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# THE HOSOYA, SCHULTZ AND MODIFIED SCHULTZ POLYNOMIALS OF A CLASS OF DUTCH WINDMILL GRAPH $D_n^{(m)}$, $\forall n, m \in \mathbb{N}$ & $n \geq 4, m \geq 2$

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**ABSTRACT:** It's shown by a large number of studies that strong inner bound exists between the chemical characteristics of nano structures and their molecular structures. The material scientists can achieve a better knowledge of their chemical and biological features, if they take the degree-based topological indices on chemical molecular structures into consideration, which will also contribute to making up the shortage of chemical experiments. In this paper, by means of structure analysis and mathematical derivation, the Hosoya, Schultz and modified Schultz polynomials of Dutch windmill graph $D_n^{(m)}$ are obtained.

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**Key Words:** Wiener Index, topological index, Hosoya polynomial, Schultz polynomial, Dutch windmill graph

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## 1. INTRODUCTION

Strong inner bound between the chemical characteristics of nano structures and their molecular structures is researched and confirmed by a big mass of studies. Considering the degree-based topological indices on chemical molecular structures, the material scientists can better understand their chemical and biological features, which can help to make up the shortage of chemical experiments.

Traditionally, we consider the nano structure as a graph, the atoms as vertexes on it, and the chemical bounds between two atoms as the edges of it. Furthermore, $G$ is taken as a (molecular) graph with vertex set $V(G)$ and edge set $E(G)$. Then, the real-valued function $f: G \rightarrow \mathbb{R}^+$ could be a substance of a topological index and it helps to map each nano structure to a real number. Topological indices act as abundant descriptors of the molecular structure in the relevant nano structures. For this reason, it is widely accepted in nano engineering, such as QSPR/QSAR study. The structural features of nano molecules were measured by means of harmonic index, Wiener index, Sum-connectivity index over the past few years. Some of the insightful papers have pushed the progress to determine the topological indices of special molecular graph in chemical engineering (See Yan et al. [1], Gao et al. [2-4], Gao and Farahani [5-6], Gao and Shi [7], and Gao and Wang [8-10] for more detail). The used notations and terminologies without being clearly defined in the paper can be found in book [11] by Bondy and Murty.

As a famous topological index, the Wiener index was introduced as

$$W(G)=\sum_{\{u,v\}\subseteq V(G)} d(u,v),$$

where $d(u,v)$ is the distance between vertices $u$ and $v$ in $G$ . The Hosoya polynomial related on Wiener index was defined by

$$W(G,x)=\sum_{\{u,v\}\subseteq V(G)} x^{d(u,v)}.$$

Some conclusions on Hosoya polynomial can be found in Klavar et al. [12], Deutsch and 1Klavar [13], Deutsch et al. [14] and Dehmer et al. [15]. The Schultz index (also, called degree distance) and modified Schultz index (or, called Gutman index) are denoted by

$$Sc(G)=\sum_{\{u,v\}\subseteq V(G)} (d(u)+d(v))d(u,v),$$

and

$$Sc*(G)=\sum_{\{u,v\}\subseteq V(G)} d(u)d(v)d(u,v)$$

respectively. The corresponding Schultz polynomial and modified Schultz polynomial are denoted as

$$Sc(G,x)=\sum_{\{u,v\}\subseteq V(G)} (d(u)+d(v))x^{d(u,v)},$$

and

$$Sc*(G,x)=\sum_{\{u,v\}\subseteq V(G)} d(u)d(v)x^{d(u,v)}$$

respectively. Several contributions on the Schultz polynomial and modified Schultz polynomial can refer to Eliasi and Taeri [16-17], Alizadeh et al. [18], Farahani [19-30].

The Dutch windmill graph is denoted by $D_n^{(m)}$ and it is the graph obtained by taking $m$ copies of the cycle $C_n$ with a vertex in common. Some advances have been witnessed in distance-based indices of molecular graphs these years, but the degree-based indices research for special nano molecular structures still waits for being furthered in a long time. What's more, one of the most widespread and critical nano structures, Dutch windmill graph is widely borrowed in medical science and material field. As a consequence, it's necessary to express the above mentioned degree-based indices for Dutch windmill graph clearly.

## 2. MAIN RESULTS

In this section, we aim to present the main results of our paper.

**Denotation 1.** To compute the Wiener index and Hosoya polynomial of a graph $G$, we first introduce some notions, which are useful to aims in this paper. The number of unordered pairs of vertices $x$ and $y$ of $G$ in which distance $d(x,y)=k$ is denoted by $d(G,k)$. Obviously $1 \leq k \leq d(G)$ (the diameter $d(G)$ of $G$ is the maximum eccentricity). Thus we redefine the Hosoya polynomial and Wiener index as follows:

$$H(G,x) = \sum_{k=1}^{d(G)} d(G,k)x^k$$

and

$$W(G) = \sum_{k=1}^{d(G)} d(G,k) \times k.$$

**Theorem 1.** *Consider the graph of Dutch windmill graph* $D_4^{(n)} (\forall n \in \mathbb{N} - \{1\})$. *Then:*

*(1) The Hosoya polynomial of* $D_4^{(n)}$ *is equal to:*

$$H(D_4^{(n)}, x) = 4nx^1 + n(2n+1)x^2 + 2n(n+1)x^3 + \frac{n(n-1)x^4}{2}.$$

*(2) The Schultz polynomial of* $D_4^{(n)}$ *is equal to:*

$$Sc(D_4^{(n)}, x) = (2n^2 + 12n)x^1 + (9n^2 + 2n)x^2 + 8n(n-1)x^3 + 2n(n-1)x^4.$$

*(3) The modified Schultz polynomial of* $D_4^{(n)}$ *is equal to:*

$$Sc*(D_4^{(n)}, x) = (4n^2 + 8n)x^1 + (10n^2)x^2 + 8n(n-1)x^3 + 2n(n-1)x^4.$$

**Proof.** Suppose $D_4^{(n)}$ denotes the Dutch windmill graph for all positive integer number $n \geq 2$. According to Figure 1, we see that $D_4^{(n)}$ has $3n+1$vertices (a centre vertex has degree $2n$ and other $3n$ vertices have degree 2.) Also, one can see that the diameter $d(D_4^{(n)})$ of the Dutch windmill graph is equal to 4.

From the edge set of the Dutch windmill graph $D_4^{(n)}$, we see that there are $2n$ 1-edge-paths between the only centre vertex and all other vertices of $D_4^{(n)}$ and there are $2n$ 1-edge-paths between all vertices with degree 2 of$V(D_4^{(n)})$ . So, the coefficient of the first term of the Hosoya, Schultz and modified Schultz polynomials of the Dutch windmill graph $D_4^{(n)}$are equal to $(2n+2n)$,$(4\times 2n+(n+2)\times 2n)$ and$(4\times 2n+2n\times 2n)$, respectively.

In case of $d(D_4^{(n)},2)$: By Figure 1, we see that there are $n$ 2-edge-paths between centre vertex and all other vertices of $D_4^{(n)}$ and there are $n+\frac{2n(2n-1)}{2}$=$2n^2$ 2-edge-paths between all vertices with degree 2 of$D_4^{(n)}$. Thus the second term of the Hosoya, Schultz and modified Schultz polynomials of $D_4^{(n)}$are equal to $(2n^2+n)x^2$, $(4\times 2n^2+(n+2)\times n)x^2$ and$(4\times 2n^2+2n\times n)x^2$, respectively.

In case of $d(D_4^{(n)},3)$: We see that there exist $n(2n-2)$=$2n(n-1)$ 3-edge-paths between all pair of vertices $v,u\in V(D_4^{(n)})$ with degree 2 and the third term of the Hosoya, Schultz and modified Schultz polynomials of $D_4^{(n)}$ are equal to$2n(n-1)x^3$, $(4\times 2n(n-1))x^3$ and$(4\times 2n(n-1))x^3$, respectively.

In case of $d(D_4^{(n)},4)$ ($d(D_4^{(n)})$ =4): We have $\frac{n(n-1)}{2}$ 4-edge-paths between all pairs of vertices $v,u\in V(D_4^{(n)})$ with degree 2, and the fourth term of the Hosoya, Schultz and modified Schultz polynomials of $D_4^{(n)}$are equal to $\frac{n(n-1)x^4}{2}$, $(4\times\frac{n(n-1)}{2})x^4$ and$(4\times\frac{n(n-1)}{2})x^4$, respectively.

Now, by results achieved above, we further follow results for the Dutch Windmill graph$D_4^{(n)}$.

The Hosoya polynomial of the Dutch windmill graph $D_4^{(n)}$ is equal to:

$$H(D_4^{(n)},x)=\sum_{i=1}^{d(D_4^{(n)})} d(D_4^{(n)},i)x^i$$
$$=4nx^1+n(2n+1)x^2+2n(n+1)x^3+\frac{n(n-1)x^4}{2}.$$

The Schultz polynomial of the Dutch windmill graph $D_4^{(n)}$ is equal to:

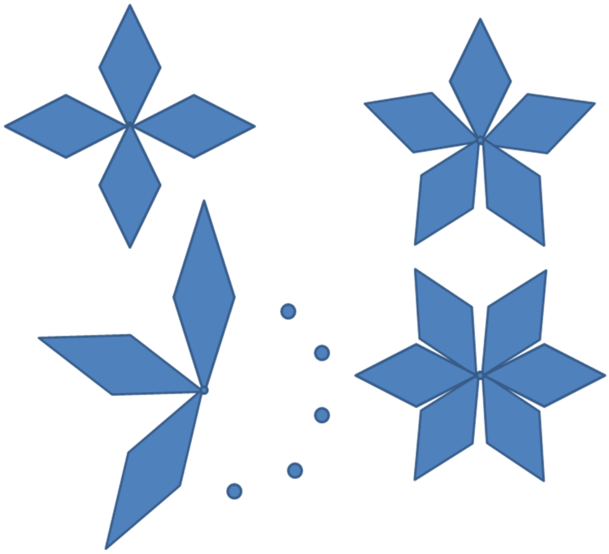

Figure 1: The representations of Dutch windmill graph $D_4^{(n)}$

$$Sc(D_4^{(n)},x)=\frac{1}{2}\sum_{u,v\in V(D_4^{(n)})}(d_u+d_v)x^{d(u,v)}$$
$$=(2n^2+12n)x^1+(9n^2+2n)x^2+8n(n-1)x^3+2n(n-1)x^4.$$

The modified Schultz polynomial of the Dutch windmill graph $D_4^{(n)}$ is equal to:

$$Sc*(D_4^{(n)},x)=\frac{1}{2}\sum_{u,v\in V(D_4^{(n)})}(d_u\times d_v)x^{d(u,v)}$$
$$=(4n^2+8n)x^1+(10n^2)x^2+8n(n-1)x^3+2n(n-1)x^4.$$

Here the proof of theorem is completed.

**Notation 2.** Let $\alpha=\frac{2n(2n-2)}{2}=2n(n-1)$ and $\beta=\frac{2n(2n-1)}{2}=n(2n-1)$; $\forall n,a,b\in\mathbb{N}$.

**Position 1.** *The Hosoya, Schultz and modified Schultz polynomials of the Dutch windmill graph* $D_4^{(n)}(\forall n\in\mathbb{N}-\{1\})$ *are equal to:*

$$H(D_4^{(n)},x)=4nx^1+(2n^2+n)x^2+(2n^2+2n)x^3+\frac{\alpha x^4}{4},$$

$$Sc(D_4^{(n)},x)=(2n^2+12n)x^1+(9n^2+2n)x^2+4\alpha x^3+\alpha x^4,$$

$$Sc*(D_4^{(n)},x) = (4n^2+8n)x^1 + (10n^2)x^2 + 4\alpha x^3 + \alpha x^4.$$

**Theorem 2.** *Consider the Dutch windmill graph* $D_5^{(n)}(\forall n \in \mathbb{N} - \{1\})$*. Then:*

*(1) The Hosoya polynomial of* $D_5^{(n)}$ *is equal to:*

$$H(D_5^{(n)},x) = 5nx^1 + (2n^2+3n)x^2 + 4n(n-1)x^3 + 2n(n-1)x^4.$$

*(2) The Schultz polynomial of* $D_5^{(n)}$ *is equal to:*

$$Sc(D_5^{(n)},x) = (2n^2+16n)x^1 + (10n^2+8n)x^2 + 16n(n-1)x^3 + 8n(n-1)x^4.$$

*(3) The modified Schultz polynomial of* $D_5^{(n)}$ *is equal to:*

$$Sc*(D_5^{(n)},x) = (4n^2+12n)x^1 + (12n^2+4n)x^2 + 16n(n-1)x^3 + 8n(n-1)x^4.$$

**Proof.** Let $D_5^{(n)}$denote the Dutch windmill graph$\forall n \in \mathbb{N} - \{1\}$, with $4n+1$ vertices and $5n$ edges, where a center vertex has degree $2n$ and other $4n$ vertices have degree 2, and there are $2n$ edges consisting of a center vertex and there are $3n$ edges $\forall u,v \in V(D_5^{(n)})$ &$uv \in E(D_5^{(n)})$, that $d_u + d_v = 2$.

By analyzing Figure 2, one can see that the diameter of $D_5^{(n)}$ is $d(D_5^{(n)}) = 4$. And obviously the first term of the Hosoya, Schultz and modified Schultz polynomials of $D_5^{(n)}$are equal to$(3n+2n)x$, $(4\times 3n + (n+2)\times 2n)x$ and$(4\times 3n + 2n\times 2n)x$, respectively.

In case of $d(D_5^{(n)},2)$: We see that there are $2n$ 2-edge-paths between center vertex and all other vertices of $D_5^{(n)}$and there are$2n+\beta = 2n + \frac{2n(2n-1)}{2} = 2n^2+n$ 2-edge-paths between all vertices with degree 2 of$D_5^{(n)}$ and the second term of the Hosoya, Schultz and modified Schultz polynomials of $D_5^{(n)}$will be $(4n+\beta = 2n^2+3n)x^2$, $(4\times(2n^2+n)+(n+2)\times 2n)x^2$ and$(4\times(2n^2+n)+2n\times 2n)x^2$, respectively.

In case of $d(D_5^{(n)},3)$: From Figure 2, we have $0+2\alpha = 2n(2n-2)$ 2-edge-paths between all vertices with degree 2 of$D_5^{(n)}$, thus the third term of the Hosoya, Schultz and modified Schultz polynomials of $D_5^{(n)}$are$(2\alpha = 4n(n-1))x^3$, $(4\times 2\alpha = 4\times 4n(n-1))x^3$ and$16n(n-1)x^3$, respectively.

In case of $d(D_5^{(n)}$,4) & $d(D_5^{(n)}) = 4$: In terms of Figure 2, we see that there are $\frac{2n(2n-2)}{2} = \alpha$ 4-edge-paths between all vertices with degree 2 of$D_5^{(n)}$

and we have three terms $\alpha x^4$, $4\alpha x^4$ and $4\alpha x^4$ of the Hosoya, Schultz and modified Schultz polynomials of$D_5^{(n)}$, respectively. These imply the following results$\forall n \in \mathbb{N} - \{1\}$:

The Hosoya polynomial of the Dutch windmill graph $D_5^{(n)}$ is equal to:

$$H(D_5^{(n)}, x) = \sum_{i=1}^{d(D_5^{(n)})} d(D_4^{(n)}, i)x^i$$
$$= 5nx^1 + (2n^2 + 3n)x^2 + 4n(n-1)x^3 + 2n(n-1)x^4.$$

The Schultz polynomial of the Dutch windmill graph $D_5^{(n)}$ is equal to:

$$Sc(D_5^{(n)}, x) = \frac{1}{2} \sum_{u,v \in V(D_5^{(n)})} (d_u + d_v)x^{d(u,v)}$$
$$= (2n^2 + 16n)x^1 + (10n^2 + 8n)x^2 + 16n(n-1)x^3 + 8n(n-1)x^4.$$

The modified Schultz polynomial of the Dutch windmill graph $D_5^{(n)}$ is equal to:

$$Sc*(D_5^{(n)}, x) = \frac{1}{2} \sum_{u,v \in V(D_5^{(n)})} (d_u \times d_v)x^{d(u,v)}$$
$$= (4n^2 + 12n)x^1 + (12n^2 + 4n)x^2 + 16n(n-1)x^3 + 8n(n-1)x^4.$$

**Position 2.** By Denotation 2, we can see that the Hosoya, Schultz and modified Schultz polynomials of the Dutch windmill graph $D_5^{(n)}(\forall n \in \mathbb{N} - \{1\})$ are equal to:

$$H(D_5^{(n)}, x) = 5nx^1 + (2n^2 + 3n)x^2 + 2\alpha x^3 + \alpha x^4$$

$$Sc(D_5^{(n)}, x) = (2n^2 + 16n)x^1 + (10n^2 + 8n)x^2 + 8\alpha x^3 + 4\alpha x^4$$

and

$$Sc*(D_5^{(n)}, x) = (4n^2 + 12n)x^1 + (12n^2 + 4n)x^2 + 8\alpha x^3 + 4\alpha x^4.$$

**Theorem 3.** *Let $D_6^{(n)}$be the $4^{th}$ member of Dutch windmill graphs;$\forall n \in \mathbb{N} - \{1\}$. Then:*

*(1) The Hosoya polynomial of $D_6^{(n)}$ is equal to:*

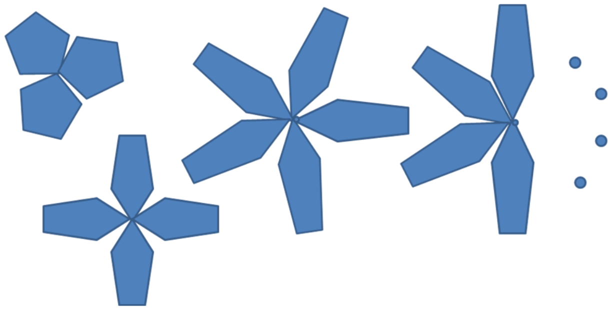

Figure 2: The representations of Dutch windmill graph $D_5^{(n)}$

$$H(D_6^{(n)},x) = 6nx^1 + (2n^2+4n)x^2 + (4n^2-n)x^3 + 4n(n-1)x^4 + 2n(n-1)x^5 + \frac{n(n-1)x^6}{2}.$$

*(2) The Schultz polynomial of* $D_6^{(n)}$ *is equal to:*

$$Sc(D_6^{(n)},x) = (2n^2+20n)x^1 + (10n^2+12n)x^2 + (17n^2-6n)x^3 + 16n(n-1)x^4 + 8n(n-1)x^5 + 2n(n-1)x^6.$$

*(3) The modified Schultz polynomial of* $D_6^{(n)}$ *is equal to:*

$$Sc*(D_6^{(n)},x) = (4n^2+16n)x^1 + (12n^2+8n)x^2 + (18n^2-2n)x^3 + 16n(n-1)x^4 + 8n(n-1)x^5 + 2n(n-1)x^6.$$

**Proof.** Consider the Dutch windmill graph $D_6^{(n)}$with $5n+1$ vertices and $6n$ edges deposit in Figure 3. Similar to above proofs, it is easy to see that the first term of the Hosoya, Schultz and modified Schultz polynomials of $D_6^{(n)}$are equal to$(4n+2n)x$,$(4\times 4n+(n+2)\times 2n)x$ and$(4\times 4n+2n\times 2n)x$, respectively. Also, by the structure of the Dutch windmill graph$D_6^{(n)}$, we conclude $d(D_6^{(n)}) = 2\left\lfloor \frac{6}{2} \right\rfloor = 6$.

In case of $d(D_6^{(n)},2)$: We have $2n$ 2-edge-paths between centre vertex and all other vertices of $D_6^{(n)}$and there are $\left\lceil \frac{6}{2} \right\rceil n + b = 3n + \frac{2n(2n-1)}{2} = 2n^2+2n$ 2-edge-paths between all vertices with degree 2 of$D_6^{(n)}$. Thus the second term of the Hosoya, Schultz and modified Schultz polynomials of $D_6^{(n)}$will be$(2n+2n^2+2n)x^2$,$(4(2n^2+2n)+(n+2)\times 2n)x^2$ and$(4(2n^2+2n)+2n\times 2n)x^2$, respectively.

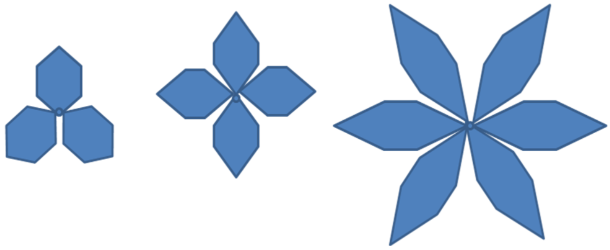

Figure 3: The representations of Dutch windmill graph $D_6^{(n)}$

In case of $d(D_6^{(n)}, 3)$: By means of Figure 3, one can see that there are $n$ 3-edge-paths consisting of a centre vertex. And there are $(\lceil \frac{6}{2} \rceil - 1)n + 2a = 2n + 2n(2n-2) = 4n^2 - 2n$ 3-edge-paths between all vertices with degree 2 of$D_6^{(n)}$. And obviously, the third terms of our polynomials of $D_6^{(n)}$will be$(n+4n^2-2n)x^3$,$(4\times 2\beta+(n+2)\times n)x^3$ and$(4\times 2\beta+2n\times n)x^3$, respectively.

In case of $d(D_6^{(n)}, 4)$: There are only $a+a = 4n(n-1)$ 4-edge-paths between all vertices with degree 2 of $D_6^{(n)}$.

In cases of $d(D_6^{(n)}, 5)$& $d(D_6^{(n)}, 6)$: From Figure 3, we see that there are $\alpha$ 5-edge-paths and $\frac{\alpha}{4}$ 6-edge-paths between all vertices with degree 2 of$D_6^{(n)}$. Thus, we have three terms of the Hosoya, Schultz and modified Schultz polynomials of $D_6^{(n)}$ as

$$(2\alpha + 4n(n-1))x^4 + (\alpha + 2n(n-1))x^5 + (\frac{\alpha}{4} + \frac{n(n-1)}{2})x^6,$$

$$(8\alpha + 16n(n-1))x^4 + (4\alpha + 8n(n-1))x^5 + (\alpha + 2n(n-1))x^6,$$

and $16n(n-1)x^4 + 8n(n-1)x^5 + 2n(n-1)x^6$, respectively.

Here, we will have following formulas for the Hosoya, Schultz and modified Schultz polynomials of the Dutch windmill graph$D_6^{(n)}$:

The Hosoya polynomial of $D_6^{(n)}$is equal to:

$$H(D_6^{(n)}, x) = 6nx^1 + (2n^2+4n)x^2 + (4n^2-n)x^3 + 4n(n-1)x^4 + 2n(n-1)x^5 + \frac{n(n-1)}{2}x^6.$$

The Schultz polynomial of $D_6^{(n)}$is equal to:

$$Sc(D_6^{(n)}, x) = (2n^2+20n)x^1 + \left(10n^2+12n\right)x^2$$

$$+ (17n^2 - 6n)x^3 + 16n(n-1)x^4 + 8n(n-1)x^5 + 2n(n-1)x^6.$$

The modified Schultz polynomial of $D_6^{(n)}$is equal to:

$$Sc * (D_6^{(n)}, x) = (4n^2 + 16n)x^1 + (12n^2 + 8n)x^2 + (18n^2 - 2n)x^3 + 16n(n-1)x^4 + 8n(n-1)x^5 + 2n(n-1)x^6.$$

**Position 3.** By Denotation 2 and Theorem 3, we have following results for the Dutch windmill graph$D_6^{(n)}$:

$$H(D_6^{(n)}, x) = 6nx^1 + (2n^2 + 4n)x^2 + (4n^2 - n)x^3 + 2\alpha x^4 + \alpha x^5 + \frac{\alpha x^6}{4},$$

$$Sc(D_6^{(n)}, x) = (2n^2+20n)x^1+(10n^2+12n)x^2+(17n^2-6n)x^3+8\alpha x^4+4\alpha x^5+\alpha x^6,$$

$$Sc*(D_6^{(n)}, x) = (4n^2+16n)x^1+(12n^2+8n)x^2+(18n^2-2n)x^3+8\alpha x^4+4\alpha x^5+\alpha x^6.$$

**Theorem 4.** Consider the graph of Dutch windmill graph $D_7^{(n)}$depicted in Figure4, $\forall n \in \mathbb{N} - \{1\}$. Then:

The Hosoya polynomial of $D_7^{(n)}$ is equal to:

$$H(D_7^{(n)}, x) = 6nx^1 + (2n^2 + 4n)x^2 + 4n(n-1)x^4 + 2n(n-1)x^5 + \frac{n(n-1)x^6}{2}.$$

The Schultz polynomial of $D_7^{(n)}$ is equal to:

$$Sc(D_7^{(n)}, x) = (2n^2 + 20n)x^1 + (10n^2 + 12n)x^2 + 16n(n-1)x^4 + 8n(n-1)x^5 + 2n(n-1)x^6.$$

The modified Schultz polynomial of $D_7^{(n)}$ is equal to:

$$Sc * (D_7^{(n)}, x) = (4n^2 + 16n)x^1 + (12n^2 + 8n)x^2 + 16n(n-1)x^4 + 8n(n-1)x^5 + 2n(n-1)x^6.$$

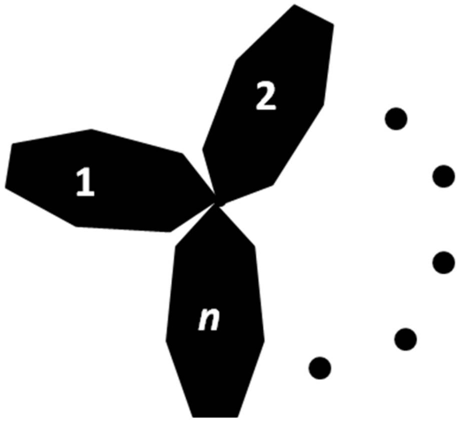


Figure 4: A general representations of Dutch windmill graph $D_7^{(n)}$ .

**Proof 4.** For the graph Dutch windmill graph $D_7^{(n)}$with $6n+1$ vertices and $7n$ edges, the formulas of the Hosoya, Schultz and modified Schultz polynomials can be deduced to:

In cases of $d(D_7^{(n)},1)$& $d(D_7^{(n)},2)$: Similar to above proofs, the first term of polynomials of $D_7^{(n)}$can be concluded by the edges set $E(D_7^{(n)})$. There are $2n$ 2-edge-paths between centre vertex and all other vertices of $D_7^{(n)}$and there are $\lceil\frac{6}{2}\rceil n+\beta{=}4n+\frac{2n(2n-1)}{2}{=}2n^2+3n$ 2-edge-paths between all vertices with degree 2 of$D_7^{(n)}$. So, we have the first and second terms of the Hosoya, Schultz and modified Schultz polynomials of $D_7^{(n)}$as $(5n{+}2n)x{+}(2n{+}2n^2{+}3n)x^2$,$(4\times 5n+(n+2)\times 2n)x+(4(2n^2+3n)+(n+2)\times 2n)x^2$ and$(4\times 5n+(n+2)\times 2n)x+(4(2n^2+3n)+2n\times 2n)x^2$, respectively.

In case of $d(D_7^{(n)},3)$ $(3=\lfloor\frac{6}{2}\rfloor)$: From Figure 4, for the Dutch windmill graph$D_7^{(n)}$, We have $2n$ 3-edge-paths between centre vertex and all other vertices of $D_7^{(n)}$and there are $(\lceil\frac{6}{2}\rceil-1)n+2\beta{=}3n+2n\,(2n-1){=}4n^2+n$ 3-edge-paths between all vertices with degree 2 of $D_7^{(n)}$. Thus the $3^{rd}$ term of the Hosoya, Schultz and modified Schultz polynomials of $D_7^{(n)}$will be$(2n+4n^2+n)x^3$, $(4(4n^2+n)+(n+2)\times 2n)x^3$ and$(4(4n^2+n)+2n\times 2n)x^3$, respectively.

In case of $d(D_7^{(n)},4)$: In view of Figure 4, we see that there are $2n(2n-2){+}\frac{2n(2n-2)}{2}{=}3\alpha$ 4-edge-paths between all vertices with degree 2 of$D_7^{(n)}$ and we have three terms $3\alpha x^4$, $12\alpha x^4$ and $12\alpha x^4$ of the Hosoya, Schultz and modified Schultz polynomials of $D_7^{(n)}$, respectively.

In cases of $d(D_7^{(n)},5)$ & $d(D_7^{(n)},6)$: There are only $\alpha+\alpha=4n(n-1)$ and

$\frac{2n(2n-2)}{2} = \alpha$ 5-edge-paths and 6-edge-paths between all vertices with degree 2 of$D_7^{(n)}$ (see Figure 4). From the structure of the Dutch windmill graph$D_7^{(n)}$ one can see that $d(D_7^{(n)})$=2 $\lfloor \frac{7}{2} \rfloor$=6. Thus, we have three last terms of the Hosoya, Schultz and modified Schultz polynomials of $D_7^{(n)}$are equal to, respectively:

$$(3\alpha + 6n(n-1))x^4 + (2\alpha + 4n(n-1))x^5 + (\alpha + 2n\,(n-1))x^6,$$

$$(12\alpha + 24n(n-1))x^4 + (8\alpha + 16n(n-1))x^5 + (4\alpha + 8n(n-1))x^6,$$

$$(12\alpha + 24n(n-1))x^4 + (8\alpha + 16n(n-1))x^5 + (4\alpha + 8n(n-1))x^6.$$

Finally, we have: The Hosoya polynomial of $D_7^{(n)}$ is equal to:

$$H(D_7^{(n)}, x) = 7nx^1 + (2n^2 + 5n)x^2 + (4n^2 + 3n)x^3 + 6n(n-1)x^4 + 4n(n-1)x^5 + 2n(n-1)x^6.$$

The Schultz polynomial of $D_7^{(n)}$is equal to:

$$Sc(D_7^{(n)}, x) = (2n^2 + 24n)x^1 + \left(10n^2 + 16n\right)x^2 + (18n^2 + 8n)x^3 + 24n(n-1)x^4 + 16n(n-1)x^5 + 8n(n-1)x^6.$$

The modified Schultz polynomial of $D_7^{(n)}$is equal to:

$$Sc*(D_7^{(n)}, x) = (4n^2 + 20n)x^1 + \left(12n^2 + 12n\right)x^2 + (20n^2 + 4n)x^3 + 24n(n-1)x^4 + 16n(n-1)x^5 + 8n(n-1)x^6.$$

**Position 4.** The Hosoya, Schultz and modified Schultz polynomials of Dutch windmill graph $D_7^{(n)}$are equal to:

$$\begin{aligned} H(D_7^{(n)}, x) =& 7nx^1 + (2n^2 + 5n)x^2 + (4n^2 + 3n)x^3 + 3\alpha x^4 \\ &+ 2\alpha x^5 + \alpha x^6, \\ Sc(D_7^{(n)}, x) =& (2n^2 + 24n)x^1 + (10n^2 + 12n)x^2 + (18n^2 + 8n)x^3 \\ &+ 12\alpha x^4 + 8\alpha x^5 + 4\alpha x^6, \\ Sc*(D_7^{(n)}, x) =& (4n^2 + 20n)x^1 + (12n^2 + 8n)x^2 \\ &+ (20n^2 + 4n)x^3 + 12\alpha x^4 + 8\alpha x^5 + 4\alpha x^6. \end{aligned}$$

**Theorem 5.** *Let $D_m^{(n)}$be the $m^{th}$member of Dutch windmill graphs; $\forall n, m \in \mathbb{N} - \{1\}$ & $\forall m = 2q \geq 6$. Then:*

$$H(D_m^{(n)}, x) = 7nx^1 + (2n + \beta + (m-3)n + 2n^2 + (m-2)n)x^2$$

$$
\begin{aligned}
&+ (2n + 2\alpha + (m-4)n + 4n^2 + (m-4)n)x^3 \\
&+ ... + (2n + (q+1)n + 2\alpha + 4n^2 + (q-1)n)x^{q-2} \\
&+ (2n + qn + 2a + 4n^2 + (q-2)n)x^{q-1} \\
&+ (n + (q-1)n + 2\alpha + 4n^2 + (q-4)n)x^q + (q-1)\alpha x^{q+1} \\
&+ ... + 2\alpha x^{2q-2} + \alpha x^{2q-1} + x^{2q};
\end{aligned}
$$

$$
\begin{aligned}
Sc(D_m^{(n)}, x) =&(2n^2 + 4(m-1)n)x^1 + (10n^2 + 4(m-3)n)x^2 \\
&+ ((n+2) \times 2n + 4 \times (2\alpha + (m-4)n) + 18n^2 + 4(m-7)n)x^3 \\
&+ ... + ((n+2) \times 2n + 4 \times ((q+1)n + 2\alpha) \\
&+ 18n^2 + 4(q-2)n)x^{q-2} + ((n+2) \times 2n + 4 \times (qn + 2\alpha) \\
&+ 18n^2 + 4(q-3)n)x^{q-1} + ((n+2) \times n + 4 \times ((q-1)n + 2\alpha) \\
&+ 18n^2 + 4(q-4)n)x^q + 4(q-1)\alpha x^{q+1} + ... + 8\alpha x^{2q-2} \\
&+ 4\alpha x^{2q-1} + \alpha x^{2q};
\end{aligned}
$$

$$
\begin{aligned}
Sc*(D_m^{(n)}, x) =&(4n^2 + 4(m-2)n)x^1 + (12n^2 + 4(m-4)n)x^2 \\
&+ (2n \times 2n + 4 \times (2\alpha + (m-4)n) + 20n^2 + 4(m-8)n)x^3 \\
&+ ... + (2n \times 2n + 4 \times ((q+1)n + 2\alpha) + 20n^2 + 4(q-3)n)x^{q-2} \\
&+ (2n \times 2n + 4 \times (qn + 2\alpha) + 20n^2 + 4(q-4)n)x^{q-1} \\
&+ (2n \times n + 4 \times ((q-1)n + 2\alpha) + 20n^2 + 4(q-5)n)x^q \\
&+ 4(q-1)\alpha x^{q+1} + ... + 8\alpha x^{2q-2} + 4\alpha x^{2q-1} \\
&+ \alpha x^{2q}.
\end{aligned}
$$

**Proof.** For the Dutch windmill graph $D_m^{(n)}$ ($\forall m = 2q \geq 6$ or $\lfloor \frac{m}{2} \rfloor = q \geq 3$), the formulas of the Hosoya, Schultz and modified Schultz polynomials can be deduced. Since the diameter $d(D_{2q}^{(n)})$of the Dutch windmill graph is equal to $2\lfloor \frac{m}{2} \rfloor = 2q$.

From the above proofs, we see that there are only $\frac{\alpha}{4}$ 2q-edge-paths between all vertices with degree 2 of$D_{2q}^{(n)}$. And there are $da\,(q-d)$- edge-paths between all vertices with degree 2 of$D_{2q}^{(n)}$ for all $d = 1, 2, \ldots,\ q-1$.

In case of $d(D_{2q}^{(n)}, q)$ ($q = \lfloor \frac{m}{2} \rfloor$): For the Dutch windmill graph$D_{2q}^{(n)}$, we have $n$ 3-edge-paths between center vertex and all other vertices of $D_{2q}^{(n)}$and there are $(\lfloor \frac{m}{2} \rfloor - 1)n + 2\alpha$=$(q-1)n + 2n(2n-2)q$ -edge-paths between all vertices with degree 2 of$D_{2q}^{(n)}$.

In case of $d(D_{2q}^{(n)}, d)\forall d = 3, ..., q-1$: From the definition of Dutch windmill graph$D_m^{(n)}$, We have $2n$ d-edge-paths between centre vertex and all other vertices and there are$(2q-d-1)n + 2\alpha$=$4n^2 + (m-5-d)n$ $d$ -edge-paths between all vertices with degree 2 of$D_m^{(n)}$.

Thus these complete the proof of Theorem 5, clearly.

**Theorem 6.** *Let be the mth member of Dutch windmill graphs;*$\forall n, m \in \mathbb{N} - \{1\}$ *&* $\forall m = 2q+1 \geq 5$, *Then the Hosoya, Schultz and modified Schultz polynomials of Dutch windmill graph* $D_m^{(n)}$ *are equal to:*

$$
\begin{aligned}
H(D_m^{(n)}, x) =& 7nx^1 + (2n + \beta + (m-3)n + 2n^2 + (m-2)n)x^2 \\
&+ (2n + 2\beta + (m-4)n + 4n^2 + (m-4)n)x^3 + ... \\
&+ (2n + (q+2)n + 2\beta + 4n^2 + (q+2)n)x^{q-2} \\
&+ (2n + (q+1)n + 2\beta \\
&+ 4n^2 + (q+2)n)x^{q-1} + (2n + qn + 2\beta + 4n^2 + qn)x^q + ... \\
&+ q\alpha x^{q+1} + ... + 3\alpha x^{2q-2} + 2\alpha x^{2q-1} + \alpha x^{2q}; \\
Sc(D_m^{(n)}, x) =& (2n^2 + 4(m-1)n)x^1 + (10n^2 + 4(m-3)n)x^2 \\
&+ ((n+2) \times 2n + 4 \times (2\beta + (m-4)n) + 18n^2 \\
&+ 4(m-5)n)x^3 ... + ((n+2) \times 2n + 4 \times ((q+2)n + 2\beta)) \\
&+ 18n^2 + 4(q+1)nx^{q-2} + ((n+2) \times 2n \\
&+ 4 \times ((q+1)n + 2\beta)) + 18n^2 + 4qn)x^{q-1} \\
&+ (n+2) \times 2n + 4 \times (qn + 2\beta) + 18n^2 + 4(q-1)nx^q \\
&+ 4q\alpha x^{q+1} + ... + 12\alpha x^{2q-2} + 8\alpha x^{2q-1} + 4\alpha x^{2q}; \\
Sc*(D_m^{(n)}, x) =& (4n^2 + 4(m-2)n)x^1 + (12n^2 + 4(m-4)n)x^2 \\
&+ (2n \times 2n + 4 \times (2\beta + (m-4)n) + 20n^2 \\
&+ 4(m-6)n)x^3 + ... + (2n \times 2n \\
&+ 4 \times ((q+2)n + 2\beta)) + 20n^2 + 4qn)x^{q-2} \\
&+ (2n \times 2n + 4 \times ((q+1)n + 2b) + 20n^2 + 4(q-1)n)x^{q-1} \\
&+ (2n \times 2n + 4 \times (qn + 2\beta) + 20n^2 + 4(q-2)n)x^q \\
&+ 4q\alpha x^{q+1} + ... + 12\alpha x^{2q-2} + 8\alpha x^{2q-1} + 4\alpha x^{2q}.
\end{aligned}
$$

**Proof** Consider the graph of Dutch windmill graph $D_m^{(n)} \forall n, m, q \in \mathbb{N} - \{1\}$

and $m = 2q + 1 \geq 5$. Thus, by similar arguments to above proof, we have following results.

In case of $d(D_{2q+1}^{(n)}, d) \forall d = q+1, ..., 2\left\lfloor \frac{m}{2} \right\rfloor = 2q = m-1$: From the definition of Dutch windmill graph$D_m^{(n)}$, it is easy to see that the diameter of$D_{2q+1}^{(n)}$ is equal to $2\left\lfloor \frac{m}{2} \right\rfloor = 2q$ and the number of $d$ -edge-paths between all vertices with degree 2 of $D_m^{(n)}(d = q+1, \ldots, m-1)$ are equal to $(m-d)\alpha$. Obviously the $q$ last terms of the Hosoya, Schultz and modified Schultz polynomials of Dutch windmill graph $D_m^{(n)}$ will be $q\alpha x^{q+1} + ... + 3\alpha x^{2q-2} + 2\alpha x^{2q-1} + \alpha x^{2q}$,$4q\alpha x^{q+1} + ... + 12\alpha x^{2q-2} + 8\alpha x^{2q-1} + 4\alpha x^{2q}$ , and$4q\alpha x^{q+1} + ... + 12\alpha x^{2q-2} + 8\alpha x^{2q-1} + 4\alpha x^{2q}$, respectively.

In case of $d(D_{2q}^{(n)}, d) \forall d = 3, ..., q(= \left\lfloor \frac{m}{2} \right\rfloor)$: From the definition of Dutch windmill graph $D_m^{(n)}$, $2n$ d-edge-paths start from centre vertex of $D_m^{(n)}$and the number of $d$ -edge-paths between other vertices are$(2q-d)n + 2\beta = 4n^2 + (m-3-d)n$ .

These imply that the Hosoya, Schultz and modified Schultz polynomials of Dutch windmill graph $D_m^{(n)}$for $m = 2q+1$and the proof is completed.

## 3. CONCLUSION

In the paper, we propose the Hosoya, Schultz and modified Schultz polynomials of Dutch windmill graph $D_n^{(m)}$ by using nano molecular graph structural analysis, distance computation and mathematical derivation. The results obtained here shows great applications and promising prospects in chemical engineering and nano material manufacturing.

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