AN ERROR ESTIMATION IN BACKUS-GILBER MOVING LEAST-SQUARES APPROXIMATION

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ABSTRACT: Our goal is to establish a bound of the error in moving least-squares approximation in terms of coefficients of the approximation and error of nearest data points.

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1. STATEMENT

Let us remind the definition of moving least-squares approximation and a basic result.

Let:

1. $\mathcal{D}$ be a compact in $\mathbb{R}^d$.

2. $x_i \in \mathcal{D}$, $i = 1, \ldots, m$; $x_i \neq x_j$, if $i \neq j$.

3. $f : \mathcal{D} \to \mathbb{R}$ be a continuous function.

4. $p_i : \mathcal{D} \to \mathbb{R}$ be continuous functions, $i = 1, \ldots, l$. The functions $\{p_1, \ldots, p_l\}$ are linearly independent in $\mathcal{D}$ and let $\mathcal{P}_i$ be their linear span.
5. $W: [0, \infty) \to (0, \infty)$ be a continuous and positive function.

Usually the basis in $P_l$ is constructed by monomials. For example: $p_l(x) = x_1^{k_1} \cdots x_d^{k_d}$, where $x = (x_1, \ldots, x_d)$, $k_1, \ldots, k_d \in \mathbb{N}$, $k_1 + \cdots + k_d \leq l - 1$. In this case, we will use the notation $x^k$, $|k| \leq l - 1$. In the case $d = 1$, the standard basis is $\{1, x, \ldots, x^{l-1}\}$.

Following [3], we will use the following definition. The moving least-squares approximation of order $l$ at a fixed point $x$ is the value of $p^*(x)$, where $p^* \in P_l$ is minimizing the least-squares error

$$\sum_{i=1}^{m} W(\|x - x_i\|) (p(x) - f(x_i))^2$$

among all $p \in P_l$.

The approximation is “local” if weight function $W$ is fast decreasing as its argument tends to infinity and interpolation is achieved if $W(0) = \infty$. We define additional function $w: [0, \infty) \to [0, \infty)$, such taht:

$$w(r) = \begin{cases} 
\frac{1}{W(r)}, & \text{if } r > 0, \\
\frac{1}{W(r)} - 1, & \text{if } r = 0 \text{ and } W(0) < \infty, \\
0, & \text{if } r = 0 \text{ and } W(0) = \infty.
\end{cases}$$

Some examples of $W(r)$ and $w(r)$, $r \geq 0$:

$$W(r) = e^{-\alpha^2 r^2} \quad \text{exp-weight},$$
$$W(r) = r^{-\alpha^2} \quad \text{Shepard weights},$$
$$W(r) = r^2 e^{-\alpha^2 r^2} \quad \text{McLain weight},$$
$$w(r) = e^{\alpha^2 r^2} - 1 \quad \text{see Levin’s works}.$$

Below: $\| \cdot \| = \| \cdot \|_2$ is the 2-norm, $\| \cdot \|_1$ is 1-norm in $\mathbb{R}^d$; the superscript $^t$ denotes transpose of real matrix; $I$ is the identity matrix. Let

$$E = \begin{pmatrix} 
   p_1(x_1) & p_2(x_1) & \cdots & p_l(x_1) \\
   p_1(x_2) & p_2(x_2) & \cdots & p_l(x_2) \\
   \vdots & \vdots & \ddots & \vdots \\
   p_1(x_m) & p_2(x_m) & \cdots & p_l(x_m) 
\end{pmatrix}.$$
Through the article, we assume the following conditions (H1):

(H1.1) $1 \in \mathcal{P}_l$.

(H1.2) $1 \leq l \leq m$.

(H1.3) $\text{rank}(E) = l$.

(H1.4) $w$ is a smooth function.

(H1.5) The basis in $\mathcal{P}_l$ is

$$
\left\{ x^k : |k| \leq l - 1 \right\} = \left\{ x_1^{k_1} \ldots x_d^{k_d} : k_1 + \cdots + k_d \leq l - 1 \right\}.
$$

The equivalent statement of the moving least-squares minimization problem is the following constrained problem:

Minimum of quadratic form $Q = \sum_{i=1}^{m} w(x, x_i) a_i^2$, \hspace{1cm} (1.1)

subject to $\sum_{i=1}^{m} a_i p_j(x_i) = p_j(x), \ j = 1, \ldots, l$. \hspace{1cm} (1.2)

**Theorem 1.1** (see [3]). Let the conditions (H1.1)-(H1.4) hold true.

Then:

1. The matrix $E^t D^{-1} E$ is non-singular.

2. The approximation defined by the moving least-squares method is

$$
\hat{L}(f) = \langle a, F \rangle = \sum_{i=1}^{m} a_i f(x_i), \hspace{1cm} (1.3)
$$
where
\[ a = A_0 c \quad \text{and} \quad A_0 = D^{-1} E (E^t D^{-1} E)^{-1}. \] (1.4)

3. If \( w(0) = 0 \), then the approximation is interpolatory.

**Remark 1.1.** Let us mark: \( D, c, a \) depend on \( x \); the matrix \( E \) and the vector \( F \) do not depend \( x \).

## 2. APPROXIMATION ORDER

Let the hypothesis (H1) be valid, \( x \) be a fixed point, \( x \neq x_i, i = 1, \ldots, m \), and let an integer \( k_0 \) (not unique in general case) be choosen such that
\[ h = \|x - x_{k_0}\| = \min \{\|x - x_i\| : i = 1, \ldots, m\}. \]

**Remark 2.1.** Let us mark the following reproduction property: for all \( p \in \mathcal{P}_l \), we have
\[ \sum_{i=1}^{m} p(x_i)a_i = p(x). \]

or if we set \( P = \begin{pmatrix} p(x_1) & p(x_2) & \cdots & p(x_m) \end{pmatrix}^t \) then
\[ p(x) = \langle a, P \rangle. \]

Indeed, let \( p \in \mathcal{P}_l \). Then there exist constants \( b_1, \ldots, b_l \) such that
\[ p(x) = \sum_{k=1}^{l} b_k p_k(x) \]

and using (1.2), we receive
\[ p(x) = \sum_{k=1}^{l} b_k \sum_{i=1}^{m} a_i p(x_i) = \sum_{i=1}^{m} a_i \sum_{k=1}^{l} b_k p_k(x_i) \]
\[ = \sum_{i=1}^{m} a_i p(x_i) = \langle a, P \rangle. \]
Let \( p \in \mathcal{P}_l \). Combining the previous remark and Cauchy-Schwarz inequality, we obtain

\[
|f(x) - \hat{L}(f)(x)| = |f(x) - p(x) + p(x) - \langle a, F \rangle| \\
\leq |f(x) - p(x)| + |\langle a, P \rangle - \langle a, F \rangle| \\
= |f(x) - p(x)| + |\langle a, P - F \rangle| \\
\leq |f(x) - p(x)| + \|a\| \|P - F\|.
\]

Introducing the classical infinity norm (\( D \) is a compact subset in \( \mathbb{R}^d \)):

\[
\|f - p\|_\infty = \max \{|f(x) - p(x)| : x \in D\},
\]

we have:

\[
|f(x) - p(x)| \leq \|f - p\|_\infty,
\]

\[
\|P - F\|_2^2 = \sum_{i=1}^l (p(x_i) - f(x_i))^2 \\
\leq m \max \{|(p(x_i) - f(x_i))^2 : i = 1, \ldots, l\} \\
= m \left( \max \{|p(x_i) - f(x_i)| : i = 1, \ldots, l\} \right)^2 \\
= m \|f - p\|_\infty^2.
\]

Therefore, for any \( p \in \mathcal{P}_l \) we have

\[
|f(x) - \hat{L}(f)(x)| \leq (1 + \sqrt{m} \|a\|) \|f - p\|_\infty, \quad (2.1)
\]

or we have to analyze the following two terms:

1. The infinity norm of \( f - p \). In Subsection 2.1, we will construct an upper bound of \( \|f - p\|_\infty \), following [1].

2. The norm of Lebesgue function \( \|a\|_1 \). In Subsection 2.2 we will construct an upper estimate for the Lebesgue function \( \|a\|_1 \) or for the equivalent norm \( \|a\|_2 \) in the case of Backus-Gilber approach.

### 2.1. The Norm \( \|f - p\|_\infty \)

Let \( f \) be \((N + 1)\)-times continuously differentiable function in \( D \) and let all derivatives of \( f \) be bounded functions in \( D \).
The inequality (2.1) is valid for any polynomial \( p \in \mathcal{P}_l \). Then it is valid for the Taylor’s polynomial for \( f \) at \( x \) of degree \( N \) on the one dimensional segment with endpoints \( x \) and \( x_1 \). Hence there exists a constant \( C_{k_0} > 0 \) such that

\[
\|f - p\|_\infty \leq C_{k_0} \|x - x_{k_0}\|^{N+1}.
\]

Then

\[
|f(x) - \hat{L}(f)(x)| \leq (1 + \sqrt{m} \|\alpha\|) C_{k_0} h^{N+1}.
\] (2.2)

Obviously, we may use

\[
C_{k_0} = \frac{1}{(N+1)!} \max \left\{ \left| f^{(N+1)}_\alpha (\alpha x + (1 - \alpha)x_{k_0}) \right| : \alpha \in [0, 1] \right\},
\]

but in this case \( C_{k_0} \) depends on \( x \). To avoid this, let \( C_{k_0} = \max \{ C_{k_0}(x) : x \in \mathcal{D} \} \).

### 2.2. BACKUS-GILBER APPROACH

The Backus-Gilber approach of moving-least squares method is exactly the constrained problem (1.1), (1.2) with \( l = 1 \) and \( p_1(x) = 1 \), see [3, Section 2].

In this case, we have the following result.

**Theorem 2.1.** Let the conditions (H1) hold. Let \( f \) be \((N+1)\)-times continuously differentiable function in \( \mathcal{D} \) and let all derivatives of \( f \) be bounded functions in \( \mathcal{D} \).

Then for any \( x \in \mathcal{D} \) the following inequality holds

\[
|f(x) - \hat{L}(f)(x)| \leq (1 + \|c\|) C_{k_0} h^{N+1}.
\] (2.3)

**Proof.** Let

\[
A = D^{-1}E \left( E^tD^{-1}E \right)^{-1} E^t = A_0E^t.
\]

Then (using [2, P15-SVD], let us remind that \( l \leq m \))

\[
\sigma_{\max}(A_0) \sigma_{\min}(E^t) \leq \sigma_{\max}(A_0E^t) = \sigma_{\max}(A).
\]

But (see [4])

\[
\sigma_{\max}(A) \leq \sqrt{m}.
\]
On the other hand, it is not hard to calculate the singular value decomposition of the vector $E$:

$$E = UV^t,$$

where:

$$U = \left( e \ 	ilde{U} \right), \quad e = \frac{E}{\|E\|_2} = \frac{1}{\sqrt{m}}E;$$

$\tilde{U}$ is the $(m \times (m - 1))$-matrix with orthonormal columns and such that $\tilde{U}^t E = 0$;

$$\Sigma = \left( \|E\|_2 \ 0 \ \cdots \ 0 \right)^t = \left( \sqrt{m} \ 0 \ \cdots \ 0 \right)^t, \quad V = (1).$$

Hence $\sigma_{\min}(E^t) = \sigma_{\min}(E) = \sqrt{m}$ and

$$\|A_0\| = \sigma_{\max}(A_0) \leq \frac{\sqrt{m}}{\sigma_{\min}(E^t)} = 1.$$

Therefore

$$\|a\| = \|A_0c\| \leq \|A_0\| \|c\| = \|c\|.$$

The inequality (2.3) follows from (2.2). \hfill \Box

REFERENCES


