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THIRD-ORDER DIFFERENTIAL SANDWICH-TYPE RESULT OF MEROMORPHIC p-VALENT FUNCTIONS ASSOCIATED WITH A CERTAIN LINEAR OPERATOR

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ABSTRACT: By utilizing a certain linear operator considered on meromorphic multivalent functions (MMF) in the punctured unit disk U^*. We investigate the third-order differential subordination and superordination results. The outcomes here are acquired by introducing appropriate class of admissible functions. Sufficient conditions are determined to gain the best dominant and the best subordinate, respectively. Differential sandwich-type result is also derived.

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1. INTRODUCTION

Let $H(U)$ be the set of analytic functions defined in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, the given complex number a and the positive integer k will be

$$H[a,n] = \{h : h \in H(U), h(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\}$$

$H_0 = H[0,1]$ and $H_1 = H[1,1]$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two members in the class $H(U)$, the function $\mathcal{F}_1$ is said to be subordinate to $\mathcal{F}_2$, or $\mathcal{F}_2$ is superordinate to $\mathcal{F}_1$, if there exists an analytic function $g(z)$ in the unit open disk H with $g(0) = 0$ and $|g(z)| < 1$ for all $z \in \mathcal{H}$, such that $\mathcal{F}_1(z) = \mathcal{F}_2(g(z))$. In such a case, we write $\mathcal{F}_1 \prec \mathcal{F}_2$ or $\mathcal{F}_1(z) \prec \mathcal{F}_2(z)$.

Let Σ_p denote signify the class of all meromorphic multivalent functions (MMF) of the form:

$$h(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k, \quad (p \in \mathbb{N} = \{1,2,...\},\ z \in U^*)\,, \tag{1.1}$$

which are analytic in the punctured unit disk $U^* = \{z \in \mathbb{C} :\ 0 < |z| < 1\}$, which emerges in various researchers contexts.

For functions h given by (1.1) and g given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k. \tag{1.2}$$

The convolution (or Hadamard product), denoted by $h * g$ of the functions h and g is defined by

$$h * g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k. \tag{1.3}$$

By utilizing a linear operator $\mathcal{L}_p(\ell,\mu)(\mu > 0,\ \ell \in \mathbb{N}_0 = \mathbb{N} \cup 0)$ defined by El-Ashwah et al. [1] for function $h \in \Sigma_p$ written by (1.1) as follows:

$$\mathcal{L}_p(\ell,\mu)h(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{\mu + k + p}{\mu}\right)^{\ell} a_k z^k. \tag{1.4}$$

We define the dual operator $\mathcal{X}_p(\ell,\mu):\Sigma_p \to \Sigma_p$:

$$\mathcal{X}_p(\ell,\mu)h(z)=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}\left(\frac{\mu}{\mu+k+p}\right)^{\ell}a_kz^k. \tag{1.5}$$

where h is written by (1.1).

From (1.5) presents a proof of a recurrence relation as:

$$z\left(\mathcal{X}_p(\ell+1,\mu)h(z)\right)'=\mu\left(\mathcal{X}_p(\ell,\mu)h(z)\right)-(\mu+p)\left(\mathcal{X}_p(\ell+1,\mu)h(z)\right). \tag{1.6}$$

The theory of differential subordination was discussed by many authors to deal with numerous important problems in the field of geometric function theory (GFT). Work in this direction include those of (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]). In 2011, Antonino and Miller [19] imposed the third-order differential subordination (TDSB) (which is a generalization of second-order differential subordination). In 2014, Tang et al. [20] have introduced the development of study in this field. They studied the third-order differential superordination (TDSP) as the dual problem of the differential subordination (TDSB). Later, various complex analysts attracted to use the methods of the third-order differential subordination (TDSB) and superordination (TDSP). For example, Tang and Deniz [21], Tang et al. [22], Farzana et al. [23], Ibrahim [24], Al-Janaby and Ghanim [25] and El-Ashwah and Hassan [26].

2. PRELIMINARIES

In 2011, Antonino and Miller [19] presented basic concepts in the theory of the third-order differential subordination. In this part, we will mention the fundamental definitions and theorems in the theory of third-order differential subordination and superordination that are needed in our outcomes.

Definition 2.1. ([19], *Definition* 1, p. 440) Let $\tau:\mathbb{C}^4\times U\to\mathbb{C}$ and the function $\jmath(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the following third-order differential subordination:

$$\tau(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\prec\jmath(z). \tag{2.1}$$

Then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.1). A dominant $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2.1) is said to be the best dominant.

Definition 2.2. ([19], *Definition* 2, p. 441) Let Q denote the set of functions q that are analytic and univalent on the set $\overline{U}\backslash \mathrm{E(q)}$, where

$$\mathrm{E(q)} = \{\xi : \xi \in \partial U : \lim_{z\to\xi} q(z) = \infty\},$$

is such that $\min |q'(\xi)| = \rho > 0$ for $\xi \in \partial U\backslash \mathrm{E(q)}$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) = Q_0$ and $Q(1) = Q_1$.

The following definition of the class of admissible functions (ADF) due to Antonino and Miller [19].

Definition 2.3. ([19], *Definition* 3, p. 449) Let Λ be a set in $\mathbb{C}$ and $q \in Q$ and $n \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\Psi_n[\Lambda, q]$ consists of those functions $\tau : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\tau(r, s, t, u; z) \notin \Lambda$$

whenever

$$r = q(\zeta), \;\; s = k\zeta q'(\zeta), \;\; \Re\left(\frac{t}{s} + 1\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{u}{s}\right) \geq k^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in U$, $\zeta \in \partial U\backslash E(q)$, and $k \geq n$.

Antonino and Miller [19] proposed the main theorem in the theory of third-order difference subordination.

Theorem 2.4. *([19], Theorem 1, p. 449) Let $p \in H[a, n]$ with $n \geq 2$. Also, let $q \in Q(a)$ and satisfy the following conditions:*

$$\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \; and \;\; \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq k, \tag{2.2}$$

where $z \in U$, $\zeta \in \partial U \backslash E(q)$, *and* $k \geq n$. *If* Λ *is a set in* $\mathbb{C}$, $\tau \in \Psi_n[\Lambda, q]$ *and*

$$\tau\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \in \Lambda,$$

then $p(z) \prec q(z)$ $z \in U$.

We now introduce the basic definitions and the main outcomes in the theory the third-order differential superordination that are considered by Tang et al. [20].

Definition 2.5. [20] Let $\tau : \mathbb{C}^4 \times U \to \mathbb{C}$ and the function $\jmath(z)$ be analytic in U. If the functions $p(z)$ and $\tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ are univalent in U and satisfy the following third-order differential superordination:

$$\jmath(z) \prec \tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), \tag{2.3}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination, or, more simply, a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (2.3). A univalent subordinant $\widetilde{q}(z)$ that satisfies the condition $q(z) \prec \widetilde{q}(z)$ for all subordinants $q(z)$ of (2.3) is said to be the best subordinant.

Definition 2.6. [20] Let Λ be a set in $\mathbb{C}$, $q \in H\,[a, n]$ and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Lambda, q]$ consists of those functions $\tau : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\tau(r, s, t, u; \zeta) \in \Lambda$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{u}{s}\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in U$, $\zeta \in \partial U$, and $m \geq n \geq 2$.

Theorem 2.7. *[20] Let* $q \in H[a,n]$ *and* $\tau \in \Psi'_n[\Lambda, q]$*. If* $\tau(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ *is univalent in* U *and* $p \in Q(a)$ *satisfy the following conditions:*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geq 0, \quad \textit{and} \quad \left|\frac{\zeta p'(\zeta)}{q'(z)}\right| \leq m, \tag{2.4}$$

where $z \in U$*,* $\zeta \in \partial U$*, and* $m \geq n \geq 2$*, then*

$$\Lambda \subset \left\{\tau\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) : z \in U\right\}$$

implies that

$$q(z) \prec p(z) \quad (z \in U).$$

3. SUBORDINATION ASSOCIATED WITH THE LINEAR OPERATOR $\mathcal{X}_P(\ell, \mu)$

In this section, by using the remarkable results for the third order differential subordination (TDSB) due to Antonino and Miller [19] in the open unit disk U, we pose a suitable class of (ADF) which is a first step required to prove the differential subordination outcomes for meromorphically multivalent functions associated with the operator $\mathcal{X}_p(\ell, \mu)h(z)$ as given in (1.5).

Definition 3.1. Let Λ be a set in $\mathbb{C}$ and $q \in Q_1 \cap H_1$. The class of (ADF) $\Gamma_{H,1}[\Lambda, q]$ consists of those functions $\omega : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\omega(a, b, c, d; z) \notin \Lambda$$

whenever

$$a = q(\zeta),\ b = \frac{\mu q(\zeta)}{\mu - k\zeta q'(\zeta)},\ \Re\left(\frac{\mu\left[c\left(\frac{2a}{b} - 1\right) - a\right]}{c(b-a)}\right) \geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{\mu^3}{a^2 b}\left[\frac{a^3(d-c)}{c^2 d} - \left[\frac{2a^2(c-b)}{b^2 c} + \frac{a}{b} - 1\right] - a\left[\frac{a(2c-b)}{bc} - 1\right]^2\right]\right.$$

$$-\frac{3\mu}{(b-a)}\left[\frac{a\,(2c-b)}{bc}-1\right]-4\Bigg)\geq k^{2}\Re\left(\frac{\zeta^{2}q'''(\zeta)}{q'(\zeta)}\right),$$

where $z\in U$, $\zeta\in\partial U\backslash E(q)$, and $k\geq n$.

We now state and prove our main third-order differential subordinations results.

Theorem 3.2. *Let $\omega\in\Gamma_{H,1}[\Lambda,q]$. If the function $h\in\Sigma_{p}$ and $q\in Q_{1}$ satisfy the following conditions*

$$\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\}\geq 0,\qquad \left|\frac{(\mathcal{X}_{p}(\ell+3,\mu)\,h(z))}{(\mathcal{X}_{p}(\ell+2,\mu)\,h(z))\,q'(\zeta)}\right|\leq k \tag{3.1}$$

and

$$\left\{\omega\left(\begin{array}{l}\frac{(\mathcal{X}_{p}(\ell+4,\mu)h(z))}{(\mathcal{X}_{p}(\ell+3,\mu)h(z))},\ \frac{(\mathcal{X}_{p}(\ell+3,\mu)h(z))}{(\mathcal{X}_{p}(\ell+2,\mu)h(z))},\ \frac{(\mathcal{X}_{p}(\ell+2,\mu)h(z))}{(\mathcal{X}_{p}(\ell+1,\mu)h(z))}\\ ,\ \frac{(\mathcal{X}_{p}(\ell+1,\mu)h(z))}{(\mathcal{X}_{p}(\ell,\mu)h(z))};z\end{array}\right):z\in U\right\}\subset\Lambda. \tag{3.2}$$

Then

$$\frac{(\mathcal{X}_{p}(\ell+4,\mu)\,h(z))}{(\mathcal{X}_{p}(\ell+3,\mu)\,h(z))}\prec q(z).$$

Proof. Define the analytic function $p(z)$ in the unit disk U by

$$p(z)=\frac{(\mathcal{X}_{p}(\ell+4,\mu)\,h(z))}{(\mathcal{X}_{p}(\ell+3,\mu)\,h(z))}. \tag{3.3}$$

Because of the relation (1.6), we have

$$z\,(\mathcal{X}_{p}(\ell+4,\mu)\,h(z))'$$
$$=\mu\,(\mathcal{X}_{p}(\ell+3,\mu)\,h(z))-(\mu+p)\,(\mathcal{X}_{p}(\ell+4,\mu)\,h(z)). \tag{3.4}$$

Using equations (3.3) and (3.4), it follows that

$$\frac{(\mathcal{X}_{p}(\ell+3,\mu)\,h(z))}{(\mathcal{X}_{p}(\ell+2,\mu)\,h(z))}=\frac{\mu\ p(z)}{\mu-z\ p'(z)}. \tag{3.5}$$

Further computations show that

$$\frac{(\mathcal{X}_{p}(\ell+2,\mu)\,h(z))}{(\mathcal{X}_{p}(\ell+1,\mu)\,h(z))}=\frac{\mu}{\frac{[\mu-2z\ p'(z)]}{p(z)}-\left[\frac{z^{2}\ p''(z)+z\ p'(z)}{\mu-z\ p'(z)}\right]} \tag{3.6}$$

and

$$\frac{(\mathcal{X}_p(\ell+1,\mu)\,h(z))}{(\mathcal{X}_p(\ell,\mu)\,h(z))}= \tag{3.7}$$

$$\frac{\mu}{\left[\begin{array}{l}\frac{[\mu-2zp'(z)]}{p(z)}-\left[\frac{z^2\,p''(z)+z\,p'(z)}{\mu-z\,p'(z)}\right]- \\ \left[\frac{\left[\begin{array}{l}(\mu-z\,p'(z))^2\left[2p(z)\left(z^2p''(z)+zp'(z)\right)+z\,p'(z)(\mu-2z\,p'(z))\right] \\ +p^2(z)\left[(\mu-z\,p'(z))\left(z^3p'''(z)+3z^2p''(z)+zp'(z)\right)\right]+\left(z^2p''(z)+zp'(z)\right)^2\end{array}\right]}{p(z)(\mu-z\,p'(z))[(\mu-z\,p'(z))(\mu-2z\,p'(z))-p(z)(z^2p''(z)+zp'(z))]}\right]\end{array}\right]}.$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u)=r,\qquad b(r,s,t,u)=\frac{\mu r}{\mu-s},$$

$$c(r,s,t,u)=\frac{\mu}{\frac{[\mu-2s]}{r}-\left[\frac{t+s}{\mu-s}\right]}, \tag{3.8}$$

and

$$d(r,s,t,u)$$
$$=\frac{\mu}{\frac{[\mu-2s]}{r}-\left[\frac{t+s}{\mu-s}\right]-\left[\frac{(\mu-s)^2[2r(t+s)+s(\mu-2s)]+r^2\left[(\mu-s)(u+3t+s)+(t+s)^2\right]}{r(\mu-s)[(\mu-s)(\mu-2s)-r(t+s)]}\right]}.$$

Let

$$\tau(r,s,t,u;z)=\omega(a,b,c,d;z)$$
$$=\omega\left(\begin{array}{c} r,\ \frac{\mu r}{\mu-s},\ \frac{\mu}{\frac{[\mu-2s]}{r}-\left[\frac{t+s}{\mu-s}\right]}, \\ \\ \frac{\mu}{\frac{[\mu-2s]}{r}-\left[\frac{t+s}{\mu-s}\right]-\left[\frac{(\mu-s)^2[2r(t+s)+s(\mu-2s)]+r^2\left[(\mu-s)(u+3t+s)+(t+s)^2\right]}{r(\mu-s)[(\mu-s)(\mu-2s)-r(t+s)]}\right]}.\end{array}\right). \tag{3.9}$$

The proof will make use of Theorem 2.4. Using equations (3.3), (3.5), (3.6) and (3.7), we have from the equation (3.9) that

$$\tau\left(p(z),\ zp'(z),\ z^2p''(z),z^3p'''(z);z\right)$$

$$= \omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right). \quad (3.10)$$

Hence equation (3.2) becomes

$$\tau\left(p(z),\ zp'(z),\ z^2p''(z), z^3p'''(z); z\right) \in \Lambda.$$

Utilizing (3.8), then we have

$$\frac{t}{s}+1 = \frac{\mu\left[c\left(\frac{2a}{b}-1\right)-a\right]}{c(b-a)}$$

and

$$\frac{u}{s} = \frac{\mu^3}{a^2b}\left[\frac{a^3(d-c)}{c^2d} - \left[\frac{2a^2(c-b)}{b^2c}+\frac{a}{b}-1\right] - a\left[\frac{a(2c-b)}{bc}-1\right]^2\right] - \frac{3\mu}{(b-a)}\left[\frac{a(2c-b)}{bc}-1\right] - 4,$$

Thus, the admissibility condition for $\omega \in \Gamma_{H,1}[\Lambda, q]$ in Definition 3.1 is equivalent to the admissibility condition for $\tau \in \Psi_n[\Lambda, q]$ as given in Definition 2.3. Therefore, by using (3.1) and Theorem 2.4, we have $p(z) \prec q(z)$ or equivalently,

$$\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))} \prec q(z).$$

The proof of Theorem 3.2 is completed. □

The following result is an extension of Theorem 3.2 to the case where the behaviour of $q(z)$ on ∂U is not known.

Corollary 3.3. *Let $\Lambda \subset \mathbb{C}$ and let the function q be univalent in U with $q(0) = 1$. Suppose also that $\omega \in \Gamma_{H,1}[\Lambda, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the functions $h \in \Sigma_p$ and q_ρ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta q''_\rho(\zeta)}{q'_\rho(\zeta)}\right\} \geq 0,$$

$$\left|\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))\,q'_\rho(\zeta)}\right| \leq k \qquad (z \in U;\ \zeta \in \partial U \setminus E(q_\rho)),$$

and

$$\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))};z\right)\in\Lambda.$$

Then

$$\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}\prec q(z).$$

Proof. From Theorem 3.2, we have $\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}\prec q_\rho(z)$. This outcome follows easily from the subordination property $q_\rho(z)\prec q(z)$. □

If $\Lambda\neq\mathbb{C}$ is a simply connected domain, then $\Lambda=\jmath(U)$, for some conformal mapping $\jmath(z)$ of U onto Λ in this case, the class $\Gamma_{H,1}[\jmath(U),q]$. is written as $\Gamma_{H,1}[\jmath,q]$. The following result follows immediately consequence from Theorem 3.2.

Theorem 3.4. *Let $\omega\in\Gamma_H[\jmath,q]$. If the function $h\in\Sigma_p$ and $q\in Q_1$ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\}\geq 0,\qquad\left|\frac{(\mathcal{X}_p(n+3,\mu)h(z))}{(\mathcal{X}_p(n+2,\mu)h(z))q'(\zeta)}\right|\leq k$$

$$\left\{\omega\left(\begin{array}{l}\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))},\\ \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))};z\end{array}\right)\right\}\prec\jmath(z). \tag{3.11}$$

Then

$$\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}\prec q(z).$$

The next result is an immediate consequence of Corollary 3.3.

Corollary 3.5. *Let $\Lambda\subset\mathbb{C}$ and let the function q be univalent in U with $q(0)=1$. Suppose also that $\omega\in\Gamma_{H,1}[\jmath,q_\rho]$ for some $\rho\in(0,1)$, where $q_\rho(z)=q(\rho z)$. If the functions $h\in\Sigma_p$ and q_ρ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta q_\rho''(\zeta)}{q_\rho'(\zeta)}\right\}\geq 0,$$

$$\left| \frac{(\mathcal{X}_p(\ell+3,\mu)\, h(z))}{(\mathcal{X}_p(\ell+2,\mu)\, h(z))\, q'_\rho(\zeta)} \right| \leq k \qquad (z \in U;\ \zeta \in \partial U \setminus E(q_\rho)),$$

and

$$\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)\, h(z))}{(\mathcal{X}_p(\ell+3,\mu)\, h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)\, h(z))}{(\mathcal{X}_p(\ell+2,\mu)\, h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)\, h(z))}{(\mathcal{X}_p(\ell+1,\mu)\, h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)\, h(z))}{(\mathcal{X}_p(\ell,\mu)\, h(z))}; z \right) \prec \jmath(z),$$

then

$$\frac{(\mathcal{X}_p(\ell+4,\mu)\, h(z))}{(\mathcal{X}_p(\ell+3,\mu)\, h(z))} \prec q(z).$$

The following result yields the best dominant of the differential subordination (3.9).

Theorem 3.6. *Let the function $\jmath$ be univalent in unit open disk U. Also, let the function $\omega : \mathbb{C}^4 \times U \to \mathbb{C}$ and τ be given by (3.9). Suppose that the differential equation*

$$\tau\left(p(z),\ zp'(z),\ z^2p''(z), z^3p'''(z); z\right) = \jmath(z) \tag{3.12}$$

has a solution $q(z)$ with $q(0) = 1$, which satisfies the condition (3.1). If the function $h \in \Sigma_p$ satisfies the condition (3.11) and the function

$$\left(\frac{(\mathcal{X}_p(\ell+4,\mu)\, h(z))}{(\mathcal{X}_p(\ell+3,\mu)\, h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)\, h(z))}{(\mathcal{X}_p(\ell+2,\mu)\, h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)\, h(z))}{(\mathcal{X}_p(\ell+1,\mu)\, h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)\, h(z))}{(\mathcal{X}_p(\ell,\mu)\, h(z))}; z \right)$$

is analytic in U, then

$$\frac{(\mathcal{X}_p(\ell+4,\mu)\, h(z))}{(\mathcal{X}_p(\ell+3,\mu)\, h(z))} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. In view of Theorem 3.2, we deduce that q is a dominant of (3.11). Since q satisfies (3.12), it is also a solution of (3.11) and therefore q will be dominated by all dominates. Hence q is the best dominant. □

In view of Definition 3.1, in the special case $q(z) = 1 + Mz,\ M > 0$, the class (ADF) $\Gamma_{H,1}[\Lambda, q]$, denoted by $\Gamma_{H,1}[\Lambda, M]$, is expressed as follows:

Definition 3.7. Let Λ be a set in $\mathbb{C}$ and $M > 0$. The class $\Gamma_{H,1}[\Lambda, M]$ of admissible functions consists of those functions $\omega : \mathbb{C}^4 \times U \to \mathbb{C}$ such that

$$\omega\left(\begin{array}{l} 1+Me^{i\theta},\ \frac{\mu\left(1+Me^{i\theta}\right)}{\mu-kMe^{i\theta}},\ \frac{\mu\left(1+Me^{i\theta}\right)\left(\mu-kMe^{i\theta}\right)}{\left(\mu-kMe^{i\theta}\right)\left(\mu-2kMe^{i\theta}\right)-\left(1+Me^{i\theta}\right)\left(L-kMe^{i\theta}\right)}, \\ \frac{\mu\left(1+Me^{i\theta}\right)\left(\mu-kMe^{i\theta}\right)\left[\left(\mu-kMe^{i\theta}\right)\left(\mu-2kMe^{i\theta}\right)-\left(1+Me^{i\theta}\right)\left(L+kMe^{i\theta}\right)\right]}{\left[\begin{array}{l}\left[\left(\mu-kMe^{i\theta}\right)\left(\mu-2kMe^{i\theta}\right)-\left(1+Me^{i\theta}\right)\left(L+kMe^{i\theta}\right)\right]^2- \\ \left[\begin{array}{l}\left(\mu-kMe^{i\theta}\right)^2\left[2\left(1+Me^{i\theta}\right)+kMe^{i\theta}\left(\mu-2kMe^{i\theta}\right)\right]+ \\ \left(1+Me^{i\theta}\right)^2\left[\left(\mu-kMe^{i\theta}\right)\left(\mathcal{N}+3L+kMe^{i\theta}\right)+\left(L-kMe^{i\theta}\right)^2\right]\end{array}\right]\end{array}\right]};z \end{array}\right) \notin \Lambda \quad (3.13)$$

whenever $z \in U$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ and $\Re(N e^{-i\theta}) \geq 0$, for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$.

Corollary 3.8. *Let $\omega \in \Gamma_{H,1}[\Lambda, q]$. If the functions $h \in \Sigma_p$ satisfy the following conditions:*

$$\left|\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))\,q_\rho'(\zeta)}\right| \leq kM \qquad (k \in \mathbb{N}\setminus\{1\}; M > 0),$$

and

$$\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right) \in \Lambda.$$

Then

$$\left|\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))} - 1\right| < M.$$

Proof. The result follows from Theorem 3.2 by taking $q(z) = 1 + Mz$. □

In the special case $\Lambda = q(U) = \{\nu : |\nu - 1| < M\}$, the class of admissible functions $\Gamma_{H,1}[\Lambda, M]$ is simply denoted by $\Gamma_{H,1}[M]$.

Corollary 3.9. *Let $\omega \in \Gamma_{H,1}[\Lambda, q]$. If the functions $h \in \Sigma_p$ satisfy the following conditions:*

$$\left|\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}\right| \leq kM \qquad (k \in \mathbb{N}\setminus\{1\}; M > 0),$$

and

$$\left|\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))},\frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))};z\right)-1\right|<M,$$

then

$$\left|\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}-1\right|<M.$$

Proof. By setting $\Lambda=q(U)$ in Corollary 3.8. This result is gained. □

Example 3.10. Let $k\in\mathbb{N}\setminus\{1\}$, and $M>0$. If $h\in\Sigma_p$ satisfies the following conditions:

$$\left|\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}\right|\le kM \quad\text{and}\quad \left|\frac{\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}}{\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}}-1\right|\le\frac{M}{\mu},$$

then

$$\left|\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}-1\right|<M.$$

Proof. Take $\omega(a,b,c,d;z)=\frac{a}{b}-1$, and $\Lambda=\jmath(U)$, where $\jmath(z)=\frac{2M}{\mu}z,\ M>0$. Hence

$$\omega\left(\begin{array}{c}1+Me^{i\theta},\ \frac{\mu\left(1+Me^{i\theta}\right)}{\mu-kMe^{i\theta}},\frac{\mu\left(1+Me^{i\theta}\right)\left(\mu-kMe^{i\theta}\right)}{(\mu-kMe^{i\theta})(\mu-2kMe^{i\theta})-(1+Me^{i\theta})(L-kMe^{i\theta})},\\ \frac{\mu\left(1+Me^{i\theta}\right)\left(\mu-kMe^{i\theta}\right)\left[\left(\mu-kMe^{i\theta}\right)\left(\mu-2kMe^{i\theta}\right)-\left(1+Me^{i\theta}\right)\left(L+kMe^{i\theta}\right)\right]}{\left[\begin{array}{c}\left[\left(\mu-kMe^{i\theta}\right)\left(\mu-2kMe^{i\theta}\right)-\left(1+Me^{i\theta}\right)\left(L+kMe^{i\theta}\right)\right]^2-\\ \left[\begin{array}{c}\left(\mu-kMe^{i\theta}\right)^2\left[2\left(1+Me^{i\theta}\right)+kMe^{i\theta}\left(\mu-2kMe^{i\theta}\right)\right]+\\ \left(1+Me^{i\theta}\right)^2\left[\left(\mu-kMe^{i\theta}\right)\left(\mathcal{N}+3L+kMe^{i\theta}\right)+\left(L-kMe^{i\theta}\right)^2\right]\end{array}\right]\end{array}\right]};z\end{array}\right)$$

$$=\left|\frac{1+Me^{i\theta}}{\frac{\mu\left(1+Me^{i\theta}\right)}{\mu-Mke^{i\theta}}}-1\right|>\frac{2M}{\mu}$$

whenever $z\in U$, $\Re(Le^{-i\theta})\ge(k-1)kM$ and $\Re(\mathbb{N}\,e^{-i\theta})\ge0$, for all $\theta\in\mathbb{R}$ and $k\in\mathbb{N}\setminus\{1\}$. Hence $\omega\in\Gamma_{H,1}[\Lambda,M]$, that is the admissibility condition (3.13) is satisfied. Therefore, we get the required result from Corollary 3.8. □

4. SUPERORDINATION ASSOCIATED WITH THE LINEAR OPERATOR $\mathcal{X}_P(\ell, \mu)$

Definition 4.1. Let Λ be a set in $\mathbb{C}$ and $q \in H_1$ with $q'(z) \neq 0$. The class of (ADF) $\Gamma'_{H,1}[\Lambda, q]$ consists of those functions $\Omega : \mathbb{C}^4 \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\omega(a, b, c, d; z) \notin \Lambda$$

whenever

$$a = q(z), \; b = \frac{m\mu q(z)}{m\mu - zq'(z)}, \; \Re\left\{\frac{\mu\left[c\left(\frac{2a}{b} - 1\right) - a\right]}{c(b-a)}\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

and

$$\Re\left\{\frac{\mu^3}{a^2 b}\left[\frac{a^3(d-c)}{c^2 d} - \left[\frac{2a^2(c-b)}{b^2 c} + \frac{a}{b} - 1\right] - a\left[\frac{a(2c-b)}{bc} - 1\right]^2\right] - \frac{3\mu}{(b-a)}\left[\frac{a(2c-b)}{bc} - 1\right] - 4\right\} \leq \frac{1}{m^2}\Re\left\{\frac{z^2 q'''(z)}{q'(z)}\right\},$$

where $z \in U$, $\zeta \in \partial U$ and $m \in \mathbb{N} \setminus \{1\}$.

Theorem 4.2. *Let $\omega \in \Gamma'_{H,1}[\Lambda, q]$. If the function $h \in \Sigma_p$, $\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))} \in Q_1$ and $q \in H_1$ with $q'(z) \neq 0$ satisfy the following conditions:*

$$\Re\left\{\frac{zq''(z)}{q'(z)}\right\} \geq 0, \qquad \left|\frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))\,q'(z)}\right| \leq m, \tag{4.1}$$

$$\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right)$$

is univalent, then

$$\Lambda \subset \left\{\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right)\right\} \tag{4.2}$$

implies that

$$q(z) \prec \frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}.$$

Proof. Let the function $p(z)$ be defined by (3.3) and τ by (3.9). Since $\omega \in \Gamma'_{H,1}[\Lambda, q]$, (3.10) and (4.2), we have

$$\Lambda \subset \left\{\tau\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z \in U\right)\right\}.$$

We note from (3.8) that the admissibility condition for $\omega \in \Gamma'_{H,1}[\Lambda, q]$ in Definition 4.1 is equivalent to the admissibility condition for $\tau \in \Psi'_n[\Lambda, q]$ as given in Definition 2.6. Hence, by utilizing (4.1) and utilizing Theorem 2.7, we have

$$q(z) \prec p(z), \quad \text{or} \quad q(z) \prec \frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}. \qquad \square$$

If $\Lambda \neq \mathbb{C}$ is a simply connected domain, then $\Lambda = \jmath(U)$, for some conformal mapping $\jmath(z)$ of U onto Λ in this case, the class $\Gamma'_{H,1}[\jmath(U), q]$ is written as $\Gamma'_H[\jmath, q]$. The following result follows immediately from Theorem 4.2 .

Theorem 4.3. *Let $\omega \in \Gamma'_{H,1}[\Lambda, q]$. Also, let the function $\jmath$ be analytic in U. If the function $h \in \Sigma_p$, $\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))} \in Q_1$ satisfy condition (4.1) and*

$$\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right)$$

is univalent in U, then

$$\jmath(z) \prec \left\{\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right)\right\} \tag{4.3}$$

implies that

$$q(z) \prec \frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}.$$

The next outcome proves the existence of the best subordinant of (4.3) for appropriate chosen ω.

Theorem 4.4. *Let the function J be analytic in U, and let $\Omega : \mathbb{C}^4 \times \bar{U} \to \mathbb{C}$ and τ be given by (3.9). Suppose that the differential equation*

$$\tau\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) = \jmath(z)$$

has a solution $q(z) \in Q_1$*. If the function* $h \in \Sigma_p$, $\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))} \in Q_1$ *and* $q \in H_1$ *with* $q'(z) \neq 0$ *satisfy condition (4.1) and*

$$\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right)$$

is univalent in U*, then*

$$\jmath(z) \prec \left\{\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right)\right\}$$

implies that

$$q(z) \prec \frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}.$$

and q *is the best subordinant.*

Proof. The proof of this result resembles that of Theorem 4.3 and it has been omitted here. □

Combining Theorems 3.4 and 4.3, we yield the following sandwich-type theorem.

Corollary 4.5. *Let the functions* $\jmath_1$ *and* q_1 *be analytic functions in* U, $\jmath_2$ *be univalent function in* U, $q_2 \in Q_1$ *with* $q_1(0) = q_2(0) = 1$, $\omega \in \Gamma_{H,1}[\jmath_1, q] \cap \Gamma'_{H,1}[\jmath_2, q]$*. If the function* $h \in \Sigma_p$, $\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))} \in Q_1 \cap H_1$ *and*

$$\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+1,\mu)h(z))}{(\mathcal{X}_p(\ell,\mu)h(z))}; z\right)$$

is univalent in U*, and the condition (3.1) and (4.2) are satisfied, then*

$$\jmath_1(z) \prec \left\{\omega\left(\frac{(\mathcal{X}_p(\ell+4,\mu)h(z))}{(\mathcal{X}_p(\ell+3,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+3,\mu)h(z))}{(\mathcal{X}_p(\ell+2,\mu)h(z))}, \frac{(\mathcal{X}_p(\ell+2,\mu)h(z))}{(\mathcal{X}_p(\ell+1,\mu)h(z))},\right.\right.$$

$$\left.\left.\frac{(\mathcal{X}_p\,(\ell+1,\mu)\,h\,(z))}{(\mathcal{X}_p\,(\ell,\mu)\,h\,(z))};z\right)\right\} \prec \jmath_2(z)$$

implies that

$$q_1\,(z) \prec \frac{(\mathcal{X}_p\,(\ell+4,\mu)\,h\,(z))}{(\mathcal{X}_p\,(\ell+3,\mu)\,h\,(z))} \prec q_2(z).$$

5. CONCLUSION

In this study, we have studied and discussed third-order differential subordination and differential superordination results involving a certain linear operator on the class meromorphically multivalent functions in the punctured unit disk. These outcomes are gained by introducing appropriate classes of admissible functions. Furthermore, we have determined sufficient conditions to yeild the best dominant and the best subordinant, respectively. In addition, we have acquired sandwich-type result.

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