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BEST PROXIMITY POINTS FOR GENERALIZED CYCLIC CONTRACTION MAPPINGS

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ABSTRACT: In this paper, by using the concept of MT-function, a notion of generalized cyclic contraction mapping is introduced and best proximity point theorems for such mapping is established in the framework of metric spaces. We also provide example illustrating the obtained results.

This paper is dedicated to Professor A. A. Tijjani

Key Words: best proximity point, contraction mapping, cyclic mappings, MT-function

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1. INTRODUCTION AND PRELIMINARIES

The fact that, fixed point theory gives a cohesive treatment and is a fundamental tool for solving equations of the form $Tx = x$ where T is a self-mapping defined on a subset of a normed space, metric space, topological space or some appropriate space, leads to the importance of the area under consideration.

But, almost all such results depend upon the existence of a fixed point for self-mappings. Indeed, if T is a non-self mapping, the equation $Tx = x$ does not necessarily have a solution.

In this case, Fan [1] introduced the concept of best approximation theorems, which investigate the existence of an approximate solution, that is, if A is a non-empty subset of a considered space X and $T : A \to X$, then we can find a point $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Although best approximation theorems generate an approximate solution to the equation $Tx = x$, they do not give an approximate solution that is optimal. In contrast, best proximity point theorems give an approximate solution that is optimal, that is, if A and B are nonempty subsets of a considered space X such that $d(x, Tx) = \text{dist}\,(A, B)$, where $\text{dist}\,(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

An exciting class of mappings for considering best proximity point theorems is a concept of cyclic map, that is, let A and B are nonempty subsets of a nonempty set X, a map $T : A \cup B \to A \cup B$, is a cyclic map if $T(A) \subset B$ and $T(B) \subset A$. Kick [2] introduced the notion of contractions under cyclic conditions. Actually in this case the problem is solved under the hypothesis that the intersection of the sets involved in the cyclic contraction is nonempty, that is $A \cap B$. Moreover, the fixed points are situated in the intersection set.

In [3] Eldreed and Veeramani introduced and proved the following exciting best proximity point theorem for a class of cyclic map.

Definition 1.1. [3] Let A and B be nonempty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is called a *cyclic contraction map*, if $\exists k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y) + (k - 1)\text{dist}\,(A, B)$, for all $x \in A$ and $y \in B$.

Theorem 1.2. *[3] Let A and B be nonempty closed subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Fix $x_1 \in A$ and define $X_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n-1}\}$ has a convergent subsequence in A. Then $\exists\, x \in A$ such that $d(x, Tx) = \text{dist}\,(A, B)$.*

Remark 1.3. If A and B are nonempty subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ is a cyclic contraction, and $A \cap B \neq \emptyset$ then $d(A, B) = 0$, subsequently T is a contraction on the complete metric space $(A \cap B, d)$. Hence, applying the Banach contraction principle by the above theorem we know that T has a uniqued fixed point in $A \cap B$

For more details about best proximity point theorems the reader may consult [4, 5, 6, 7, 8, 9, 10]

In [11], Sadiq et al. introduce the concepts of $K-$Cyclic and $C-$Cyclic for two mappings $T : A \to B$ and $S : B \to A$, as follows:

Definition 1.4. [11] A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a $K-$Cyclic mapping between A and B if there exists a nonnegative real number $k < \frac{1}{2}$ such that

$$d(Tx, Sy) = k[d(x, Tx) + d(y, Sy)] + (1 - 2k)\text{dist}\,(A, B)$$

for all $x \in A$ and $y \in B$.

Definition 1.5. [11] A pair of mappings $T : A \to B$ and $S : B \to A$ is said to form a $C-$Cyclic mapping between A and B if there exists a nonnegative real number $k < \frac{1}{2}$ such that

$$d(Tx, Sy) = k[d(x, Sy) + d(y, Tx)] + (1 - 2k)\text{dist}\,(A, B),$$

for all $x \in A$ and $y \in B$.

Let us recall that a function $\varphi : [0, \infty) \to [0, 1)$ is said to be an MT-*function* if $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$. It is clear that if $\varphi : [0, \infty) \to [0, 1)$ is a non-decreasing function or non-increasing function, then φ is an MT-function. So the set of MT-function is a rich class.

By using MT-function, Du and Lakzian [12] introduced the following concept.

Definition 1.6. Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies

(MT1) $T(A) \subset B$ and $T(B) \subset A$,

(MT2) $\exists$ an $\mathbb{R}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + (1 - \varphi(d(x, y)))\text{dist}\,(A, B),$$

for any $x \in A$ and $y \in B$.

Then T is called an MT-cyclic contraction with respect to φ on $A \cup B$.

Theorem 1.7. *[12] Let A and B be nonempty subsets of a metric space (X,d) and $T : A \cup B \to A \cup B$ be an $\mathbb{R}$-cyclic contraction with respect to φ. Let $x_1 \in A$ be given and define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} = Tx_n$. Suppose that $\{x_{2n-1}\}$ has a convergent subsequence in A. Then there exists $v \in A$ such that $d(v, Tv) = \operatorname{dist}(A, B)$.*

In 2012, Du and Lakzian [12] first proved some characterizations of MT-functions.

Theorem 1.8. *[12] Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent*

(a) φ is an MT-function

(b) For each $t \in [0,\infty)$, there exists $r_t^{(1)} \in [0,\infty)$ and $\epsilon_t^{(1)} > 0$ such that $\varphi(t) \le r_t^{(1)}$ for all $s \in (t, t+\epsilon_t^{(1)})$.

(c) For each $t \in [0,\infty)$, there exists $r_t^{(2)} \in [0,\infty)$ and $\epsilon_t^{(2)} > 0$ such that $\varphi(t) \le r_t^{(2)}$ for all $s \in [t, t+\epsilon_t^{(2)}]$.

(d) For each $t \in [0,\infty)$, there exists $r_t^{(3)} \in [0,\infty)$ and $\epsilon_t^{(3)} > 0$ such that $\varphi(t) \le r_t^{(3)}$ for all $s \in (t, t+\epsilon_t^{(3)}]$.

(e) For each $t \in [0,\infty)$, there exists $r_t^{(4)} \in [0,\infty)$ and $\epsilon_t^{(4)} > 0$ such that $\varphi(t) \le r_t^{(4)}$ for all $s \in [t, t+\epsilon_t^{(4)})$.

(f) For any non-increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have

$$0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1.$$

(g) φ is a function of contractive factor [13]; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

The main objective of this article is to prove best proximity point theorems for generalize MT- cyclic mappings in the frame works of Metric space.

2. MAIN RESULTS

We start this section by presenting some required definitions to prove our new results.

Definition 2.1. Let A and B be two non-empty subsets of a metric space (X,d). A cyclic map $T : A \cup B \to A \cup B$ is called generalized cyclic MT-contraction mapping if there exists an MT-function φ such that

$$\begin{aligned} d(Tx,Ty) \leq &\varphi(d(x,y)) \max[d(x,y), d(y,Tx), d(x,Tx) + d(y,Ty) - d(Tx,Ty), \\ &d(x,Ty), d(y,Ty)] + (1 - \varphi(d(x,y)))d(A,B) \end{aligned}$$

for all $x \in A$ and $y \in B$.

First, we prove in what follows an existence theorem that relate to generalized cyclic MT-contraction mapping.

Theorem 2.2. *Let A and B be two non-empty subsets of a metric space (X,d) and $T : A \cup B \to A \cup B$ be generalized $R-$contraction mapping.Suppose for a fixed element $x_1 \in A$, define an iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ $\forall n \in \mathbb{N}$. Then $d(x_n, x_{n+1}) \to d(A,B)$ as $n \to \infty$.*

Proof. Since T is a generalized cyclic MT-contraction mapping, we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) =&d(Tx_{n+1}, Tx_n) \\ \leq&\varphi(d(x_{n+1}, x_n)) \max[d(x_{n+1}, x_n), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_{n+1}) \\ &+ d(x_n, Tx_n) - d(Tx_{n+1}, Tx_n), d(x_{n+1}, Tx_n), d(x_n, Tx_n)] \\ &+ (1 - \varphi(d(x_{n+1}, x_n)))d(A,B) \\ \leq&\varphi(d(x_{n+1}, x_n)) \max[d(x_{n+1}, x_n), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+2}) \\ &+ d(x_n, x_{n+1}) - d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_{n+1}), d(x_n, x_{n+1})] \\ &+ (1 - \varphi(d(x_{n+1}, x_n)))d(A,B) \\ \leq&\varphi(d(x_{n+1}, x_n)) \max[d(x_{n+1}, x_n), d(x_n, x_{n+2}), d(x_n, x_{n+1}), \\ &d(x_n, x_{n+1}) + (1 - \varphi(d(x_{n+1}, x_n)))d(A,B) \\ \leq&\varphi(d(x_{n+1}, x_n))\max[d(x_{n+1}, x_n), d(x_n, x_{n+2})]+ \\ &(1 - \varphi(d(x_{n+1}, x_n)))d(A,B) \\ \leq&\varphi(d(x_{n+1}, x_n))d(x_{n+1}, x_n) + (1 - \varphi(d(x_{n+1}, x_n)))d(A,B) \end{aligned}$$

$$
\begin{aligned}
&\leq \varphi(d(x_{n+1}, x_n))d(x_{n+1}, x_n) + (1 - \varphi(d(x_{n+1}, x_n)))d(x_{n+1}, x_n) \\
&= d(x_{n+1}, x_n)
\end{aligned}
$$

This shows that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence in $(0, \infty)$. If there exists $j \in \mathbb{N}$ such that $x_j = x_{j+1} \in A \in B$, then by definition $Tx_j = x_{j+1} = x_j$. Also $Tx_{j+2} = T(Tx_{j+1}) = Tx_j = x_j$, so $x_j = x_{j+1} = x_{j+2} = ...$ and therefore

$$
\lim_{n\to\infty} d(x_{n+1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n+1}, x_n) = \text{dist}\,(A, B) = 0,
$$

and proof is completed. So it remain to consider the case $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.

Since the sequence $\{d(x_{n+1}, x_n)\}$ is a non-increasing in $(0, \infty)$, by the property of MT-function, we have

$$
0 \leq \varphi(d(x_{n+1}, x_n)) \leq \lambda < 1.
$$

for all $n \in \mathbb{N}$ where $\lambda := \sup_{n\in\mathbb{N}} \varphi(d(x_{n+1}, x_n))$.

For $x_1 \in A$, we have $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in \mathbb{N}$. Since T is an MT- contraction mapping, we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \varphi(d(x_{n-1}, x_n))d(x_{n-1}, x_n) + \{1 - \varphi(d(x_{n-1}, x_n))\}\text{dist}\,(A, B) \\
&\leq \lambda d(x_{n-1}, x_n) + \text{dist}\,(A, B),
\end{aligned}
$$

and

$$
\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + \{1 - \varphi(d(x_n, x_{n+1}))\}\text{dist}\,(A, B) \\
&\leq \varphi(d(x_n, x_{n+1}))[\lambda d(x_{n-1}, x_n) + \text{dist}\,(A, B)] \\
&\quad + \{1 - \varphi(d(x_n, x_{n+1}))\}\text{dist}\,(A, B) \\
&= \varphi(d(x_n, x_{n+1}))\lambda d(x_{n-1}, x_n) + \varphi(d(x_n, x_{n+1}))\text{dist}\,(A, B) \\
&\quad + \text{dist}\,(A, B) - \varphi(d(x_n, x_{n+1}))\text{dist}\,(A, B) \\
&= \lambda\varphi(d(x_n, x_{n+1}))d(x_{n-1}, x_n) + \text{dist}\,(A, B) \\
&\leq \lambda^2 d(x_{n-1}, x_n) + \text{dist}\,(A, B).
\end{aligned}
$$

Similarly

$$
d(x_{n+2}, x_{n+3}) = d(Tx_{n+1}, Tx_{n+2})
$$

$$
\begin{aligned}
&\leq \varphi(d(x_{n+1}, x_{n+2}))d(x_{n+1}, x_{n+2}) \\
&\quad + \{1 - \varphi(d(x_{n+1}, x_{n+2}))\}\text{dist}\,(A, B) \\
&\leq \varphi(d(x_{n+1}, x_{n+2}))[\lambda^2 d(x_{n-1}, x_n) + \text{dist}\,(A, B)] \\
&\quad + \{1 - \varphi(d(x_{n+1}, x_{n+2}))\}\text{dist}\,(A, B) \\
&= \varphi(d(x_{n+1}, x_{n+2}))\lambda^2 d(x_{n-1}, x_n) + \varphi(d(x_{n+1}, x_{n+2}))\text{dist}\,(A, B) \\
&\quad + \text{dist}\,(A, B) - \varphi(d(x_{n+1}, x_{n+2}))\text{dist}\,(A, B) \\
&= \lambda^2 \varphi(d(x_{n+1}, x_{n+2}))d(x_{n-1}, x_n) + \text{dist}\,(A, B) \\
&\leq \lambda^3 d(x_{n-1}, x_n) + \text{dist}\,(A, B).
\end{aligned}
$$

On the other hand, if $x_1 \in B$,then $x_{2n} \in A$ and $x_{2n-1} \in B$ $\forall n \in \mathbb{N}$. Applying MT-cyclic contraction again, we also have

$$\text{dist}\,(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) + \text{dist}\,(A, B)$$

and

$$\text{dist}\,(x_{n+1}, x_{n+2}) \leq \lambda^2 d(x_{n-1}, x_n) + \text{dist}\,(A, B)$$

Hence, continuing in this fashion, we obtained

$$d(A, B) \quad \leq \quad d(x_{n+1}, x_{n+2}) \leq \lambda^n d(x_n, x_{n++1}) + d(A, B) \tag{2.1}$$

$\forall n \in \mathbb{N}$. Since $\lambda \in [0, 1)$, we have $\lim_{n\to\infty} \lambda^n = 0$. By inequality (2.1) and nonincreasing nature of $\{d(x_n, x_{n+1})\}$, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = d(A, B).$$

□

Here we give an existence theorem for best proximity points

Theorem 2.3. *Let A and B be two nonempty subsets of a metric space (X,d) and $T : A \cup B \to A \cup B$ be a generalized cyclic MT-contraction. suppose for a fixed element $x_1 \in A$, define an iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$. Suppose further that $\{x_{n-1}\}$ has a convergent subsequence in A, then there exists a point $x \in A$ such that $d(x, Tx) = d(A, B)$.*

Proof. Let $\{x_{2n_k-1}\}$ be a subsequence of $\{x_{2n-1}\}$ converging to a point $x \in A$. Then by triangle inequality, we have

$$d(A, B) \quad \leq \quad d(x, x_{2n_k}) \leq d(x_n, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) \tag{2.2}$$

$\forall n \in \mathbb{N}$. Thus, by $\lim_{n\to\infty} d(x, x_{2n_k-1}) = 0$ from theorem 2.2 and inequality (2.2) we have (taken limit as $k \to \infty$)

$$d(A,B) \leq \lim_{n\to\infty} d(x_n, x_{2n_k}) \leq \lim_{n\to\infty} d(x_{2n_k-1}, x_{2n_k}) = d(A,B)$$

this implies

$$d(A,B) \leq \lim_{n\to\infty} d(x_n, x_{2n_k}) \leq d(A,B),$$

hence, $\lim_{n\to\infty} d(x_n, x_{2n_k}) = d(A,B)$.

On the other hand, since T is a generalized MT-cyclic contraction mapping we have

$$\begin{aligned} d(A,B) &\leq d(Tx, x_{2n_k+1}) \\ &\leq max[d(x, x_{2n_k}), d(x_{2n_k}, Tx), d(x, Tx) + d(x_{2n_k}, Tx_{2n_k}) \\ &\quad -d(Tx, x_{2n_k+1}), d(x, x_{2n_k+1}), d(x_{2n_k}, x_{2n_k+1})] \\ &\leq max[d(x, x_{2n_k}), d(x_{2n_k}, Tx), d(x, Tx) + d(Tx, x_{2n_k+1}) \\ &\quad -d(Tx, x_{2n_k+1}), d(x, x_{2n_k+1}), d(x_{2n_k}, x_{2n_k})] \\ &\leq max[d(x, x_{2n_k}), d(x_{2n_k}, Tx), d(x, Tx), d(x, x_{2n_k+1})] \\ &\leq max[d(x, x_{2n_k}), d(x_{2n_k}, Tx), d(x, Tx)]. \end{aligned}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned} d(A,B) &\leq d(Tx, x) \\ &\leq max[d(A,B), d(A,B), d(x, Tx)] \\ &\leq max[d(A,B), d(x, Tx)]. \end{aligned}$$

Now we consider the following two cases:

Case I: If $\max\{\text{dist}(A,B), d(x,Tx)\} = \text{dist}(A,B)$, then we see that

$$\text{dist}(A,B) \leq d(Tx,x) \leq \text{dist}(A,B).$$

This implies, $\text{dist}(A,B) = d(Tx,x)$.

Case II: If $\max\{\text{dist}(A,B), d(Tx,x)\} = d(Tx,x)$. Then we have

$$\text{dist}(A,B) \leq d(Tx,x) \leq d(Tx,x)$$

and it follows that $d(Tx,x) = \text{dist}(A,B)$.

Hence the result. □

Corollary 2.4. *Let A and B be two nonempty subsets of a metric space (X,d) and $T : A \cup B \to A \cup B$ satisfy the condition*

$$\begin{aligned} d(Tx,Ty) \quad &\leq \quad \lambda \{\max \{d(x,y), d(y,Tx), d(x,Tx) + d(y,Ty) - d(Tx,Ty)\ , \\ &\qquad d(x,Ty), d(y,Ty)\}\} + (1-\lambda)d(A,B), \end{aligned}$$

for all $x \in A$ and $y \in B$. let $x_1 \in A$ be arbitrary given. Define an iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$. Suppose further that $\{x_{n-1}\}$ has a convergent subsequence in A, then there exists a point $x \in A$ such that $d(x,Tx) = d(A,B)$.

Here we give example which shows that Theorem 2.3 is a genuine generalization of Theorem 1.7.

Example 2.5. Let $X = \mathbb{R}$ and d is a usual metric on $\mathbb{R}$. Let us consider $A = [-2,-1]$, $B = [2,3]$ and cyclic mapping $T : A \cup B \to A \cup B$ which is defined by

$$Tx = \begin{cases} 2 & \text{if } x \in [-2,-1], \\ -1 & \text{if } x \in [2,3), \\ -2 & \text{if } x = 3. \end{cases}$$

Now, we will show that T is a generalized cyclic MT-contraction mapping with respect to an MT-function φ defined by

$$\varphi(t) = \frac{2}{3} \text{ for all } t \in (0,\infty).$$

We consider Nine cases.

Case 1: If $x, y \in [-2,-1]$ assume w.o.l.g $x > y$, we see that:

$$\begin{aligned} d(Tx,Ty) = 0, &\quad d(x,Tx) = x - 2, \\ d(y,Ty) = y - 2, &\quad d(x,y) = x - y, \\ d(x,Ty) = x - 2, &\quad d(y,Tx) = y - 2, \end{aligned}$$

and $d(A,B) = 3$.

Then it follows that

$$\begin{aligned} &\varphi(d(x,y)) \max[d(x,y), d(y,Tx), d(x,Tx) + d(y,Ty) - d(Tx,Ty), \\ &\qquad\qquad d(x,Ty), d(y,Ty)] + (1 - \varphi(d(x,y)))d(A,B) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}\max\{x-y, y-2, x-2+y-2-0, x-2, y-2\} + (1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{x-y, y-2, x-2, x+y-4\} + (1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{x-y, y-2, x-2, x+y-4\} + 1\\
&= \frac{2}{3}(x-y) + 1\\
&> 0\\
&= d(Tx, Ty).
\end{aligned}
$$

Case 2: If $x, y \in [2, 3)$ assume w.o.l.g $x > y$, we see that:

$$
\begin{aligned}
d(Tx, Ty) = 0, \quad d(x, Tx) = x + 1,\\
d(y, Ty) = y + 1, \quad d(x, y) = x - y,\\
d(x, Ty) = x + 1, d(y, Tx) = y + 1,
\end{aligned}
$$

and $d(A, B) = 3$.

Then it follows that

$$
\begin{aligned}
\varphi(d(x,y))\max[d(x,y), d(y,Tx), d(x,Tx) + d(y,Ty) - d(Tx,Ty),\\
d(x,Ty), d(y,Ty)] + (1-\varphi(d(x,y)))d(A,B)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{3}\max\{x-y, y+1, x+1+y+1-0, x+1, y+1\} + (1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{x-y, y+1, x+y+2, x+1\} + (1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{x-y, y+1, x+y+2, x+1\} + 1\\
&= \frac{2}{3}(x-y) + 1\\
&> 0\\
&= d(Tx, Ty).
\end{aligned}
$$

Case 3: If $x = y = 3$ assume w.o.l.g $x > y$, we see that:
$d(Tx, Ty) = 0$, $d(x, Tx) = 5$

$d(y, Ty) = 5$, $d(x, y) = 0$

$d(x,Ty)=5,\ d(y,Tx)=5$

and $d(A,B)=3$.
Then it follows that

$$\begin{aligned}\varphi(d(x,y))\max[d(x,y),d(y,Tx),d(x,Tx)+d(y,Ty)-d(Tx,Ty),\\ d(x,Ty),d(y,Ty)]+(1-\varphi(d(x,y)))d(A,B)\end{aligned}$$

$$\begin{aligned}&= \frac{2}{3}\max\{0,5,5+5-0,5,5\}+(1-\frac{2}{3})3\\ &= \frac{2}{3}\max\{0,5,10,5,5\}+(1-\frac{2}{3})3\\ &= \frac{2}{3}\max\{0,5,10\}+1\\ &= \frac{2}{3}(10)+1\\ &> 0\\ &= d(Tx,Ty).\end{aligned}$$

Case 4: If $x\in[-2,-1]$ and $y\in[2,3)$, we see that:
$d(Tx,Ty)=3,\ d(x,Tx)=x-2$

$d(y,Ty)=y-2,\ d(x,y)=x-y$

$d(x,Ty)=x-2,\ d(y,Tx)=y-2$

and $d(A,B)=3$.
It then follows that

$$\begin{aligned}\varphi(d(x,y))\max[d(x,y),d(y,Tx),d(x,Tx)+d(y,Ty)-d(Tx,Ty),\\ d(x,Ty),d(y,Ty)]+(1-\varphi(d(x,y)))d(A,B)\end{aligned}$$

$$\begin{aligned}&= \frac{2}{3}\max\{y-x,y-2,x-2+y+1-3,x+1,y+1\}+(1-\frac{2}{3})3\\ &= \frac{2}{3}\max\{y-x,y-2,x+y-4,x+1,y+1\}+(1-\frac{2}{3})3\\ &= \frac{2}{3}\max\{y-x,y-2,x+y-4,x+1,y+1\}+1\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}(y-x)+1 \\
&\geq 3 \\
&= d(Tx,Ty).
\end{aligned}
$$

Case 5: If $x \in [-2,-1]$ and $y = 3$, we see that:
$d(Tx,Ty) = 4$, $d(x,Tx) = 2-x$

$d(y,Ty) = 5$, $d(x,y) = 3-x$

$d(x,Ty) = x+2$, $d(y,Tx) = 1$

and $d(A,B) = 3$.
Then it follows that

$$
\varphi(d(x,y))\max[d(x,y), d(y,Tx), d(x,Tx)+d(y,Ty)-d(Tx,Ty), \\
d(x,Ty), d(y,Ty)] + (1-\varphi(d(x,y)))d(A,B)
$$

$$
\begin{aligned}
&= \frac{2}{3}\max\{3-x, 1, 2-x+5-4, x+2, 5\} + (1-\frac{2}{3})3 \\
&= \frac{2}{3}\max\{3-x, 1, 3-x, x+2, 5\} + (1-\frac{2}{3})3 \\
&= \frac{2}{3}\max\{3-x, 1, x+2, 5\} + 1 \\
&= \frac{2}{3}(5)+1 \\
&> 4 \\
&= d(Tx,Ty).
\end{aligned}
$$

Case 6: If $x \in [2,3)$ and $y \in [-2,-1]$, we see that:
$d(Tx,Ty) = 3$, $d(x,Tx) = x+1$

$d(y,Ty) = 2-y$, $d(x,y) = x-y$

$d(x,Ty) = x-2$, $d(y,Tx) = y+1$

and $d(A,B) = 3$.

It then follows that

$$\varphi(d(x,y))\max[d(x,y),d(y,Tx),d(x,Tx)+d(y,Ty)-d(Tx,Ty),\\ d(x,Ty),d(y,Ty)]+(1-\varphi(d(x,y)))d(A,B)$$

$$\begin{aligned}
&= \frac{2}{3}\max\{x-y,y+1,x+1+2-y-3,x-2,2-y\}+(1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{x-y,y+1,x-y,x-2,2-y\}+(1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{x-y,y+1,x-2,2-y\}+1\\
&= \frac{2}{3}(x-y)+1\\
&\geq 3\\
&= d(Tx,Ty).
\end{aligned}$$

Case 7: If $x\in[2,3)$ and $y=3$, we see that:
$d(Tx,Ty)=1$, $d(x,Tx)=x+1$

$d(y,Ty)=y+2$, $d(x,y)=3-x$

$d(x,Ty)=x+2$, $d(y,Tx)=4$

and $d(A,B)=3$.
Then it follows that

$$\varphi(d(x,y))\max[d(x,y),d(y,Tx),d(x,Tx)+d(y,Ty)-d(Tx,Ty),\\ d(x,Ty),d(y,Ty)]+(1-\varphi(d(x,y)))d(A,B)$$

$$\begin{aligned}
&= \frac{2}{3}\max\{3-x,4,x+1+y+2-1,x+2,y+2\}+(1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{3-x,4,x+y+2,x+2,y+2\}+(1-\frac{2}{3})3\\
&= \frac{2}{3}\max\{3-x,4,x+y+2,x+2,y+2\}+1\\
&= \frac{2}{3}(3-x)+1\\
&> 1\\
&= d(Tx,Ty).
\end{aligned}$$

Case 8: If $y \in [2, 3)$ and $x \in [-2, -1]$, we see that:
$d(Tx, Ty) = 3$, $d(x, Tx) = 2 - x$

$d(y, Ty) = y + 1$, $d(x, y) = y - x$

$d(x, Ty) = x + 1$, $d(y, Tx) = y - 2$

and $d(A, B) = 3$.
Then it follows that

$$\begin{aligned}\varphi(d(x,y)) \max[d(x,y), d(y,Tx), d(x,Tx) + d(y,Ty) - d(Tx,Ty),\\ d(x,Ty), d(y,Ty)] + (1 - \varphi(d(x,y)))d(A,B)\end{aligned}$$

$$\begin{aligned} &= \frac{2}{3}\max\{y - x, y - 2, 2 - x + y + 1 - 3, x + 1, y + 1\} + (1 - \frac{2}{3})3 \\ &= \frac{2}{3}\max\{y - x, y - 2, y - x, x + 1, y + 1\} + (1 - \frac{2}{3})3 \\ &= \frac{2}{3}\max\{y - x, y - 2, x + 1, y + 1\} + 1 \\ &= \frac{2}{3}(y - x) + 1 \\ &\geq 3 \\ &= d(Tx, Ty).\end{aligned}$$

Case 9: If $y \in [2, 3)$ and $x = 3$, we see that:
$d(Tx, Ty) = 1$, $d(x, Tx) = x + 22$

$d(y, Ty) = y + 1$, $d(x, y) = 3 - y$

$d(x, Ty) = 4$, $d(y, Tx) = y + 2$

and $d(A, B) = 3$.
Then it follows that

$$\begin{aligned}\varphi(d(x,y)) \max[d(x,y), d(y,Tx), d(x,Tx) + d(y,Ty) - d(Tx,Ty),\\ d(x,Ty), d(y,Ty)] + (1 - \varphi(d(x,y)))d(A,B)\end{aligned}$$

$$= \frac{2}{3}\max\{3 - y, y + 2, x + 2 + y + 1 - 1, 4, y + 1\} + (1 - \frac{2}{3})3$$

$$\begin{aligned}
&= \frac{2}{3}\max\{3-y, y+2, x+y+2, 4, y+1\} + (1-\frac{2}{3})3 \\
&= \frac{2}{3}\max\{3-y, y+2, x+y+2, 4, y+1\} + 1 \\
&= \frac{2}{3}(3-y) + 1 \\
&> 1 \\
&= d(Tx, Ty).
\end{aligned}$$

Hence, T is a generalized cyclic $\mathbb{R}$-contraction on $A \cup B$.

On the other hand, let us consider when $x = -1$ and $y = 3$. For any $\mathbb{R}-$function φ, we see that

$$\begin{aligned}
d(Tx, Ty) &= 4 > 4\varphi(d(x,y)) + (3 - 3\varphi(d(x,y))) \\
&= 4\varphi(d(x,y)) + (1 - \varphi(d(x,y)))3 \\
&= d(x,y)\varphi(d(x,y)) + (1 - \varphi(d(x,y)))d(A,B).
\end{aligned}$$

This means that T is not a cyclic $\mathbb{R}-$ contraction mapping on $A \cup B$.

Moreover, one can see that $x = -1 \in A$ and $y = 2 \in B$ are two best proximity points for T.

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