big\|\frac{u_k\Delta^n_{(m)}x_k}{\rho},z_1,...,z_{n-1}\Big\|\Big)\Big]<\infty, \text{ for some } \rho>0\Big\}.$$

If we take $\mathcal{M}(x)=x$ and $u=e=(1,1,1,\cdots,1)$ then these spaces reduces to

$$w_0^\lambda(A,\Delta^n_{(m)},Q,p,\|.,...,.\|)=\Big\{x=(x_k)\in w(X):\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[q_k\Big(\Big\|\frac{\Delta^n_{(m)}x_k}{\rho},z_1,...,z_{n-1}\Big\|\Big)\Big]^{p_k}\to 0 \text{ as } n\to\infty, \text{ for some } \rho>0\Big\},$$

$$w^\lambda(A,\Delta^n_{(m)},Q,u,p,\|.,...,.\|)=\Big\{x=(x_k)\in w(X):\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\Big[q_k\Big(\Big\|\frac{\Delta^n_{(m)}x_k-L}{\rho},z_1,...,z_{n-1}\Big\|\Big)\Big]^{p_k}\to 0 \text{ as } n\to\infty, \text{ for some } \rho>0 \text{ and } L\in X\Big\},$$

$$w_\infty^\lambda(A,\Delta^n_{(m)},Q,u,p,\|.,...,.\|)=\Big\{x=(x_k)\in w(X):\sup_n\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\Big[q_k\Big(\Big\|\frac{\Delta^n_{(m)}x_k}{\rho},z_1,...,z_{n-1}\Big\|\Big)\Big]^{p_k}<\infty, \text{ for some } \rho>0\Big\}.$$

Throughout the paper, we shall use the following inequality. If $0<p_k\leq\sup p_k=H$, $K=\max(1,2^{H-1})$, then

$$|a_k+b_k|^{p_k}\quad\leq\quad K\{|a_k|^{p_k}+|b_k|^{p_k}\} \tag{1.3}$$

for all k and $a_k,b_k\in\mathbb{C}$. Also, $|a|^{p_k}\leq\max(1,|a|^H)$, for all $a\in\mathbb{C}$.

The main purpose of this paper is to introduce vector-valued sequence spaces by using infinite matrix, seminorm and a sequence of Orlicz functions. We show that these spaces are complete paranormed spaces when the base space is n-Banach space and investigate these spaces for solidity, symmetricity, monotonicity and sequence algebras. We also make an effort to obtain some relation between these spaces as well as prove some inclusion results. Finally, we study statistical convergence of these spaces.

2. MAIN RESULTS

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, then the spaces $w_0^{\lambda}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$, $w^{\lambda}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ and $w_{\infty}^{\infty}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ are linear spaces over the real field $\mathbb{R}$.*

Proof. Let $x = (x_k),\ y = (y_k) \in w_0^{\lambda}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho_1}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \to 0 \text{ as } n \to \infty$$

and

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} y_k}{\rho_2}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_k is non-decreasing and convex by using inequality (1.3), we have

$$\begin{aligned}
\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k & \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} (\alpha x_k + \beta y_k)}{\rho_3}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
& \le K \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho_1}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
& \quad + K \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} y_k}{\rho_2}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
& \le K \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho_1}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k}
\end{aligned}$$

$$+ K\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}y_k}{\rho_2}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_k}$$
$$\to 0 \text{ as } n\to\infty.$$

Thus, $\alpha x + \beta y \in w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. Hence, $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ is a linear space. Similarly, we can prove $w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ and $w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ are linear spaces over the real field $\mathbb{R}$. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ is a paranormed space with the paranorm*

$$g(x) = \sum_{i=1}^{m} q(x_i) + \inf\Bigg\{(\rho)^{\frac{p_k}{H}} : \sup_{n\geq 1}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, \textit{ for some } \rho > 0\Bigg\},$$

where $H = \max(1, \sup_k p_k) < \infty$.

Proof. (i) Clearly, $g(x) \geq 0$, for $x = (x_k) \in w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. Since $M_k(\theta) = 0$, we get $g(\theta) = 0$.

(ii) $g(-x) = g(x)$,

(iii) Let $x = (x_k), y = (y_k) \in w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ then, there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_n\left(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right) \leq 1$$

and

$$\sup_n\left(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}y_k}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by Minkowski's inequality, we have

$$\begin{aligned}
&\sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} (x_k + y_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right) \\
\le &\sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} (x_k + y_k)}{\rho_1 + \rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right) \\
\le &\left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right) \\
&+ \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} y_k}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right),
\end{aligned}$$

and thus

$$\begin{aligned}
g(x+y) =& \sum_{i=1}^{m} q(x_i + y_i) + \inf \Bigg\{ (\rho_1 + \rho_2)^{\frac{p_k}{H}} : \\
&\sup_{n \ge 1} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} (x_k + y_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{\frac{1}{H}} \\
&\le 1, \text{ for some } \rho > 0 \Bigg\} \\
\le& \sum_{i=1}^{m} q(x_i) + \inf \Bigg\{ (\rho_1)^{\frac{p_k}{H}} : \\
&\sup_{n \ge 1} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{\frac{1}{H}} \\
&\le 1, \text{ for some } \rho_1 > 0 \Bigg\} \\
&+ \sum_{i=1}^{m} q(y_i) + \inf \Bigg\{ (\rho_2)^{\frac{p_k}{H}} : \\
&\sup_{n \ge 1} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} y_k}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{\frac{1}{H}} \\
&\le 1, \text{ for some } \rho_2 > 0 \Bigg\} \\
\le& g(x) + g(y).
\end{aligned}$$

Finally, we prove that scalar multiplication is continuous. Let λ be any complex number by definition

$$g(\lambda x) = \sum_{i=1}^{m} q(\lambda x_i) + \inf\Bigg\{ (\rho)^{\frac{p_k}{H}} : \sup_{n\geq 1}\Bigg(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\Bigg[q_k\Bigg(\Big\|\frac{u_k \Delta^n_{(m)}(\lambda x_k)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Bigg)\Bigg]^{p_k}\Bigg)^{\frac{1}{H}} \leq 1, \text{ for some } \rho > 0 \Bigg\}$$
$$\leq |\lambda| \sum_{i=1}^{m} q(x_i) + \inf\Bigg\{ (|\lambda| t)^{\frac{p_k}{H}} : \sup_{n\geq 1}\Bigg(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\Bigg[q_k\Bigg(\Big\|\frac{u_k \Delta^n_{(m)} x_k}{t}, z_1, \cdots, z_{n-1}\Big\|\Bigg)\Bigg]^{p_k}\Bigg)^{\frac{1}{H}} \leq 1, \text{ for some } \rho > 0 \Bigg\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda| \sup p_k)$. This completes the proof. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. If $(X, \|., ..., .\|)$ is n-Banach space, then the spaces $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$, $w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ and $w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ are complete paranormed spaces, paranormed defined by g.*

Proof. Suppose (x^n) is a Cauchy sequence in $w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$, where $x^n = (x_k^n)_{k=1}^\infty$ for all $n \in \mathbb{N}$. So that $g(x^i - x^j) \to 0$ as $i, j \to \infty$. Suppose $\epsilon > 0$ is given and let s and x_0 be such that $\frac{\epsilon}{sx_0} > 0$ and $xs > 0$. Since $g(x^i - x^j) \to 0$ as $i, j \to \infty$ which means that there exists $n_0 \in \mathbb{N}$ such that

$$g(x^i - x^j) < \frac{\epsilon}{sx_0}, \text{ for all } i, j \geq n_0.$$

This gives $g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$ and

$$\inf\Bigg\{ (\rho)^{\frac{p_k}{H}} : \sup_{n\geq 1}\Bigg(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\Bigg[q_k\Bigg(\Big\|\frac{u_k \Delta^n_{(m)}(x_k^i - x_k^j)}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Bigg)\Bigg]^{p_k}\Bigg)^{\frac{1}{H}} \leq 1, \text{ for some } \rho > 0 \Bigg\} < \frac{\epsilon}{sx_0}. \quad (2.1)$$

It shows that (x_1^i) is a Cauchy sequence in X. Therefore (x_1^i) is convergent in X because X is complete. Suppose $\lim_{i\to\infty} x_1^i = x_1$ then $\lim_{j\to\infty} g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$, we get

$$g(x_1^i - x_1) < \frac{\epsilon}{sx_0}.$$

Now from (2.1), we have

$$\left(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}(x_k^i - x_k^j)}{g(x^i - x^j)}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right) \le 1.$$

This implies that

$$\left(\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}(x_k^i - x_k^j)}{g(x^i - x^j)}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right) \le 1 \le M_k\left(\frac{sx_0}{2}\right),$$

and thus

$$\|u_k\Delta^n_{(m)}x_k^i - u_k\Delta^n_{(m)}x_k^j, z_1, ..., z_{n-1}\| \le \left(\frac{sx_0}{2}\right)\left(\frac{\epsilon}{sx_0}\right) = \frac{\epsilon}{2}$$

which shows that $(u_k\Delta^n_{(m)}x_k^i)$ is a Cauchy sequence in n-Banach space X for all $k \in \mathbb{N}$. Therefore, $(u_k\Delta^n_{(m)}x_k^i)$ converges in X. Suppose $\lim_{i\to\infty} u_k\Delta^n_{(m)}x_k^i = y_k$ for all $k \in \mathbb{N}$.

Also, we have $\lim_{i\to\infty} \Delta^n_{(m)}x_2^i = y_1 - x_1$. On repeating the same procedure, we obtain $\lim_{i\to\infty} \Delta^n_{(m)}x_{k+1}^i = y_k - x_k$ for all $k \in \mathbb{N}$. Therefore, by continuity of M_k, we get

$$\lim_{j\to\infty}\sup_{n\ge 1}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}(x_k^i - x_k^j)}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \le 1.$$

Let $i \ge n_0$ and taking infimum of each ρ's, we have

$$g(x^i - x) < \epsilon.$$

So $(x^i - x) \in w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. Hence, $x = x^i - (x^i - x) \in w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$, since $w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ is a linear space. Hence, $w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ is a complete paranormed space. Similarly, we can prove the spaces $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ and $w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ are complete paranormed spaces. □

Theorem 2.4. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then*

$$w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) \subseteq w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|).$$

Proof. Let $x = (x_k) \in w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$, then there exists positive number ρ_1 and $z_1, ..., z_{n-1} \in X$ such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Define $\rho = 2\rho_1$. Since M_k is non-decreasing and convex so by using inequality (1.3), we have

$$\begin{aligned}
&\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
&= \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k + L - L}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
&\leq K \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho_1}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
&\quad + K \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} a_{nk} M_k \left[q_k \left(\left\| \frac{L}{\rho_1}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
&\leq K \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho_1}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
&\quad + K \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{L}{\rho_1}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \\
&< \infty.
\end{aligned}$$

Hence, $x = (x_k) \in w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. ☐

Theorem 2.5. *Let $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ and $\mathcal{M} = (M_k), \mathcal{M}' = (M'_k)$ be two sequences of Orlicz functions satisfying Δ_2-condition, then we have*

(i) $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$
$\subset w_0^\lambda(\mathcal{M} \circ \mathcal{M}', A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|);$

(ii) $w^{\lambda}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$
$\subset w^{\lambda}(\mathcal{M} \circ \mathcal{M}', A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$;

(iii) $w^{\lambda}_{\infty}(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$
$\subset w^{\lambda}_{\infty}(\mathcal{M} \circ \mathcal{M}', A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$.

Proof. Let $x = (x_k) \in w^{\lambda}_0(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$, then we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \le t \le \delta$. Let $y_k = a_{nk} M'_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]$ for all $k \in \mathbb{N}$. We can write

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k[y_k]^{p_k} = \frac{1}{\lambda_n}\sum_{k\in I_n, y_k\le\delta} a_{nk} M_k[y_k]^{p_k} + \frac{1}{\lambda_n}\sum_{k\in I_n, y_k>\delta} a_{nk} M_k[y_k]^{p_k}.$$

So we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n, y_k\le\delta} a_{nk} M_k[y_k]^{p_k} \le [M_k(1)]^H \frac{1}{\lambda_n}\sum_{k\in I_n, y_k\le\delta} a_{nk} M_k[y_k]^{p_k} \quad (2.2)$$

$$\le [M_k(2)]^H \frac{1}{\lambda_n}\sum_{k\in I_n, y_k\le\delta} a_{nk} M_k[y_k]^{p_k}.$$

For $y_k > \delta, y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Since $M'_k s$ are non-decreasing and convex, it follows that

$$M_k(y_k) < M_k\Big(1 + \frac{y_k}{\delta}\Big) < \frac{1}{2}M_k(2) + \frac{1}{2}M_k\Big(\frac{2y_k}{\delta}\Big).$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we can write

$$M_k(y_k) < \frac{1}{2}T\frac{y_k}{\delta}M_k(2) + \frac{1}{2}T\frac{y_k}{\delta}M_k(2) = T\frac{y_k}{\delta}M_k(2).$$

Hence,

$$\frac{1}{\lambda_n}\sum_{k\in I_n, y_k>\delta} a_{nk} M_k[y_k]^{p_k}$$
$$\le \max\Big(1, \Big(T\frac{M_k(2)}{\delta}\Big)^H\Big)\frac{1}{\lambda_n}\sum_{k\in I_n, y_k>\delta} a_{nk}[y_k]^{p_k}. \quad (2.3)$$

From equation (2.2) and (2.3) we have $x = (x_k) \in w_0(\mathcal{M}\circ\mathcal{M}', \lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. This completes the proof of (i). Similarly, we can prove that $w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda(\mathcal{M}\circ\mathcal{M}', A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ and $w^\lambda_\infty(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda_\infty(\mathcal{M}\circ\mathcal{M}', A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. □

Theorem 2.6. *Let $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. Then for a sequence of Orlicz functions $\mathcal{M} = (M_k)$ which satisfies Δ_2-condition, we have*

(i) $w_0(\lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda_0(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$;

(ii) $w(\lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$;

(iii) $w_\infty(\lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda_\infty(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$.

Proof. It is easy to prove so we omit the details. □

Theorem 2.7. *Let $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. Then for a sequence of Orlicz functions $\mathcal{M} = (M_k)$ which satisfies Δ_2-condition, we have*

(i) $w^\lambda_0(\mathcal{M}, A, \Delta^{n-1}_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda_0(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$;

(ii) $w^\lambda(\mathcal{M}, A, \Delta^{n-1}_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$;

(iii) $w^\lambda_\infty(\mathcal{M}, A, \Delta^{n-1}_{(m)}, Q, u, p, \|., ..., .\|) \subset w^\lambda_\infty(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$.

Proof. Here we prove the result for $w^\lambda_0(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ and for other cases it will follow on applying similar arguments. Let $x = (x_k) \in w^\lambda_0(\mathcal{M}, A, \Delta^{n-1}_{(m)}, Q, u, p, \|., ..., .\|)$. Then there exists $\rho > 0$ and $z_1, ..., z_{n-1} \in X$ such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\left[q_k\left(\left\|\frac{u_k\Delta^{n-1}_{(m)}x_k}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \to 0 \text{ as } n \to \infty. \quad (2.4)$$

On considering 2ρ, by the convexity of Orlicz function we have

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\left[q_k\left(\left\|\frac{u_k\Delta^{n}_{(m)}x_k}{2\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] \\
&\qquad \le \frac{1}{2}\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\left[q_k\left(\left\|\frac{u_k\Delta^{n-1}_{(m)}x_k}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] \\
&\qquad\quad + \frac{1}{2}\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk} M_k\left[q_k\left(\left\|\frac{u_k\Delta^{n-1}_{(m)}x_{k-m}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\tfrac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{2\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \\
&\quad \le \quad K\left\{\frac{1}{2}\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right. \\
&\qquad\qquad \left. +\frac{1}{2}\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right\}.
\end{aligned}$$

Then using (2.4), we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{2\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \to 0 \text{ as } n\to\infty.$$

Thus, $w_0^\lambda(\mathcal{M}, A, \Delta^{n-1}_{(m)}, Q, u, p, \|., ..., .\|) \subset w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. □

Theorem 2.8. *Let $0 \le p_k \le s_k$ for all k and let $(\frac{s_k}{p_k})$ be bounded. Then*

$$w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, s, \|., ..., .\|) \subseteq w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|).$$

Proof. Let $x = (x_k) \in w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, s, \|., ..., .\|)$, write

$$t_k = a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{s_k}$$

and $\mu_k = \frac{p_k}{s_k}$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \le 1$ for all $k \in \mathbb{N}$. Take $0 < \mu \le \mu_k$ for $k \in \mathbb{N}$. Define sequences (v_k) and (w_k) as follows:

For $t_k \ge 1$, let $v_k = t_k$ and $w_k = 0$ and for $t_k < 1$, let $v_k = 0$ and $w_k = t_k$. Then clearly for all $k \in \mathbb{N}$, we have

$$t_k = v_k + w_k, \quad t_k^{\mu_k} = v_k^{\mu_k} + w_k^{\mu_k}.$$

Now it follows that $v_k^{\mu_k} \le v_k \le t_k$ and $w_k^{\mu_k} \le w_k^{\mu}$. Therefore,

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n} t_k^{\mu_k} &= \frac{1}{\lambda_n}\sum_{k\in I_n}(v_k^{\mu_k} + w_k^{\mu_k}) \\
&\le \frac{1}{\lambda_n}\sum_{k\in I_n} t_k + \frac{1}{\lambda_n}\sum_{k\in I_n} w_k^{\mu}.
\end{aligned}$$

Now for each k,

$$\frac{1}{\lambda_n}\sum_{k\in I_n} w_k^{\mu} = \sum_{k\in I_n}\left(\frac{1}{\lambda_n}w_k\right)^{\mu}\left(\frac{1}{\lambda_n}\right)^{1-\mu}$$

$$\leq \Big(\sum_{k\in\lambda_n}\Big[\Big(\frac{1}{\lambda_n}w_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{k\in I_n}\Big[\Big(\frac{1}{\lambda_n}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu}$$
$$= \Big(\frac{1}{\lambda_n}\sum_{k\in I_n} w_k\Big)^{\mu}$$

and so
$$\frac{1}{\lambda_n}\sum_{k\in I_n} t_k^{\mu_k} \leq \frac{1}{\lambda_n}\sum_{k\in I_n} t_k + \Big(\frac{1}{\lambda_n}\sum_{k\in I_n} w_k\Big)^{\mu}.$$

Hence, $x=(x_k)\in w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,p,\|.,...,.\|)$. This completes the proof of the theorem. □

Theorem 2.9. *(i) If* $0<\inf p_k \leq p_k \leq 1$ *for all* $k\in\mathbb{N}$, *then*

$$w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,p,\|.,...,.\|) \subseteq w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,\|.,...,.\|).$$

(ii) If $1\leq p_k \leq \sup p_k = H < \infty$, *for all* $k\in\mathbb{N}$, *then*

$$w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,\|.,...,.\|) \subseteq w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,p,\|.,...,.\|).$$

Proof. (i) Let $x=(x_k)\in w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,p,\|.,...,.\|)$, then

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\Bigg[q_k\Bigg(\Bigg\|\frac{u_k\Delta^n_{(m)}x_k-L}{\rho},z_1,...,z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k}\to 0 \text{ as } n\to\infty.$$

Since $0<\inf p_k\leq p_k\leq 1$. This implies that
$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\Bigg[q_k\Bigg(\Bigg\|\frac{u_k\Delta^n_{(m)}x_k-L}{\rho},z_1,...,z_{n-1}\Bigg\|\Bigg)\Bigg]$$
$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\Bigg[q_k\Bigg(\Bigg\|\frac{u_k\Delta^n_{(m)}x_k-L}{\rho},z_1,...,z_{n-1}\Bigg\|\Bigg)\Bigg]^{p_k}.$$

Thus, $x=(x_k)\in w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,\|.,...,.\|)$

(ii) Let $p_k\geq 1$ for each k and $\sup_k p_k<\infty$. Let $x=(x_k)\in w^{\lambda}(\mathcal{M},A,\Delta^n_{(m)},Q,u,\|.,...,.\|)$. Then for each $0<\epsilon<1$ there exists a positive integer N and $z_1,...,z_{n-1}\in X$ such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\Bigg[q_k\Bigg(\Bigg\|\frac{u_k\Delta^n_{(m)}x_k-L}{\rho},z_1,...,z_{n-1}\Bigg\|\Bigg)\Bigg]\leq\epsilon<1 \text{ for all } n\geq N.$$

This implies that

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k - L}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_k}$$
$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}M_k\left[q_k\left(\left\|\frac{u_k\Delta^n_{(m)}x_k - L}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right].$$

Therefore, $x = (x_k) \in w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. This completes the proof. □

Theorem 2.10. *If $0 < \inf p_k \leq p_k \leq \sup p_k = H < \infty$, for all $k \in \mathbb{N}$, then*

$$w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) = w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|).$$

Proof. It is easy to prove so we omit the details. □

Theorem 2.11. *The spaces $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|)$, $w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|)$ and $w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|)$ are not monotone and as such are not solid in general.*

The proof follows from the following example.

Example 2.12. Let $n = 2$, $m = 3$, $p_k = 1$ for all k odd and $p_k = 2$ for all k even, $u_k = 1$, $q_k = |x|$, $\lambda_n = (1, 2, ..., n)$ for all $n \in \mathbb{N}$ and $M_k(x) = x^2$ for all $k \geq 1$ and for all $x \in [0, \infty)$. Consider the n-normed space as defined in Example 1.1. Then $\Delta^2_{(3)} = x_k - 2x_{k-3} + x_{k-6}$ for all $k \in \mathbb{N}$. Consider the J^{th} step space of a sequence space E defined as, for (x_k), $(y_k) \in E^J$ implies that $y_k = x_k$ for all k odd and $y_k = 0$ for k even. Consider $x = k$. Then $x \in w^\lambda(\mathcal{M}, A, \Delta^2_{(3)}, Q, u, \|., ..., .\|)$ but its J^{th} canonical pre-image does not belong to $w^\lambda(\mathcal{M}, A, \Delta^2_{(3)}, Q, u, \|., ..., .\|)$. Hence, the space is not monotone and as such are not solid in general. Similarly, for the other spaces.

Theorem 2.13. *The spaces $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|)$, $w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|)$ and $w_\infty^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, \|., ..., .\|)$ are not symmetric in general.*

To show that the spaces are not symmetric in general, consider the following example.

Example 2.14. Let $n = 2$, $m = 2$, $p_k = 2$ for all k odd and $p_k = 3$ for all k even, $u_k = 1$, $q_k = |x|$, $\lambda_n = (1, 2, ..., n)$ for all $n \in \mathbb{N}$ and $M_k(x) = x^2$

for all $k \geq 1$ and for all $x \in [0, \infty)$. Consider the n-normed space as defined in Example 1.1. Then $\Delta^2_{(2)} = x_k - 2x_{k-2} + x_{k-4}$ for all $k \in \mathbb{N}$. Consider the sequences $x = (x_k)$ defined as $x_k = k$ for k odd and $x_k = 0$ for k even. Then $\Delta^2_{(2)} = 0$, for all $k \in \mathbb{N}$. Hence, $(x_k) \in w^\lambda(\mathcal{M}, A, \Delta^2_{(2)}, Q, u, p, \|., ..., .\|)$. Consider the rearranged sequence, (y_k) of (x_k) defined as

$$(y_k) = (x_1, x_3, x_2, x_4, x_5, x_7, x_6, x_8, x_9, x_11, x_10, x_12, ...).$$

Then $(y_k) \notin w^\lambda(\mathcal{M}, A, \Delta^2_{(2)}, Q, u, p, \|., ..., .\|)$. Hence, $w^\lambda(\mathcal{M}, A, \Delta^2_{(2)}, Q, u, p, \|., ..., .\|)$ is not symmetric in general. Similarly, we can prove for others spaces.

Theorem 2.15. *The following spaces are not sequence algebras in general:* $w_0^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$, $w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$ *and* $w_\infty(\mathcal{M}, \lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$.

The proof follows from the following example.

Example 2.16. Let $n = 2$, $m = 1$, $p_k = 3$ for all k, $u_k = 1$, $q_k = |x|$, $\lambda_n = (1, 2, ..., n)$ for all $n \in \mathbb{N}$ and $M_k(x) = x^7$ for $k \in \mathbb{N}$ and for all $x \in [0, \infty)$. Consider the n-normed space as defined in Example 1.1. Then $\Delta^2_{(1)} = x_k - 2x_{k-1} + x_{k-2}$ for all $k \in \mathbb{N}$. Let $(x_k) = (k)$ and $(y_k) = (k)$ defined as $x_k = k$ for k odd and $x_k = 0$ for k even. Then $\Delta^2_{(1)} = 0$, for all $k \in \mathbb{N}$. Then $x, y \in w_0^\lambda(\mathcal{M}, A, \Delta^2_{(1)}, Q, u, \|., ..., .\|)$ but $\notin w^\lambda(\mathcal{M}, A, \Delta^2_{(1)}, Q, u, \|., ..., .\|)$ and $w_\infty^\lambda(\mathcal{M}, A, \Delta^2_{(1)}, Q, u, \|., ..., .\|)$. Hence, the space $w_0^\lambda(\mathcal{M}, A, \Delta^2_{(1)}, Q, u, p, \|., ..., .\|)$ is not sequence algebras in general.

3. STATISTICAL CONVERGENCE

The notion of statistical convergence was introduced by Fast [7] and Schoenberg [27] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [8], Connor [2], Mursaleen et al. [21] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists:

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

In this section we introduce $\Delta^n_{(m)}(\lambda, u_q, \|., ..., .\|)$-statistical convergent sequences and give some relations between $\Delta^n_{(m)}(\lambda, u_q, \|., ..., .\|)$-statistical convergent sequences and $w(\mathcal{M}, \lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$-summable sequences. A sequence $x = (x_k)$ is said to be $\Delta^n_{(m)}(\lambda, u_q, \|., ..., .\|)$-statistically convergent to L, if for every $\epsilon > 0$, $\delta > 0$ and $z_1, \cdots, z_{n-1} \in X$,

$$\lim_n \frac{1}{\lambda_n} \Big| \Big\{ k \in I_n : a_{nk}\Big(q_k \Big\| u_k \Delta^n_{(m)} x_k - L, z_1, \cdots, z_{n-1} \Big\|\Big) \geq \epsilon \Big\} \Big| = 0.$$

In this case we write $x_n \to L\Big(S_\lambda\big(\Delta^n_{(m)}, u_q, \|., ..., .\|\big)\Big)$. The set of all $\Delta^n_{(m)}(\lambda, u_q, \|., ..., .\|)$-statistically convergent sequences is denoted by $S_\lambda\big(\Delta^n_{(m)}, u_q, \|., ..., .\|\big)$.

Theorem 3.1. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. Then $w(\mathcal{M}, \lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|) \subset S_\lambda\big(\Delta^n_{(m)}, u_q, \|., ..., .\|\big)$.*

Proof. Suppose $x = (x_k) \in w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. Take $\epsilon > 0$ and $\epsilon_1 = \frac{\epsilon}{\rho}$. Then for each $z_1, \cdots, z_{n-1} \in X$, we obtain

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k}$$

$$= \frac{1}{\lambda_n} \sum_{k \in I_n \,\&\, \left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$+ \frac{1}{\lambda_n} \sum_{k \in I_n \,\&\, \left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| < \epsilon} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$\geq \frac{1}{\lambda_n} \sum_{k\in I_n \& \left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$\geq \frac{1}{\lambda_n} \sum_{k\in I_n} [M_k(\epsilon_1)]^{p_k} \geq \frac{1}{\lambda_n} \sum_{k\in I_n} \min([M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H)$$

$$\geq \frac{1}{\lambda_n} \left| \left\{ k \in I_n : a_{nk} \left(q_k \left\| u_k \Delta^n_{(m)} x_k - L, z_1, \cdots, z_{n-1} \right\| \right) \geq \epsilon \right\} \right| \min([M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H).$$

Hence, $x \in S_\lambda\big(\Delta^n_{(m)}, u_q, \|., ..., .\|\big)$ □

Theorem 3.2. *Let $\mathcal{M} = (M_k)$ be a bounded sequence of Orlicz functions and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_\lambda\big(\Delta^n_{(m)}, u_q, \|., ..., .\|\big) \subset w(\mathcal{M}, \lambda, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$.*

Proof. Suppose that $\mathcal{M} = (M_k)$ is bounded. Then there exists an integer K such that $M_k(t) < K$, for all $t \geq 0$. Take $\epsilon > 0$ and $\epsilon_1 = \frac{\epsilon}{\rho}$. Then

$$\frac{1}{\lambda_n} \sum_{k\in I_n} a_{nk} M_k \left[q_k \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k}$$

$$= \frac{1}{\lambda_n} \sum_{k\in I_n \& \left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon} a_{nk} M_k \left[q \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$+ \frac{1}{\lambda_n} \sum_{k\in I_n \& \left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| < \epsilon} a_{nk} M_k \left[q \left(\left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k}$$

$$\leq \frac{1}{\lambda_n} \sum_{k\in I_n \& \left\| \frac{u_k \Delta^n_{(m)} x_k - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \geq \epsilon} \max\left\{ \left(\frac{K}{\rho}\right)^h, \left(\frac{K}{\rho}\right)^H \right\} + \frac{1}{\lambda_n} \sum_{k\in I_n} [M_k(\epsilon_1)]^{p_k}$$

$$\leq \max(T^h, T^H) \frac{1}{\lambda_n} \left| \left\{ k \in I_n : a_{nk} \left(\left\| u_k \Delta^n_{(m)} x_k - L, z_1, \cdots, z_{n-1} \right\| \right) \geq \epsilon \right\} \right|$$

$$+ \max([M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H), \quad \text{where } T = \frac{K}{\rho}.$$

Hence, $x \in w^\lambda(\mathcal{M}, A, \Delta^n_{(m)}, Q, u, p, \|., ..., .\|)$. □

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