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(M,K)-QUASI CLASS Q AND (M,K)-QUASI *CLASS Q COMPOSITION OPERATORS ON WEIGHTED HARDY SPACE

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ABSTRACT: In this paper we discuss the conditions for a composition operator and a weighted composition operator to be (M,k) quasi class Q and (M,k) quasi * class Q operator and also the characterization of (M,k) quasi class Q and (M,k) quasi * class Q composition operators on weighted Hardy space.

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1. INTRODUCTION

Various properties of composition operators on weighted Hardy spaces have been studied by different authors, see [3], Cowen and Kriete obtained a nice correlation between hyponormality of composition operator on H^2. In [9], E.A. Nordgeen, studied some results on the hyponormality of composition operators and their adjoints. In [13], S. Panayappan D. Senthil kumar and Mohenraj have investigated on M-Quasihyponormality of composition operators and their adjoints. T. Veluchamy (see [15]) have investigated parahyponormal * paranormal and posinormal operators.

In this paper, we are interested in k-quasi * class Q Operators, we give a characterization of such operators and other known classes of operators.

2. PRELIMINARY NOTES

Let f be an analytic map on the open disk D given by the Taylor's series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots .$$

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$ as $n \to \infty$. The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$ is a Hilbert space of functions analytic in the unit disk with the inner product $\langle f, g\rangle_\beta = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$ for f as above and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Let D be the open unit disk in the complex plane and let $T : D \to D$ be an analytic self map of the unit disk and consider the corresponding composition operator C acting on $H^2(\beta)$ ie., $Cf = foT$, $f \in H^2(\beta)$.

Let $(x, \sum, \lambda)$ be a sigma-finite measure space and let $T : X \to X$ be a non singular measurable transformation. A bounded linear operator $Cf = foT$ on $L^2(x, \sum, \lambda)$ is said to be a composition operator induced by T when the measure λT^{-1} is absolutely continuous with respect to the measure λ and the Random-Nikodym derivative $\frac{d\lambda T^{-1}}{d\lambda} = f_0$ is essentially bounded. The random-Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by $f_0^{(k)}$ where T^k is obtained by composing T-k times. Every essentially bounded complex valued measurable function f_0 induces the bounded operatorM_{f_0} on $L^2(\lambda)$ which is defined by $M_{f_0} f = f_0 f$ for every $f \in L^2(\lambda)$. Further $C^*C = M_{f_0}$ and $C^{*2}C^2 = M_{f_0^2}$.

Let u be an essentially bounded function. Then the weighted composition operator $W(= W_{u,T})$ on the space $L^2(\mu)$ induced by u and T is given by $Wf = u.foT$ for each $f \in L^2(\mu)$.

A transformation T is measurable if $T^{-1}(A) \in \mathbf{A}$ for any $A \in \mathbf{A}$. A measurable transformation T is said to be non-singular if $\mu(T^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for $A \in \mathbf{A}$. If T is a measurable transformation then T^n

is also a measurable transformation. If T is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ and hence $\mu(T^{-1})^n$ becomes absolutely continuous with respect to μ. Hence, by Radon-Nikodym theorem there exists a unique non-negative essentially bounded measurable function h_n such that

$$\mu(T^{-1})^n(A) = \int_A h_n \, d\mu \text{ for every } A \in \mathbf{A},$$

and h_n is called the nth order radon-nikodym derivative and is denoted by $\frac{d\mu(T^{-1})^n}{d\mu}$. It can be seen that $h_n = h.hoT^{-1}.hoT^{-2}...hoT^{-(n-1)}$ and $h_n = h_{n-1}.hoT^{-(n-1)}$. Throughout this paper, we assume that u is non-negative.

Definition 2.1. [10] An operator T on a Hilbert space H is said to be k-quasi class Q if

$$T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^{*k}T^k \geq 0.$$

Definition 2.2. An operator T on a Hilbert space H is said to be (M,k)-quasi class Q if

$$M^2T^{*(k+2)}T^{k+2} - 2T^{*(k+1)}T^{k+1} + T^{*k}T^k \geq 0.$$

Definition 2.3. An operator T on a Hilbert space H is said to be M *-class Q if

$$M^2T^{*2}T^2 - 2TT^* + I \geq 0.$$

Definition 2.4. An operator T on a Hilbert space H is said to be M-quasi-*-class Q if

$$M^2T^{*3}T^3 - 2(T^*T)^2 + T^*T \geq 0.$$

Definition 2.5. An operator T on a Hilbert space H is said to be (M,k)-quasi-*-class Q if

$$M^2T^{*(k+2)}T^{k+2} - 2T^{*k}TT^*T^k + T^{*k}T^k \geq 0.$$

Proposition 1. *Change of variable: Let X be a non-empty set and let* $\boldsymbol{A}$ *be a* σ*-algebra on X. Let* μ *and* μT^{-1} *be measures on* $\boldsymbol{A}$ *let* $h : X \to [0, \infty]$ *be a measurable function. Then the following are equivalent:*

(i) μT^{-1} is absolutely continuous with respect to μ and h is Radon-nikodym derivative of μT^{-1} with respect to μ.

(ii) For every measurable function $f : X \to [0, \infty]$, the equality

$$\int_X f \, d\mu T^{-1} = \int_X f \, h \, d\mu$$

holds.

The conditional expectation operator $E(T^{-1}(\boldsymbol{A})) = E(f)$ is defined for each non-negative function f in $L^p(1 \le p < \infty)$ and is uniquely determined by the following set of conditions:

(i) $E(f)$ is $T^{-1}(\boldsymbol{A})$ measurable.

(ii) If B is any $T^{-1}(\boldsymbol{A})$ measurable set for which $\int_A f d\mu$ converges then we have

$$\int_A f \, d\mu = \int_A E(f) \, d\mu.$$

E is the projection operator onto the closure of the range of the composition operator C on $L^2(\mu)$.

Lemma 1. *[14] Let P be the projection of $L^2(X, \boldsymbol{A}, \mu)$onto $\overline{R(C)}$. Then:*

(i) $C^*Cf = hf$ and $CC^*f = (hoT)Pf$ for all $f \in L^2(\mu)$.

(ii) $\overline{R(C)} = \{f \in L^2(\mu) : f \text{ is } T^{-1}(\boldsymbol{A}) \text{ measurable}\}$.

(iii) If f is $T^{-1}(\boldsymbol{A})$ measurable and g and fg belong to $L^2(\mu)$, then $P(fg) = fP(g)$, (f need not be in $L^2(\mu)$).

In [12] Senthikumar has proved the conditions for composition and weighted composition operators to be k-quasi paranormal operator. A. Gupta and N. Bhatia [1] also proved the conditions for composition and weighted composition operators to be (n,k)-quasi paranormal and (n,k)-quasi -*-paranormal operators. In this paper we obtain the conditions for composition and weighted composition operators to be k quasi * class Q operator and quasi * class Q operators in terms of expectation operator and Radon-Nikodyn derivative h(or h_n)

3. COMPOSITION OPERATOR

Let C be the composition operator and C^* be its adjoint which is given by

$$C^* f = h.E(f)oT^{-1}.$$

Proposition 2. *[1] For every $n \in N$:*
(i) $(C^*C)^n f = h^n f$.
(ii) $(CC^*)^n f = (hoT)^n P(f)$.
(iii) E is the identity operator on $L^2(\mu)$ iff $T^{-1}(\boldsymbol{A}) = \boldsymbol{A}$.

Theorem 3.1. *Let C be a composition operator on $L^2(\mu)$. Then C is (M,k)-quasi class Q if and only if $M^2h^{k+2} - 2h^{k+1} + h^k \geq 0$.*

Proof. Suppose C is (M,k)-quasi class Q operator. Then for every $f \in L^2(\mu)$,

$$\left\langle (M^2C^{*(k+2)}C^{k+2} - 2C^{*(k+1)}C^{k+1} + C^{*k}C^k)f, f \right\rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Therefore

$$\begin{aligned} &\left\langle (M^2C^{*(k+2)}C^{k+2} - 2C^{*(k+1)}C^{k+1} + C^{*k}C^k)\chi_A, \chi_A \right\rangle \geq 0 \\ &\qquad \Leftrightarrow \int ((M^2h^{k+2} - 2h^{k+1} + h^k)\,\chi_A)\,d\mu \geq 0 \\ &\qquad \Leftrightarrow \int (M^2h^{k+2} - 2h^{k+1} + h^k)\,d\mu \geq 0 \\ &\qquad \Leftrightarrow M^2h^{k+2} - 2h^{k+1} + h^k \geq 0. \end{aligned}$$

□

Theorem 3.2. *Let C be a composition operator on $L^2(\mu)$.Then C is (M,k)-quasi * class Q if and only if $M^2h^{k+2} - 2h_k.E(h)oT^{-1} + h^k \geq 0$.*

Proof. Consider

$$\begin{aligned} C^{*k}CC^*C^kf &= C^{*k}CC^*(foT^k) \\ &= C^{*k}C(h.E(foT^k)oT^{-1}) \\ &= C^{*k}(hoT^{-1}.foT^k)oT \\ &= h_k.E(h)oT^{-1}.f. \end{aligned}$$

C is (M,k)-quasi * class Q if and only if for every $f \in L^2(\mu)$ and $\lambda > 0$,

$$\left\langle (M^2C^{*(k+2)}C^{k+2} - 2C^{*k}CC^*C^k + C^{*k}C^k)f, f \right\rangle \geq 0.$$

$$\Leftrightarrow (M^2h^{k+2} - 2h_k.E(h)oT^{-1} + h^k)f \geq 0$$
$$\Leftrightarrow M^2h^{k+2} - 2h_k.E(h)oT^{-1} + h^k \geq 0. \qquad \square$$

Example 3.1. Consider the space $l^2(\omega) = L^2(N, 2^N, \mu)(\omega)$ where $\omega = \langle m_n \rangle_{n=1}^{\infty}$ is a sequence of positive real numbers. μ is a measure given by $\mu(n) = m_n$. Let $T : N \to N$ be a non-singular measurable transformation. Then T^n is also a non-singular measurable transformation for $n \in N$. Now,

$$h_k(s) = \frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j,$$
$$h^k(s) = \frac{1}{m_s^k} \left(\sum_{j \in T^{-1}(s)} m_j \right)^k,$$
$$E(f)(k) = \frac{\sum_{j \in T^{-1}T(k)} f_j m_j}{\sum_{j \in T^{-1}T(k)} m_j},$$

for all non-negative sequence $f =< f_n >_{n=1}^{\infty}$ and $s, k \in N$. by theorem (3.1), C is (M,k)-quasi class Q if and only if

$$M^2 \frac{1}{m_s^{k+2}} \left(\sum_{j \in T^{-1}(s)} m_j \right)^{k+2} - 2\frac{1}{m_s^{k+1}} \left(\sum_{j \in T^{-1}(s)} m_j \right)^{k+1} + \frac{1}{m_s^k} \left(\sum_{j \in T^{-1}(s)} m_j \right)^k \geq 0.$$

for all non-negative sequence $f =< f_n >_{n=1}^{\infty}$ and $s, k \in N$. by theorem (3.2), C is (M,k)-quasi * class Q if and only if

$$M^2 \frac{1}{m_s^{k+2}} \left(\sum_{j \in T^{-1}(s)} m_j \right)^{k+2} - 2\frac{1}{m_s} \sum_{j \in T^{-k}(s)} m_j \frac{1}{m_{T^{-k}(s)}} \sum_{j \in T^{-k+1}(s)} m_j + \frac{1}{m_s^k} \left(\sum j \in T^{-1}(s) m_j \right)^k \geq 0.$$

Proposition 3. *If P denote the projection of $L^2(\mu)$ on $\overline{R(C)}$, then $C^*Cf = f_0 f$ and $CC^*f = (f_0 oT)Pf$. For all $f \in L^2(\mu)$, where P denote the projection of L^2 on $\overline{R(C)}$ and*

$$\overline{R(C)} = \left\{ f \in L^2 : f \text{ is } T^{-1} \sum \text{ measurable} \right\}.$$

Theorem 3.3. *Let $C \in B(L^2(\mu))$. Then C is of M quasi * class Q if and only if $M^2 f_o^{(3)} - 2(f_0)^2 + f_0 \geq 0$.*

Proof. Let $C \in B(L^2(\mu))$. Then C is of M quasi * class Q operator

$$\Leftrightarrow M^2 C^{*3} C^3 - 2(C^* C)^2 + C^* C \geq 0$$
$$\Leftrightarrow \left\langle (M^2 C^{*3} C^3 - 2(C^* C)^2 + C^* C)\chi_E, \chi_E \right\rangle \geq 0,$$

for every characteristic function χ_E of E in $\sum$ such that $\lambda(E) \prec \infty$.

Since, $C^{*2} C^2 = M_{f_0^{(2)}}$, $C^* C = M f_0$,

$$\Leftrightarrow \left\langle (M^2 M_{f_0^{(3)}} - 2(M_{f_0})^2 + M_{f_0})\chi_E, \chi_E \right\rangle \geq 0$$
$$\Leftrightarrow \int_E (M^2 f_0^{(3)} - 2(f_0)^2 + f_0)\, d\mu \geq 0$$

for every E in $\sum$.

Hence C is M-quasi * class Q operator if and only if $M^2 f_o^{(3)} - 2(f_0)^2 + f_0 \geq 0$. □

Corollary 1. *Let $C \in B(L^2(\mu))$ with dense range. Then $C \in$ M-quasi * class Q if and only if $M^2 f_o^{(3)} - 2(f_0)^2 + f_0 \geq 0$.*

Theorem 3.4. *Let $C \in B(L^2(\mu))$. Then $C^* \in$ M- quasi * class Q if and only if*

$$M^2 (f_0 oT)^3 P_1 - 2((f_0 oT) P_1)^2 + (f_0 oT) P_1 \geq 0,$$

where P_1 is the projection of L^2 on to $\overline{R(C)}$.

Proof. Let C^* is of M-quasi * class Q operator if and only if

$$M^2 C^3 C^{*3} - 2(CC^*)^2 + CC^* \geq 0,$$
$$\left\langle \left(C^3 C^{*3} - 2(CC^*)^2 + CC^*\right) f, f \right\rangle \geq 0,$$

for every $f \in L^2$. $< CC^* f, f > = < (f_0 oT) P_1 f, f >$ where P_1 is the projections of L^2 on to $\overline{R(C)}$. Thus C^* is of M-quasi * class Q Operator if and only if

$$\left\langle M^2 (f_0 oT)^3 P_1 f, f \right\rangle - 2 \left\langle ((f_0 oT) P_1)^2 f, f \right\rangle + \langle (f_0 oT) P_1 f, f \rangle \geq 0$$

for every $f \in L^2$,

$$M^2 (f_0 oT)^3 P_1 - 2((f_0 oT) P_1)^2 + (f_0 oT) P_1 \geq 0.$$ □

Corollary 2. *Let $C \in B(L^2(\mu))$ with dense range. Then $C \in$ M-quasi * class Q operator if and only if* $M^2(f_0oT)^3P_1 - 2((f_0oT)P_1)^2 + (f_0oT)P_1 \geq 0$

4. WEIGHTED COMPOSITION OPERATORS

Let W be the weighted composition operator on $L^2(\mu)$. Let W^* be its adjoint which is given by $W^*f = h.E(u.f)oT^{-1}$ for $f \in L^2(\mu)$. For a positive integer n, $u_n = u.(uoT)^2...(uoT)^{(n-1)}$. For $f \in L^2(\mu)$, $W^nf = u_n.foT^{-n}$ and $W^{*n}f = h_n.E(u_n.f)oT^{-n}$.

Proposition 4. *[2] For $u \geq 0$:*

(i) $W^*Wf = hE[(u^2)]oT^{-1}f$.

(ii) $WW^*f = u(hoT)E(uf)$.

Theorem 4.1. *Let W be a weighted composition operator on $L^2(\mu)$. Then W is (M,k)-quasi class Q operator if and only if*

$$M^2h_{k+2}.E(u_{k+2}^2)oT^{-(2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-1} + h_k.E(u_k^2) \geq 0.$$

Proof. Suppose W is (M,k)-quasi class Q operator. Then for $f \in L^2(\mu)$.

$$\left\langle (M^2W^{*k}W^{*2}W^2W^k - 2W^{*k}W^*WW^k + W^{*k}W^k)f, f \right\rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Then:

$$\left\langle (M^2W^{*k}W^{*2}W^2W^k - 2W^{*k}W^*WW^k + W^{*k}W^k)\chi_A, \chi_A \right\rangle \geq 0$$

$$\Leftrightarrow \langle (M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k})\chi_A, \chi_A \rangle \geq 0$$

$$\Leftrightarrow \int (M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k})\chi_A d\mu \geq 0$$

$$\Leftrightarrow \int M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k} d\mu \geq 0$$

$$\Leftrightarrow M^2h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_{k+1}.E(u_{k+1}^2)oT^{-(k+1)} + h_k.E(u_k^2)oT^{-k} \geq 0.$$

□

Corollary 3. *If W be a weighted composition operator on $L^2(\mu)$ and $T^{-1}(\boldsymbol{A}) = \boldsymbol{A}$. Then W is (M,k)-quasi class Q operator if and only if*

$$M^2 h_{k+2}.u_{k+2}^2 oT^{-2} - 2h_{k+1}.u_{k+1}^2 oT^{-1} + h_k.u_k^2 \geq 0$$

Theorem 4.2. *Let W be a weighted composition operator on $L^2(\mu)$. Then W is (M,k)-quasi * class Q operator if and only if*

$$M^2 h_{k+2}.E(u_{k+2}^2)oT^{-2} - 2h_k.hoT^{-(k-1)}.E(u_{k+1}^2) + h_k.E(u_k^2) \geq 0.$$

Proof. Suppose W is (M,k)-quasi * class Q operator. Then for $f \in L^2(\mu)$.

$$\Big\langle (M^2 W^{*k} W^{*2} W^2 W^k - 2W^{*k} W W^* W^k + W^{*k} W^k) f, f \Big\rangle \geq 0.$$

Let $f = \chi_A$ with $\mu(A) < \infty$. Then

$$\begin{aligned}
&\Big\langle (M^2 W^{*k} W^{*2} W^2 W^k - 2W^{*k} W W^* W^k + W^{*k} W^k) \chi_A, \chi_A \Big\rangle \geq 0 \\
&\Leftrightarrow \langle (M^2 h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_k.E(u_{k+1}(hoT)E(u_{k+1}))oT^{-k} \\
&\qquad\qquad + h_k.E(u_k^2)oT^{-k})\chi_A, \chi_A \rangle \geq 0 \\
&\Leftrightarrow \int (M^2 h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_k.E(u_{k+1}(hoT)E(u_{k+1}))oT^{-k} \\
&\qquad\qquad + h_k.E(u_k^2)oT^{-k})\chi_A d\mu \geq 0 \\
&\Leftrightarrow \int_A M^2 h_{k+2}.E(u_{k+2}^2)oT^{-(k+2)} - 2h_k.E(u_{k+1}(hoT)E(u_{k+1}))oT^{-k} \\
&\qquad\qquad + h_k.E(u_k^2)oT^{-k} d\mu \geq 0 \\
&\Leftrightarrow M^2 h_{k+2}.E(u_{k+2}^2)oT^{-(2)} - 2h_k.hoT^{-(k-1)}.E(u_{k+1}^2) + h_k.E(u_k^2) \geq 0. \quad \square
\end{aligned}$$

Corollary 4. *If W be a weighted composition operator on $L^2(\mu)$ and $T^{-1}(\boldsymbol{A}) = \boldsymbol{A}$. Then W is (M,k)-quasi * class Q operator if and only if*

$$M^2 h_{k+2}.u_{k+2}^2 oT^{-2} - 2h_k.hoT^{-(k-1)}.u_{k+1}^2 + h_k.u_k^2 \geq 0$$

Theorem 4.3. *Let $T^{-1}\sum = \sum$, $W \in B(L^2(\mu))$. Then W is of M-quasi * class Q if and only if*

$$M^2 f_0^{(3)} E(\pi_3^2)oT^{-3} - 2(f_0^{(1)} E(\pi_1^2)oT^{-1})^2 + f_0^{(1)} E(\pi_1^2)oT^{-1} \geq 0$$

Proof. Since W is weighted composition operator with weight $\pi = (\frac{f_0}{f_0 oT})^{\frac{s}{2}}$ it follows that W is of M-quasi * class Q operator if and only if $M^2 f_0^{(3)} E(\pi_3^2)oT^{-3} - 2(f_0^{(1)} E(\pi_1^2)oT^{-1})^2 + f_0^{(1)} E(\pi_1^2)oT^{-1} \geq 0$. □

The second Aluthge transformation of T described by B.P.Duggal is given by

$$\tilde{T} = \left|\tilde{T}\right|^{\frac{1}{2}} V \left|\tilde{T}\right|^{\frac{1}{2}},$$

where $\tilde{T} = V\left|\tilde{T}\right|$ is the polar decomposition of $\tilde{T}$, $\tilde{C} = |W|^{\frac{1}{2}}$, where $W = V|W|$ is the polar decomposition of the generalized Aluthge transformation $W : 0 < s < 1$ is a weighted composition operator with weight

$$\omega' = J^{\frac{1}{4}}\pi\left(\frac{\chi_{supJ}}{J^{\frac{1}{4}}} oT\right),$$

where $J = f_0 E(\pi^2)oT^{-1}$.

Corollary 5. *Let $T^{-1}\sum = \sum$, $W \in B(L^2(\mu))$. Then W is of quasi * class Q if and only if $f_0^{(3)}E(\omega_3'^2)oT^{-3} - 2(f_0^{(1)}E(\omega_1'^2)oT^{-1})^2 + f_0^{(1)}E(\omega_1'^2)oT^{-1} \geq 0$.*

5. (M,K)-QUASI -*-CLASS Q COMPOSITION OPERATOR ON WEIGHTED HARDY SPACE

The operator C_T are not necessarily defined on all of $H^2(\beta)$. They are ever where defined in some special cases in the classical Hardy spaces H^2 (the case when $\beta_n = 1$ for all n).

Let ω be a point on the open disk. Define $k_\omega^\beta(z) = \sum_{n=0}^{\infty} \frac{z^n\bar{\omega}^n}{\beta_n^2}$. Then the function k_ω^β is a point evaluation for $H^2(\beta)$. Then k_ω^β is in $H^2(\beta)$ and $\left\|k_\omega^\beta\right\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{\beta_n^2}$. Thus, $\|k_\omega\|$ is an increasing function of $|\omega|$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $\left\langle f, k_\omega^\beta\right\rangle_\beta = f(\omega)$ for all f and k_ω^β is known as the point evaluation kernel at ω. It can be easily shown that $C_T^* k_\omega^\beta = k_{T(\omega)}^\beta$ and $k_0^\beta = 1$ (the function identically equal to 1).

Now we introduce the class of (M,k) -quasi -*- class Q operators, which is a common generalization of M-* class Q and M-quasi *-class Q operators, defined as follows: An operator $T \in B(H)$ is said to be (M,k) -quasi-*-class Q operator if $M^2T^{*k}(T^{*2}T^2 - 2TT^* + I)T^k \geq 0$, where k is a natural number.

Theorem 5.1. *If C_T is a (M,k)-quasi * class Q operator in $H^2(\beta)$ then*

$$M^2 C_T^{*(k+2)} C_T^{k+2} - 2C_T^{*k} C_T C_T^* C_T^k + C_T^{*k} C_T^k \geq 0.$$

Proof. An operator C_T is (M,k)-quasi * class Q, then

$$M^2 C_T^{*(k+2)} C_T^{k+2} - 2C_T^{*k} C_T C_T^* C_T^k + C_T^{*k} C_T^k \geq 0,$$

$$\left\langle (M^2 C_T^{*(k+2)} C_T^{k+2} - 2C_T^{*k} C_T C_T^* C_T^k + C_T^{*k} C_T^k) f, f \right\rangle \geq 0,$$

$$\left\langle (M^2 C_T^{*(k+2)} C_T^{k+2}) f, f \right\rangle - 2 \left\langle (C_T^{*k} C_T C_T^* C_T^k) f, f \right\rangle + \left\langle (C_T^{*k} C_T^k) f, f \right\rangle \geq 0,$$

$$M^2 \left\| C_T^{k+2} f \right\|^2 - 2 \left\| C_T^k C_T^* f \right\|^2 + \left\| C_T^k f \right\|^2 \geq 0,$$

$$M^2 \left\| C_T^{k+1} (C_T f) \right\|^2 - 2 \left\| C_T^k (C_T^* f) \right\|^2 + \left\| C_T^{k-1} (C_T f) \right\|^2 \geq 0.$$

Let $f = k_0^\beta$, then

$$M^2 \left\| C_T^{k+1} (C_T k_0^\beta) \right\|^2 - 2 \left\| C_T^k (C_T^* k_0^\beta) \right\|^2 + \left\| C_T^{k-1} (C_T k_0^\beta) \right\|^2 \geq 0,$$

$$M^2 \left\| C_T^{k+1} (k_0^\beta) \right\|^2 - 2 \left\| C_T^k (k_{T(0)}^\beta) \right\|^2 + \left\| C_T^{k-1} (k_0^\beta) \right\|^2 \geq 0.$$

Repeating the steps for k times and $T(0) = 0$ we get

$$M^2 \left\| k_0^\beta \right\|^2 - 2 \left\| k_0^\beta \right\|^2 + \left\| k_0^\beta \right\|^2 \geq 0.$$

□

Theorem 5.2. *If C_T^* is a (M,k)-quasi * class Q operator in $H^2(\beta)$ then*

$$M^2 C_T^{k+2} C_T^{*(k+2)} - 2C_T^k C_T^* C_T C_T^{*k} + C_T^k C_T^{*k} \geq 0.$$

Proof. An operator C_T^* is (M,k)-quasi * class Q,

$$M^2 C_T^{k+2} C_T^{*(k+2)} - 2C_T^k C_T^* C_T C_T^{*k} + C_T^k C_T^{*k} \geq 0,$$

$$\left\langle (M^2 C_T^{k+2} C_T^{*(k+2)} - 2C_T^k C_T^* C_T C_T^{*k} + C_T^k C_T^{*k}) f, f \right\rangle \geq 0,$$

$$\left\langle (M^2 C_T^{k+2} C_T^{*(k+2)}) f, f \right\rangle - 2 \left\langle (C_T^k C_T^* C_T C_T^{*k}) f, f \right\rangle + \left\langle (C_T^k C_T^{*k}) f, f \right\rangle \geq 0,$$

$$M^2 \left\| C_T^{*(k+2)} f \right\|^2 - 2 \left\| C_T^{*k} C_T f \right\|^2 + \left\| C_T^{*k} f \right\|^2 \geq 0,$$

$$M^2 \left\| C_T^{*(k+1)} (C_T^* f) \right\|^2 - 2 \left\| C_T^{*k} (C_T f) \right\|^2 + \left\| C_T^{*(k-1)} (C_T^* f) \right\|^2 \geq 0.$$

Let $f = k_0^\beta$, then

$$M^2 \left\| C_T^{*(k+1)} (C_T^* k_0^\beta) \right\|^2 - 2 \left\| C_T^{*k} (C_T k_0^\beta) \right\|^2 + \left\| C_T^{*(k-1)} (C_T^* k_0^\beta) \right\|^2 \geq 0,$$

$$M^2 \left\| C_T^{*(k+1)} (k_{T(0)}^\beta) \right\|^2 - 2 \left\| C_T^{*k} (k_0^\beta) \right\|^2 + \left\| C_T^{*(k-1)} (k_{T(0)}^\beta) \right\|^2 \geq 0.$$

Repeating the steps for k times and $T(0) = 0$ we get

$$M^2 \left\| k_0^\beta \right\|^2 - 2 \left\| k_0^\beta \right\|^2 + \left\| k_0^\beta \right\|^2 \geq 0.$$

□

Example 5.1. Let $f=\sum_{n=0}^{\infty} f_n z^n \in H^2(\beta)$ and $\phi : D \to D$ be defined by $\phi(z)=\frac{e^{i\theta}}{2}z$, where $0 \le \theta \le 2\pi$ is fixed. Then

$$\begin{aligned}
(C_\phi^* f)(\omega) &=< C_\phi^* f, k_\omega > \\
&=< f, k_{\omega o \phi} > \\
&=< f, k_{\varphi(\omega)} > \text{ where } \varphi(\omega)=\frac{e^{-i\theta}}{2}\omega \\
&= f(\varphi(\omega)) \\
&= f(\frac{e^{-i\theta}}{2}\omega).
\end{aligned}$$

Now

$$\begin{aligned}
(C_\phi^* C_\phi f)(z) &= (C_\phi f)(\frac{e^{-i\theta}}{2}z) \\
&= f(\frac{e^{i\theta}}{2}.\frac{e^{-i\theta}}{2}z) \\
&= f(\frac{1}{4}z),
\end{aligned}$$

and

$$\begin{aligned}
(C_\phi C_\phi^* f)(z) &= (C_\phi^* f)(\frac{e^{i\theta}}{2}z) \\
&= f(\frac{e^{-i\theta}}{2}.\frac{e^{i\theta}}{2}z) \\
&= f(\frac{1}{4}z),
\end{aligned}$$

$$\begin{aligned}
(M^2 C_\phi^{*(k+2)} C_\phi^{k+2} - 2C_\phi^{*k} C_\phi C_\phi^* C_\phi^k + C_\phi^{*k} C_\phi^k f)(z) &\geq 0, \\
(M^2 C_\phi^{*(k+2)} C_\phi^{k+2} f)(z) - 2(C_\phi^{*k} C_\phi C_\phi^* C_\phi^k f)(z) + (C_\phi^{*k} C_\phi^k f)(z) &\geq 0, \\
(M^2 C_\phi^{k+2} f)(\frac{e^{-i(k+2)\theta}}{2^{k+2}}z) - 2(C_\phi C_\phi^* C_\phi^k f)(\frac{e^{-i(k)\theta}}{2^k}z) + (C_\phi^k f)(\frac{e^{-i(k)\theta}}{2^k}z) &\geq 0, \\
M^2 f(\frac{e^{i(k+2)\theta}}{2^{k+2}}\frac{e^{-i(k+2)\theta}}{2^{k+2}}z) - 2f(\frac{e^{i(k)\theta}}{2^k}\frac{e^{-i\theta}}{2}\frac{e^{i\theta}}{2}\frac{e^{-i(k)\theta}}{2^k}z) + f(\frac{e^{i(k)\theta}}{2^k}\frac{e^{-i(k)\theta}}{2^k}z) &\geq 0, \\
M^2 f(\frac{1}{2^{2k+4}}z) - 2f(\frac{1}{2^{2k+2}}z) + f(\frac{1}{2^{2k}}z) &\geq 0.
\end{aligned}$$

for every $f \in H^2(\beta)$ and $z \in D$. Hence C_ϕ is (M,k) quasi * class Q operator.

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