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EXISTENCE OF HOPF-BIFURCATION IN A 6-DIMENSIONAL SYSTEM

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ABSTRACT: In this paper, we discover a set of conditions for the existence of Hopf-bifurcation in a system of 6- dimensional ordinary differential equations. Here, we find these conditions in terms of the coefficients of the characteristic equation of Jacobi matrix corresponding to the equilibrium point. These conditions are also helpful in normal form theory of 6- dimensional system to study the direction of Hopf-bifurcation.

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1. INTRODUCTION

It is well known that, in a theoretical analysis of an n-dimensional differential equation system $\dot{x} = f(x, \theta)$, $(x \in R^n, \theta \in R)$ has an equilibrium point $x = x_0(\theta)$ and the Jacobi matrix $P(\theta) = Df(x_0(\theta), \theta)$, undergoes a Hopf-bifurcation at the critical point $\theta = \theta^*$, if:

H.1. All eigenvalues of the Jacobi matrix $P(\theta^*) = Df(x_0(\theta^*), \theta^*)$ corresponding to the equilibrium point $x = x_0(\theta^*)$ have negative real parts except one conjugate purely imagindary pair $\alpha(\theta^*), -\bar{\alpha}(\theta^*)$.

H.2. $\left.\dfrac{d\{\text{Re } \alpha(\theta)\}}{d\theta}\right|_{\theta=\theta^*} \neq 0$, (i.e. pair of complex conjugate eigenvalues $\alpha(\theta)$ and $\bar{\alpha}(\theta)$ crosses the imaginary axis).

As we know, if

$$X^n + c_1(\theta)X^{n-1} + c_2(\theta)X^{n-2} + \ldots\ldots + c_n(\theta) = 0 \tag{1}$$

is a characteristic equations of the Jacobi matrix $P(\theta) = Df(x_0(\theta), \theta)$ and the Routh-Hurwitz matrices

$$H_i = \begin{vmatrix} c_1 & 1 & 0 & \ldots & 0 \\ c_3 & c_2 & c_1 & \ldots & 0 \\ c_5 & c_4 & c_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{2i-1} & c_{2i-2} & c_{2i-3} & \ldots & 0 \end{vmatrix}. \tag{2}$$

Then the conditions (H.1) and (H.2) can be write as follows (using [3],[5]):

H.a. $c_i(\theta^*) > 0, \ i = 1, 2, 3, \ldots, n$.

H.b. $H_i(\theta^*) > 0, \ i = 2, 3, \ldots, n-2$.

H.c. $H_{n-1}(\theta^*) = 0$.

H.d. $-\left.\dfrac{d\{\text{Re } \alpha(\theta)\}}{d\theta}\right|_{\theta=\theta^*} = \left.\dfrac{1}{M_n}\dfrac{dH_{n-1}}{d\theta}\right|_{\theta=\theta^*} \neq 0.$

Here, value of M_n plays a very important role to determine the direction of Hopf bifurcation, it is based on the value of n in addition to different for different values of n.

The Hopf-bifurcation conditions in the terms of coefficients of characteristic equation can be discovered easily upto 4-dimensional system and some researches also find it for 5-dimensional system [4].

As we know, the use of mathematical models based on ordinary differential systems is increasing in various types of real life fields such as ecology, epidemiology, ecotoxicology, etc. To establish more realistic model, the dimension of the system also increases. But the analysis of existence and qualitative behavior of Hopf-bifurcation is limited upto 5-dimensional system. So, here we find an expression for the conditions (H.1) and (H.2) of 6- dimensional system to extend the bifurcation analysis upto 6-dimensional system.

We find these expressions with the help of Liu's theorem [5] and a new detecting method of Hopf-bifurcation [3].

2. EXPRESSION FOR THE CONDITIONS (H.1) AND (H.2) IN A 6-DIMENSIONAL DIFFERENTIAL EQUATION SYSTEM

Let

$$\dot{x} = f(x, \theta) \qquad (x \in R^6, \theta \in R) \tag{3}$$

be a 6-dimensional differential equation system and let

$$X^6 + c_1(\theta)X^5 + c_2(\theta)X^4 + c_3(\theta)X^3 + c_4(\theta)X^2 + c_5(\theta)X + c_6(\theta) = 0 \tag{4}$$

be the characteristic equation of the jacobian matrix of the system (3) at the equilibrium point $x = x_0(\theta)$.

From the conditions (H.a - H.d), system (3) shows a Hopf-bifurcation at $\theta = \theta^*$ under the following conditions:

1. $c_i(\theta^*) > 0, i = 1, 2, \ldots, 6$.
2. $H_2(\theta^*), H_3(\theta^*), H_4(\theta^*) > 0$.
3. $H_5(\theta^*) = 0$.
4. $\left.\dfrac{d\{\text{Re } \alpha(\theta)\}}{d\theta}\right|_{\theta=\theta^*} = \left.\dfrac{1}{M_6}\dfrac{dH_5}{d\theta}\right|_{\theta=\theta^*} \neq 0.$

Here, these conditions can be easily defined. But the main problem is that what are the values of $H_i (i = 2, \ldots, 5)$ and M_6. So we find these values.

Here, $H_i (i = 2, \ldots, 5)$ can be founded by taking determinant of Hurwitz matrices.

$$H_2 = \begin{vmatrix} c_1 & 1 \\ c_3 & c_2 \end{vmatrix} = c_1c_2 - c_3,$$

at $\theta = \theta^*$, $[c_1c_2 - c_3]_{\theta=\theta^*} > 0$.

$$H_3 = \begin{vmatrix} c_1 & 1 & 0 \\ c_3 & c_2 & c_1 \\ c_5 & c_4 & c_3 \end{vmatrix} = c_1c_2c_3 + c_1c_5 - c_1^2c_4 - c_3^2,$$

at $\theta = \theta^*$, $\left[c_1c_2c_3 + c_1c_5 - c_1^2c_4 - c_3^2\right]_{\theta=\theta^*} > 0$ or, $\left[c_1c_2c_3 - c_1^2c_4 - c_3^2\right]_{\theta=\theta^*} > 0$.

$$H_4 = \begin{vmatrix} c_1 & 1 & 0 & 0 \\ c_3 & c_2 & c_1 & 1 \\ c_5 & c_4 & c_3 & c_2 \\ 0 & c_6 & c_5 & c_4 \end{vmatrix} = (c_1c_2 - c_3)(c_3c_4 - c_2c_5) + (c_1c_2 - c_3)c_1c_6 - (c_1c_4 - c_5)^2,$$

at $\theta = \theta^*$, $\left[(c_1c_2 - c_3)(c_3c_4 - c_2c_5) + (c_1c_2 - c_3)c_1c_6 - (c_1c_4 - c_5)^2\right]_{\theta=\theta^*} > 0$.

$$H_5 = \begin{vmatrix} c_1 & 1 & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & 1 & 0 \\ c_5 & c_4 & c_3 & c_2 & c_1 \\ 0 & c_6 & c_5 & c_4 & c_3 \\ 0 & 0 & 0 & c_6 & c_5 \end{vmatrix}$$

$$= c_3(c_1c_2 - c_3)(c_4c_5 - c_3c_6) + c_3c_5(c_2c_5 - c_1c_6) + c_1c_3c_6(c_1c_4 - 2c_5) - c_1(c_2c_5 - c_1c_6)^2 - c_5(c_1c_4 - c_5)^2,$$

at $\theta = \theta^*$

$$\left[c_3(c_1c_2 - c_3)(c_4c_5 - c_3c_6) + c_3c_5(c_2c_5 - c_1c_6) + c_1c_3c_6(c_1c_4 - 2c_5) - c_1(c_2c_5 - c_1c_6)^2 - c_5(c_1c_4 - c_5)^2\right]_{\theta=\theta^*} = 0.$$

To construct last condition and value of M_6 in the terms of coefficients of characteristic equation (4), we use some matrix properties and detection method, see [1], [3].

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6 be the eigenvalues of the characteristic equation (4) and α_1 and α_2 be the complex conjugate eigenvalues, so we can assume

$$\alpha_1 = \lambda_1 + i\lambda_2, \qquad \alpha_2 = \lambda_1 - i\lambda_2, \tag{5a}$$

$$\implies 2\lambda_1 = \alpha_1 + \alpha_2, \quad \lambda_1^2 + \lambda_2^2 = \alpha_1\alpha_2. \tag{5b}$$

Now, let us write

$$b_1 = \alpha_3\alpha_4 + \alpha_3\alpha_5 + \alpha_3\alpha_6 + \alpha_4\alpha_5 + \alpha_4\alpha_6 + \alpha_5\alpha_6, \tag{6a}$$

$$b_2 = \alpha_3\alpha_4\alpha_5 + \alpha_3\alpha_4\alpha_6 + \alpha_3\alpha_5\alpha_6 + \alpha_4\alpha_5\alpha_6, \tag{6b}$$

$$b_3 = \alpha_3\alpha_4\alpha_5\alpha_6. \tag{6c}$$

Now, using theory of equation and keeping above in mind, we get

$$\lambda_1^2 + \lambda_2^2 - 2\lambda_1(2\lambda_1 + c_1) + b_1 = c_2, \tag{7a}$$

$$-(\lambda_1^2 + \lambda_2^2)(2\lambda_1 + c_1) + 2\lambda_1 b_1 + b_2 = -c_3, \tag{7b}$$

$$(\lambda_1^2 + \lambda_2^2)b_1 + 2\lambda_1 b_2 + b_3 = c_4, \tag{7c}$$

$$(\lambda_1^2 + \lambda_2^2)b_2 + 2\lambda_1 b_3 = -c_5, \tag{7d}$$

$$(\lambda_1^2 + \lambda_2^2)b_3 = c_6. \tag{7e}$$

Differentiating eq. (7) with respect to bifurcation parameter θ, we have

$$-2(3\lambda_1 + c_1)\frac{d\lambda_1}{d\theta} + 2\lambda_2\frac{d\lambda_2}{d\theta} + \frac{db_1}{d\theta} = -2\lambda_1\frac{dc_1}{d\theta} + \frac{dc_2}{d\theta} \tag{8a}$$

$$-2[(\lambda_1^2 + \lambda_2^2) + \lambda_1(2\lambda_1 + c_1) - b_1]\frac{d\lambda_1}{d} - 2\lambda_2(2\lambda_1 + c_1)\frac{d\lambda_2}{d\theta} + 2\lambda_1\frac{db_1}{d\theta} + \frac{db_2}{d\theta}$$
$$= (\lambda_1^2 + \lambda_2^2)\frac{dc_1}{d\theta} - \frac{dc_3}{d\theta} \tag{8b}$$

$$2(\lambda_1 b_1 + b_2)\frac{d\lambda_1}{d\theta} + 2\lambda_2 b_1\frac{d\lambda_2}{d\theta} + (\lambda_1^2 + \lambda_2^2)\frac{db_1}{d\theta} + 2\lambda_1\frac{db_2}{d\theta} + \frac{db_3}{d\theta} = \frac{dc_4}{d\theta} \tag{8c}$$

$$2(\lambda_1 b_2 + b_3)\frac{d\lambda_1}{d\theta} + 2\lambda_2 b_2\frac{d\lambda_2}{d\theta} + (\lambda_1^2 + \lambda_2^2)\frac{db_2}{d\theta} + 2\lambda_1\frac{db_3}{d\theta} = -\frac{dc_5}{d\theta} \tag{8d}$$

$$2\lambda_1 b_3\frac{d\lambda_1}{d\theta} + 2\lambda_2 b_3\frac{d\lambda_2}{d\theta} + (\lambda_1^2 + \lambda_2^2)\frac{db_3}{d\theta} = \frac{dc_6}{d\theta} \tag{8e}$$

As we know that at the critical point $\theta = \theta^*, \lambda_1 = 0$ then the equation $(7 \implies 9)$ and $(8 \implies 10)$:

$$\lambda_2^2 + b_1 = c_2, \tag{9a}$$

$$-c_1\lambda_2^2 + b_2 = -c_3, \tag{9b}$$

$$\lambda_2^2 b_1 + b_3 = c_4, \tag{9c}$$

$$\lambda_2^2 b_2 = -c_5, \tag{9d}$$

$$\lambda_2^2 b_3 = c_6, \tag{9e}$$

and

$$\left[-2c_1\frac{d\lambda_1}{d\theta} + 2\lambda_2\frac{d\lambda_2}{d\theta} + \frac{db_1}{d\theta} = \frac{dc_2}{d\theta}\right]_{\theta=\theta^*}, \tag{10a}$$

$$\left[2(b_1-\lambda_2^2)\frac{d\lambda_1}{d\theta} - 2c_1\lambda_2\frac{d\lambda_2}{d\theta} + \frac{db_2}{d\theta} = \lambda_2^2\frac{dc_1}{d\theta} - \frac{dc_3}{d\theta}\right]_{\theta=\theta^*}, \tag{10b}$$

$$\left[2b_2\frac{d\lambda_1}{d\theta} + 2\lambda_2 b_1\frac{d\lambda_2}{d\theta} + \lambda_2^2\frac{db_1}{d\theta} + \frac{db_3}{d\theta} = \frac{dc_4}{d\theta}\right]_{\theta=\theta^*}, \tag{10c}$$

$$\left[2b_3\frac{d\lambda_1}{d\theta} + 2\lambda_2 b_2\frac{d\lambda_2}{d\theta} + \lambda_2^2\frac{db_2}{d\theta} = -\frac{dc_5}{d\theta}\right]_{\theta=\theta^*}, \tag{10d}$$

$$\left[2\lambda_2 b_3\frac{d\lambda_2}{d\theta} + \lambda_2^2\frac{db_3}{d\theta} = \frac{dc_6}{d\theta}\right]_{\theta=\theta^*}. \tag{10e}$$

Equation (10) can be written in matrix form

$$\begin{bmatrix} -2c_1 & 2\lambda_2 & 1 & 0 & 0 \\ 2(b_1-\lambda_2^2) & -2c_1\lambda_2 & 0 & 1 & 0 \\ 2b_2 & 2\lambda_2 b_1 & \lambda_2^2 & 0 & 1 \\ 2b_3 & 2\lambda_2 b_2 & 0 & \lambda_2^2 & 0 \\ 0 & 2\lambda_2 b_3 & 0 & 0 & \lambda_2^2) \end{bmatrix} \begin{bmatrix} \frac{d\lambda_1}{d\theta} \\ \frac{d\lambda_2}{d\theta} \\ \frac{db_1}{d\theta} \\ \frac{db_2}{d\theta} \\ \frac{db_3}{d\theta} \end{bmatrix} = \begin{bmatrix} \frac{dc_2}{d\theta} \\ \lambda_2^2\frac{dc_1}{d\theta} - \frac{dc_3}{d\theta} \\ \frac{dc_4}{d\theta} \\ -\frac{dc_5}{d\theta} \\ \frac{dc_6}{d\theta} \end{bmatrix}. \tag{11}$$

Now using the cramer's law of simultaneous linear equations

$$\Longrightarrow \left[\frac{d\lambda_1}{d\theta}\right]_{\theta=\theta^*} = \frac{1}{2}\frac{\begin{vmatrix} \frac{dc_2}{d\theta} & 1 & 1 & 0 & 0 \\ \lambda_2^2\frac{dc_1}{d\theta} - \frac{dc_3}{d\theta} & -c_1 & 0 & 1 & 0 \\ \frac{dc_4}{d\theta} & b_1 & \lambda_2^2 & 0 & 1 \\ -\frac{dc_5}{d\theta} & b_2 & 0 & \lambda_2^2 & 0 \\ \frac{dc_6}{d\theta} & b_3 & 0 & 0 & \lambda_2^2 \end{vmatrix}_{\theta=\theta^*}}{\begin{vmatrix} -c_1 & 1 & 1 & 0 & 0 \\ b_1-\lambda_2^2 & -c_1 & 0 & 1 & 0 \\ b_2 & b_1 & \lambda_2^2 & 0 & 1 \\ b_3 & b_2 & 0 & \lambda_2^2 & 0 \\ 0 & b_3 & 0 & 0 & \lambda_2^2 \end{vmatrix}_{\theta=\theta^*}} = \frac{A}{2B} \quad \text{(let)}. \tag{12}$$

After some manipulation and using equation (9), we get

$$A = \left[\frac{1}{(c_1(c_1c_4 - c_5) - c_3(c_1c_2 - c_3))} \frac{dH_5}{d\theta} \right]_{\theta=\theta^*}, \tag{13}$$

and

$$B = \left| \begin{matrix} -c_1 & 1 & 1 & 0 & 0 \\ c_2 - 2\phi & -c_1 & 0 & 1 & 0 \\ c_1\phi - c_3 & c_2 - \phi & \phi & 0 & 1 \\ \phi^2 - c_2\phi + c_4 & c_1\phi - c_3 & 0 & \phi & 0 \\ 0 & \phi^2 - c_2\phi + c_4 & 0 & 0 & \phi \end{matrix} \right|_{\theta=\theta^*},$$

$$\text{where } \phi = \frac{c_3c_5 + c_1^2c_6 - c_1c_2c_5}{c_1^2c_4 - c_1c_5 - c_1c_2c_3 + c_3^2}. \tag{14}$$

Hence,

$$\left.\frac{d\{\text{Re } \alpha(\theta)\}}{d\theta}\right|_{\theta=\theta^*} = \left.\frac{d\lambda_1}{d\theta}\right|_{\theta=\theta^*} = \left.\frac{1}{M_6}\frac{dH_5}{d\theta}\right|_{\theta=\theta^*},$$

$$\text{where } M_6 = 2c_1(c_1c_4 - c_5) - c_3(c_1c_2 - c_3).B. \tag{15}$$

After attaining these results, we can state the following theorem.

Theorem 1. *A sixth order ordinary differential equation system undergoes a Hopf-bifurcation corresponding to the bifurcation parameter θ at θ^*, if*

1. $c_i(\theta^*) > 0, \forall\ i = 1, 2, \ldots, 6$
2. $[c_1c_2 - c_3]_{\theta=\theta^*} > 0$
3. $\left[c_1c_2c_3 - c_1^2c_4 - c_3^2\right]_{\theta=\theta^*} > 0$
4. $\left[(c_1c_2 - c_3)(c_3c_4 - c_2c_5) + (c_1c_2 - c_3)c_1c_6 - (c_1c_4 - c_5)^2\right]_{\theta=\theta^*} > 0$
5. $[H_5]_{\theta=\theta^*} = [c_3(c_1c_2 - c_3)(c_4c_5 - c_3c_6) + c_3c_5(c_2c_5 - c_1c_6)$ $+ c_1c_3c_6(c_1c_4 - 2c_5) - c_1(c_2c_5 - c_1c_6)^2 - c_5(c_1c_4 - c_5)^2]_{\theta=\theta^*} = 0.$
6. $\left.\frac{1}{M_6}\frac{dH_5}{d\theta}\right|_{\theta=\theta^*} \neq 0.$

where,

$$M_6 = 2c_1(c_1c_4 - c_5) - c_3(c_1c_2 - c_3).\begin{vmatrix} -c_1 & 1 & 1 & 0 & 0 \\ c_2 - 2\phi & -c_1 & 0 & 1 & 0 \\ c_1\phi - c_3 & c_2 - \phi & \phi & 0 & 1 \\ \phi^2 - c_2\phi + c_4 & c_1\phi - c_3 & 0 & \phi & 0 \\ 0 & \phi^2 - c_2\phi + c_4 & 0 & 0 & \phi \end{vmatrix}$$

$$and\ \phi = \frac{c_3c_5 + c_1^2c_6 - c_1c_2c_5}{c_1^2c_4 - c_1c_5 - c_1c_2c_3 + c_3^2}.$$

3. CONCLUSION

In this paper, we have extended the qualitative analysis for Hopf - bifurcation to the 6 - dimensional system. The result found here is quite useful in determining Hopf- bifurcation for mathematical models consisting of six simultaneous ordinary differential equations.

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