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EXISTENCE THEOREM FOR LINEAR NEUTRAL IMPULSIVE DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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ABSTRACT: In this paper, we consider the second order linear neutral impulsive differential equation of the form

$$\begin{cases} [y(t) - py(t-\tau)]'' + q(t)\,y(g(t)) = 0, & t \neq t_k, \\ \Delta [y(t_k) - py(t_k - \tau)]' + q_k y(g(t_k)) = 0, & \forall t = t_k, \end{cases}$$

where $p \in R$, $q_k \geq 0$, $q \in PC([t_0, \infty), R_+)$, $g \in C([t_0, \infty), R)$, $\lim_{t\to\infty} g(t) = \infty$, $\tau > 0$.

We establish conditions for the existence of a positive solution with asymptotic decay by defining a map which satisfies the assumptions of Krasnoselkii's fixed point theorem.

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1. INTRODUCTION

The asymptotic behaviour and existence of positive solutions of neutral impulsive differential equations are of both theoretical and practical importance (see [16], [9], [7], [8]). One of the major reasons may be due to the fact that equations of this type abound in networks containing lossless transmission lines. Such networks are found in high speed computers where, lossless transmission lines are used to interconnect switching circuits [11]. We must also acknowledge the role of such equations in the motion of radiating electrons, population growth, and the spread of epidemics, to mention just a few [6]. Lately, the pioneering efforts of Isaac and Lipcsey (see [10], [11], [12]) in identifying some of the essential oscillatory and non-oscillatory conditions of neutral impulsive differential equations of the first order is also worth commending. However, relatively less attention has been given to oscillations/non-oscillations of second-order neutral delay differential equations with impulses. The dearth of results in this area is attested to in the following articles (see [3], [15], [2], [14]). In this paper, we seek to fill this gap by establishing conditions for the existence of a positive solution of a class of second order neutral delay impulsive differential equations. Before the formulation of the problem considered in this study, we present some basic definitions and concepts that will be useful in our discussions throughout.

Notation 1.1. Let $J = (\alpha, \beta) \subset R, -\infty < \alpha < \beta < +\infty$ be our domain of investigation.

Definition 1.1. Let $S := \{t_k\}_{k \in E} \subset J$ be a strictly ascending sequence of the time moments of impulse effects and let E be a subscript set which can be the set of natural numbers N or the set of integers Z such that

- $t_k \to \infty$ if $k \to \infty$ and if $E = Z$, then $t_k \to -\infty$ if $k \to -\infty$;

- $t_k \geq 0$ if $k \geq 0$.

The equation under consideration then has the form

$$\begin{cases} [y(t) - py(t-\tau)]'' + q(t)\, y(g(t)) = 0, & t \geq t_0,\ t \in J \backslash S, \\ \Delta [y(t_k) - py(t_k - \tau)]' + q_k y(g(t_k)) = 0, t_k \geq t_0, & \forall t_k \in S, \end{cases} \tag{1.1}$$

where $1 \leq k \leq \infty$.

In order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows: Let $D := [T, \overline{T}) \subset J \subset R$ and let the set of impulse points S be fixed.

Definition 1.2. Let

$$PC(D, \mathrm{R}) := \{\varphi | \varphi : D \to \mathrm{R}, \varphi \in C(D \backslash S), \exists \varphi(t-0), \varphi(t+0), \forall t \in D\}.$$

From the studies in Bainov and Simeonov, Lakshmikantham et al and Isaac et al (see [1], [4], [13], [12]), we define the function space $\forall r \in N$:

Definition 1.3. Let

$$PC^r(D, R) := \left\{ \varphi | \varphi \in PC(D, R), \frac{d^j \varphi}{dt^j} \in PC(D, R), \forall 1 \leq j \leq r \right\}.$$

To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D, R; S)$ and $PC^r(D, R; S)$, $r \in N$.

Let (P) be some property of the solution of $y(t)$ of an impulsive differential equation, which can be fulfilled for some $t \in R$. Hereafter, we shall say that the function $y(t)$ enjoys the property (P) *finally*, if there exists $T \in R$ such that $y(t)$ enjoys the property (P) for all $t \geq T$, see [1].

Definition 1.4. The solution $y(t)$ of an impulsive differential equation is said to be:

i) finally positive (finally negative) if there exist $T \geq 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t \geq T$, see [10];

ii) non-oscillatory, if it is either finally positive or finally negative; and

iii) oscillatory, if it is neither finally positive nor finally negative (see [1], [11]).

2. STATEMENT OF THE PROBLEM

The equation being investigated is the second-order linear neutral delay impulsive differential equation of the form

$$\begin{cases} [y(t) - py(t-\tau)]'' + q(t)y(g(t)) = 0, & t \geq t_0,\ t \notin S, \\ \Delta\,[y(t_k) - py(t_k-\tau)]' + q(t_k)y(g(t_k)) = 0, & t_k \geq t_0,\ t_k \in S, \end{cases} \tag{2.1}$$

where $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim\limits_{k\to+\infty} t_k = +\infty$, $\Delta y^{(i)}(t_k) = y^{(i)}(t_k^+) - y^{(i)}(t_k^-)$,$i = 0, 1$ and $y(t_k^-), y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. For the sake of definiteness, we shall suppose that the functions $y(t)$ and $y'(t)$ are continuous from the left at the points t_k such that $y'(t_k^-) = y'(t_k), y(t_k^-) = y(t_k)$. Without further mentioning, we will assume throughout this paper that the conditions $p \in R$, $q_k \geq 0$, $q \in PC\left([t_0,\infty), R_+\right)$, $g \in C\left([t_0,\infty), R\right)$, $\lim\limits_{t\to\infty} g(t) = \infty$, $\tau > 0$ are fulfilled.

The system in equation (2.1) is a differential equation together with its impulsive conditions in which the second order derivative of the unknown function appears in the equation both with and without delay.

Definition 2.1. Let $\varphi(t) \in PC([t_0-\rho, t_0], R)$ be a given function, where $\rho = \max\{\tau, g(t_0)\}$ and let $y_0 \in R$. The function $y(t) \in PC([t_0-\rho,\infty), R)$ is said to be a unique solution of equation (2.1) if

$$y(t) = \varphi(t), t \in [t_0-\rho, t_0], \quad [y(t) + py(t-\tau)]''\Big|_{t=t_0} = y_0,$$

the function $y(t) + py(t-\tau)$ is twice piece-wise continuously differentiable for $t \geq t_0$ and $y(t)$ satisfies equation (2.1) for all $t \geq t_0$.

We feel it necessary at this point to remark that every solution $y(t)$ of equation (2.1) that is under consideration here continuous from the left and is nontrivial. That is, $y(t)$ is defined on some half-line $[T_y,\infty)$ and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. Such a solution is called a regular solution of equation (2.1).

Definition 2.2. Equation (2.1) is said to be oscillatory if all its solutions are oscillatory.

Remark 2.1. All functional inequalities considered in this paper are assumed to hold finally, that is they are satisfied for all t large enough.

The intention of this study is to obtain sufficient conditions that guarantee the existence of a positive solution $y(t)$ of equation (2.1). But before we continue, we proceed to present a result which will be useful in the discussion of the main results of this paper.

Consider the second order nonlinear neutral impulsive differential equation of the form

$$\begin{cases} \left[y(t) - \sum_{i=1}^{m} p_i(t)\, y(t-\tau_i)\right]'' + \sum_{j=1}^{n} f_j\left(t, y\left(g_{jl}(t), \cdots, y\left(g_{jl}\right)\right)\right) = 0, \\ t \geq t_0 \in R_+,\ t \notin S; \\ \Delta \left[y(t_k) - \sum_{i=1}^{m} p_{ik} y(t_k - \tau_i)\right]' \\ + \sum_{j=1}^{n} f_{jk}\left(t_k, y\left(g_{jl}(t_k), \cdots, y\left(g_{jl}(t_k)\right)\right)\right) = 0, \\ t_k \geq t_0 \in R_+,\ \forall t_k \in S, \end{cases} \tag{2.2}$$

subject to the following conditions:

C1: $\tau_i > 0,\ p_{ik} \geq 0, p_i \in PC^1\left([t_0, \infty), R_+\right), i = 1, 2, \cdots, m$ and there exists $\delta \in (0, 1]$ such that

$$\sum_{i=1}^{m} p_i(t) + \sum_{j=1}^{n} p_j \leq 1 - \delta, t \geq t_0 \in R_+;$$

C2: $g_{js} \in C\left([t_0, \infty), R\right), \lim_{t\to\infty} g_{js}(t) = \infty, j = 1, 2, \cdots, n, s = 1, 2, \cdots, \ell;$

C3: $f_j \in PC\left([t_0, \infty) \times R^\ell, R\right), x_1 f_j(t, x_1, \cdots, x_\ell) > 0;$

$$x_1 f_{jk}(t_k, x_1, \cdots, x_l) > 0 for x_1 x_i > 0, i = 1, 2, \cdots, \ell, j = 1, 2, \cdots, n.$$

Moreover,

$$\begin{cases} \left|f_j(t, y_1, \cdots, y_\ell)\right| \geq \left|f_j(t, x_1, \cdots, x_\ell)\right|, \\ \left|f_{jk}(t_k, y_1 \cdots, y_\ell)\right| \geq \left|f_{jk}(t_k, x_1, \cdots, x_\ell)\right|, \end{cases}$$

whenever

$$|x_i| \leq |y_i| \text{ and } y_i x_i > 0,\ i = 1, 2, \cdots, \ell,\ j = 1, 2, \cdots, n;$$

C4: Set

$$x(t) = y(t) - \sum_{i=1}^{m} p_i(t)\, y(t-\tau_i).$$

Theorem 2.1. (Extension of Theorem 4.5.5 in [4]) *Assume that*

$$\lim_{t\to\infty} \sum_{i=1}^{m} p_i(t) + \lim_{t_k\to\infty} \sum_{i=1}^{m} p_{ik} = p \in [0,1).$$

Further assume that

$$\int_{t_0}^{\infty} \left| \sum_{j=1}^{n} f_j\left(u, d_1 g_{ji}(u), ..., d_1 g_{j\ell}(u)\right) \right| du + \sum_{t_0\le t_k<\infty} \left| \sum_{j=1}^{n} f_{jk}\left(t_k, d_1 g_{j1}(t_k), ..., d_1 g_{j\ell}(t_k)\right) \right| < \infty, \quad (2.3)$$

for some $d_1 \neq 0$ *and*

$$\int_{t_0}^{\infty} u \left| \sum_{j=1}^{n} f_j(u, b_1, ..., b_1) \right| du + \sum_{t_0\le t_k \infty} t_k \left| \sum_{j=1}^{n} f_{jk}(t_k, b_1, ..., b_1) \right| = \infty \quad (2.4)$$

for some $d_1 \neq 0$*, where* $b_1 d_1 > 0$*. Then equation* (2.2) *has a non-oscillatory solution* $y(t)$.

Example 2.1. Consider

$$\begin{cases} \left[y(t) - \frac{1}{2} y(t-1)\right]'' + \frac{2(t-1)^3 - t^3}{(t-1)^6} y^3(t) = 0, \\ \Delta \left[y(t_k) - \frac{1}{2} y(t_k - 1)\right]' + \frac{2(t_k-1)^3 - t_k^3}{(t_k-1)^6} y^3(t_k) = 0, \end{cases} \quad (2.5)$$

where $q(t) = \frac{2(t-1)^3 - t^3}{(t-1)^6}$ and $q_k = \frac{2(t_k-1)^3 - t_k^3}{(t_k-1)^6}$. It is obvious that the inequality

$$\int_{t_0}^{\infty} u \left| \sum_{j=1}^{n} f_j(u, b_1, ..., b_1) \right| du + \sum_{t_0\le t_k<\infty} t_k \left| \sum_{j=1}^{n} f_{jk}(t_k, b_1, ..., b_1) \right| < \infty,$$

for $b_1 \neq 0$ holds. Therefore, equation (2.5) has a non-oscillatory solution. In fact, $y(t) = 1 - \frac{1}{t}$ is such a solution.

Example 2.2. Consider

$$\begin{cases} \left[y(t) - \frac{1}{2}y(t-1)\right]'' + q(t)\, y^{\frac{1}{3}}(t) = 0, \\ \Delta \left[y(t_k) - \frac{1}{2}y(t_k-1)\right]' + q_k y^{\frac{1}{3}}(t_k) = 0, \end{cases} \tag{2.6}$$

where

$$q(t) = \frac{1}{4}\left(t^{3/2} - \tfrac{1}{2}(t-1)^{-3/2}\right)t^{-1/6}, q_k = \frac{1}{4}\left(t_k^{-3/2} - \tfrac{1}{2}(t_k-1)^{-3/2}\right)t_k^{-1/6}.$$

For large t and t_k, $q(t) \sim Mt^{-5/3}$ and $q_k \sim Mt_k^{-5/3}$. It is obvious that conditions (2.3) and (2.4) are satisfied. From Theorem 2.1, equation (2.6) has a non-oscillatory solution $y(t)$. In fact, $y(t) = \sqrt{t}$ is such a solution of equation (2.6).

Theorem 2.2. (Krasnoselskii's Fixed Point Theorem) *Let X be a Banach space, Ω a bounded closed convex subset of X and A, B be maps of Ω into X such that $Ax + By \in \Omega$ for every pair $x, y \in \Omega$. If A is a contraction and B is completely continuous, then the equation $Ax + Bx = x$ has a solution in Ω.*

We are now ready to establish the main results of this investigation.

3. MAIN RESULTS

The following theorem and corollary are extensions of their neutral delay version as identified on page 253 of the monograph by Erbe *et al* [4].

Theorem 3.1. *Assume that*

i) $p, \tau > 0$, $q_k \geq 0$, $q \in PC([t_0, \infty), R_+)$, $g \in C([t_0, \infty), R)$, $g(t+\tau) < t$, $q(t) \geq 0$, $t \geq t_0$ *and* $\lim_{t\to\infty} g(t) = \infty$;

ii) *There exists a constant $\alpha > 0$ such that for sufficiently large t,*

$$\frac{1}{p}e^{-\alpha\tau} + \frac{1}{p}\int_{t+\tau}^{\infty}(s-t-\tau)\, q(s) \exp\left[\alpha(t-g(s))\right] ds \\ + \frac{1}{p}\sum_{t+\tau \leq t_k < \infty}(t_k - t - \tau)\, q_k \exp\left[\alpha(t - g(t_k))\right] \leq 1. \tag{3.1}$$

Then equation (2.1) *has a positive solution* $y(t)$ *that converges to zero as* t *tends to infinity.*

Proof. If the equality in equation (3.1) holds finally, then we can verify that $y\,(t) = e^{-\alpha t}$ is the expected solution. Otherwise, we assume that there exists $T \geq t_0$ such that $t+\tau \geq 0, g\,(t+\tau) \geq t_0$ for $t \geq T$, and

$$\mu = \frac{1}{p}e^{-\alpha\tau} + \frac{1}{p}\int_{T+t}^{\infty}(s-T-\tau)\,q\,(s)\exp\left[\alpha\,(T-g\,(s))\right]ds + \frac{1}{p}\sum_{\mathrm{T}+\tau\leq t_k<\infty}(t_k-T-\tau)\,q_k\exp\left[\alpha\,(T-g\,(t_k))\right] < 1 \quad (3.2)$$

and inequality (3.1) holds for $t \geq T$.

Let B_p denote the Banach space of all piece-wise continuous bounded functions defined on $[t_0, \infty)$ endowed with a sup norm. Let Ω be the subset of $B_p$defined by

$$\Omega = \left\{x \in B_p : 0 \leq x\,(t) \leq 1 fort \geq t_0\right\}.$$

Define a map $J : \Omega \to B_p$ as follows:

$$(Jx)\,(t) = (J_1x)\,(t) + (J_2x)\,(t)\,,$$

where

$$(J_1x)(t) = \begin{cases} \frac{1}{p}e^{-\alpha\tau}x\,(t+\tau)\,, & t \geq T, \\ (J_1x)\,(T) + \exp\left[\varepsilon\,(T-t)\right] - 1, & t_0 \leq t \leq T, \end{cases}$$

and

$$(J_2x)\,(t) = \begin{cases} \frac{1}{p}\int_{t+\tau}^{\infty}(s-t-\tau)\,q\,(s)\exp\left[\alpha\,(t-g\,(s))\right]x\,(g\,(s))\,ds \\ +\frac{1}{p}\sum_{t+\tau\leq t_k<\infty}(t_k-t-\tau)\,q_k\exp\left[\alpha\,(t-g\,(t_k))\right]x\,(g\,(t_k))\,, & t \geq T, \\ (J_2x)\,(T)\,, & t_0 \leq t \leq T, \end{cases}$$

where

$$\varepsilon = \frac{\ln\,(2-\mu)}{(T-t_0)}.$$

We can show that the map J satisfies all the assumptions of Krasnoselskii's fixed point theorem, and so J has a fixed point $x \in \Omega$. Clearly, $x(t) > 0$ for $t \geq t_0$. Consequently, it is easy to see that

$$y(t) = x(t)\, e^{-\alpha t}$$

is a positive solution of equation (2.1) that converges to zero as $t \to \infty$. This completes the proof of Theorem 3.1.

Corollary 3.1. *Assume that* $0 < p < 1$, $\tau > 0$ *and there exist constants* $q*, \sigma > \tau$ *such that* $0 \leq q(t) \leq q^*$, $g(t) \geq t - \sigma$ *finally. If the "majorant" equation*

$$\begin{cases} [y(t) - py(t-\tau)]'' + q^* y(t-\sigma) = 0, t \geq t_0, & t \notin S \\ \Delta [y(t_k) - py(t_k-\tau)]' + q_k^* y(t_k-\sigma) = 0, t_k \geq t_0, & \forall t_k \in S \end{cases} \tag{3.3}$$

has a positive solution, then equation (2.1) also has a positive solution.

Proof: Equation (3.3) has a positive solution if and only if the characteristic equation

$$F(\lambda) \equiv \lambda^2 \left(1 - pe^{-\lambda\tau}\left(1 + \frac{q_k^*}{q^*}\lambda\right)^{n_1}\right) + q^* e^{-\lambda\sigma}\left(1 + \frac{q_k^*}{q^*}\lambda\right)^{n_2} = 0$$

has a real root α. Clearly, α must be negative. Let $\gamma = -\lambda > 0$, then

$$1 - pe^{\gamma\tau}\left(1 - \frac{q_k^*}{q^*}\gamma\right)^{n_1} + \frac{q^*}{\gamma^2}e^{\gamma\tau}\left(1 - \frac{q_k^*}{q^*}\gamma\right)^{n_2} = 0$$

or

$$\frac{1}{p}e^{-\gamma\tau}\left(1 - \frac{q_k^*}{q*}\gamma\right)^{-n_1} + \frac{q * e^{\gamma(\sigma-\tau)}}{p\gamma^2}\left(1 - \frac{q_k^*}{q*}\gamma\right)^{n_2-n_1} = 1. \tag{3.4}$$

Obviously, there are many particular cases that could be examined. However, we shall only consider the case where $n_1 = n_2 = 0$ which concerns us. Consequently, equation (3.4) becomes

$$\frac{1}{p}e^{-\gamma\tau} + \frac{q^* e^{\gamma(\sigma-\tau)}}{p\gamma^2} = 1.$$

Hence,

$$\frac{1}{p}e^{-\gamma\tau} + \frac{1}{p}\int_{t+\tau}^{\infty}(s-t-\tau)\, q(s) \exp\left[\gamma(t-g(s))\right] ds$$

$$+\frac{1}{p}\sum_{t+\tau\le t_k<\infty}(t_k-t-\tau)\,q_k\exp\left[\gamma\left(t_k-g\left(t_k\right)\right)\right]$$
$$\le\frac{1}{p}e^{-\gamma\tau}+\frac{q^*}{p\gamma^2}e^{\gamma(\sigma-\tau)}=1,$$

that is, inequality (3.1) holds. Then, by Theorem 2.1, equation (2.1) has a positive solution.

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