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DISCRETE LEADER-FOLLOWING CONSENSUS FOR MULTI-AGENT SYSTEM WITH NON-INSTANTANEOUS IMPULSES

K. STEFANOVA[1], S. HRISTOVA[2], AND A. GOLEV[3]

[1,2,3]University of Plovdiv Paisii Hilendarski
24 Tzar Asen Str., 4000 Plovdiv, BULGARIA

ABSTRACT: The discrete-time multi-agent system with a leader and nonlinear intrinsic dynamics and non-instantaneous impulses is defined. The leader-following consensus is investigated. Some conditions ensuring the leader-following consensus are obtained. Several examples are solved by computer realization to illustrate effectiveness of the obtained results. By example the necessity and sufficiency of the obtained conditions are shown.

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1. INTRODUCTION

One of the most important topic in multi-agent systems is the consensus algorithm. It is connected with the driving a team of agents to reach an agreement on a certain issue by negotiating with their neighbors. In more details, each agent receives information from the set of other agents in the group and then all agents adjust their own information states depending on the information received from other agents. The goal is to reach an agreement. This behavior is widespread in the nature. A consensus algorithm describes the information transfers between agents and varies depending on the application and the model. In the literature, many different consensus algorithms have been proposed (see for example, [2], [3], [4], [5], [6], [7]). The virtual leader is a special agent whose motion is independent of all the other agents and thus is followed by all the other ones. Such a problem is commonly called leader-following consensus problem. Note, sometimes the multi-agent changes it behavior instantaneously, and on finite interval of time there are interactions only between the each agent and leader without interconnected interactions. Then the model is a new one. In this paper we set up the discrete model of such kind of situation and we call it, similarly to the continuous case, non-instantaneous impulsive model.

The main purpose of this paper is to study a discrete-time multi-agent system consisting of agents and the leader. For the first time it is studied the case when the control protocol is based on two interaction topologies. The first interaction topology is modeling the interactions between all agents including the leader. The second one is connecting only with the intervals on which any agent is interacting only with the leader, i.e. in the so-called intervals of non-instantaneous impulses. Sufficient conditions ensuring a leader-following consensus are obtained. By intensive application of computer simulation the influence of the impulses on the discrete leader-following consensus is illustrated and the necessity and effectiveness of the obtained conditions is showed.

2. DESCRIPTION OF THE DISCRETE MODEL WITH NON-INSTANTANEOUS IMPULSES

Let $\mathbb{Z}_+$ denote the set of all natural numbers. Let two increasing sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be given such that $n_k, m_k \in \mathbb{Z}_+$ and $m_k < n_{k+1} - n_k,\ k = 1, 2, \dots,$ be given. We denote $\mathbb{Z}[a, b] = \{z \in \mathbb{Z}_+ : a \le z \le b\},\ a, b \in \mathbb{Z}_+,\ a < b$ and $I_k = \mathbb{Z}[n_k + m_k + 1, n_{k+1}],\ k = 0, 1, 2, \dots,$ and $J_k = \mathbb{Z}[n_k + 1, n_k + m_k],\ k = 1, 2, \dots,$ where $m_0 = n_0 = 0$.

The intervals $J_k,\ k = 1, 2, \dots,$ will be called intervals of non-instantaneous impulses.

In this paper, we consider a discrete-time multi-agent system consisting of N agents and the leader. The dynamics of each agent labeled $i,\ i = 1, 2, \dots, N$, is given by the difference equation

$$x_i(n) = x_i(n-1) + f(n, x_i(n-1)) + u_i(n-1) \quad \text{for } n \in \mathbb{Z}_+,\ i = 1, 2, \dots, N, \tag{1}$$

where $x_i(n)$, $u_i(n)$ represent the state and the control input at time n, respectively. Function $f : \mathbb{Z}_+ \times \mathbb{R} \to \mathbb{R}$ describes the intrinsic, generally nonlinear, dynamics. The leader, labeled as $i = 0$, for multi-agent system (1) is an isolated agent described by

$$x_0(n) = x_0(n-1) + f(n, x_0(n-1)) \text{ for } n \in \mathbb{Z}_+. \tag{2}$$

Let the control protocol is based on two interaction topologies, $\mathcal{G}$ and $\mathcal{P}$. The graphs of both topologies represent the agent set and the edge set, respectively. The first interaction topology, $\mathcal{G}$, is modeled by a graph which is connected with time interval I_k. Any edge of the graph means that agent i receives information from agent j. Communication graph can be represented by two matrices: the first is connected with the interaction between the agents and it is a weighted adjacency matrix $A = \{a_{ij}\}$ with nonnegative entries, where $a_{ii} = 0$. The second matrix D is the leader adjacency matrix with diagonal elements d_i, $i = 1, \ldots, N$, where $d_i > 0$ if agent i receives information from the leader and $d_i = 0$, otherwise.

The second interaction topology, $\mathcal{P}$, is connected with the time intervals J_k when each agent interacts only with the leader. Communication graph can be represented by two leader adjacency matrix B_k and C. The matrix B_k has diagonal elements $B_{i,k}$, $i = 1, \ldots, N$, where $B_{i,k} > 0$ if agent i at time t_k (the point of impulse) receives information from the leader and $B_{i,k} = 0$, otherwise. The diagonal matrix C has diagonal elements c_i, $i = 1, \ldots, N$, where $c_i > 0$ if agent i at time n from the interval of impulses receives information from the leader and $c_i = 0$, otherwise.

Then the control protocol is defined by $u(0) = 0$ and:

$$
\begin{aligned}
u_i(n) = &\Big(\gamma \sum_{j=1}^{N} a_{ij}\big(x_j(n) - x_i(n)\big) + \gamma d_i(x_0(n) - x_i(n))\Big)\Delta(n)\\
&\Big(c_i(x_0(n) - x_i(n)) + B_{i,k}(x_0(n_k) - x_i(n_k)\Big)\delta(n)\\
&\quad \text{for } n \in \mathbb{Z}_+, \quad i = 1, 2, \ldots, N
\end{aligned} \tag{3}
$$

where $\delta(n) = 1$ and $\Delta(n) = 0$ for $n \in J_k,\ k = 1, 2, \ldots$ and $\delta(n) = 0$ and $\Delta(n) = 1$ for $n \notin J_k,\ k = 1, 2, \ldots$.

Then the system (1), (2), (3) could be written as a system of non-instantaneous impulsive difference equations

$$
\begin{aligned}
x_0(n) &= x_0(n-1) + f(n, x_0(n-1)) \text{ for } n \in \mathbb{Z}_+\\
x_i(n) &= x_i(n-1) + f(n, x_i(n-1))\\
&\quad + \gamma \sum_{j=1}^{N} a_{ij}\big(x_j(n-1) - x_i(n-1)\big)\\
&\quad + \gamma d_i(x_0(n-1) - x_i(n-1))\\
&\quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,\ i = 1, 2, \ldots, N,\\
x_i(n) &= x_i(n-1) + f(n, x_i(n-1))\\
&\quad + c_i(x_0(n-1) - x_i(n-1)) + B_{i,k}(x_0(n_k) - x_i(n_k))\\
&\quad \text{for } \ n \in J_k,\ k = 1, 2, \ldots,\ i = 1, 2, \ldots, N,\\
x_i(0) &= x_i^0, \quad i = 0, 1, 2, \ldots, N.
\end{aligned} \tag{4}
$$

Definition 1. Multi-agent system (1) and (2) under control law (3) (respectively, the system (4)) is said to be achieved *the leader-following consensus* if any solution to (1), (2), (3) satisfies $\lim_{n\to\infty}(x_i(n) - x_0(n)) = 0$ for $i = 1, 2, \dots, N$ for any initial values $x_i^0 \in \mathbb{R}$, $i = 0, 1, 2, \dots, N$.

Denote $e_i(n) = x_i(n) - x_0(n)$, $i = 1, 2, \dots, N$, $e_i^0 = x_i^0 - x_0^0$, and rewrite the system of difference equations (4) in the form

$$\begin{aligned} e_i(n) &= f(n, x_i(n-1)) - f(n, x_0(n-1)) \\ &\quad + \gamma \sum_{j=1}^{N} a_{ij} e_j(n-1) - e_i(n-1)\gamma \sum_{j=1}^{N} a_{ij} \\ &\quad + (1 - \gamma d_i) e_i(n-1) \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k,\ i = 1, 2, \dots, N, \\ e_i(n) &= (1 - c_i) e_i(n-1) - B_{i,k} e_i(n_k) \\ &\quad + f(n, x_i(n-1)) - f(n, x_0(n-1)), \\ &\quad \text{for } n \in J_k,\ k = 1, 2, \dots,\ i = 1, 2, \dots, N, \\ e_i(0) &= e_i^0, \quad i = 0, 1, 2, \dots, N. \end{aligned} \tag{5}$$

Introduce the quadratic n dimensional matrix $\mathcal{L}$ with elements $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij},\ i \neq j$.

2.1. LEADER-FOLLOWING CONSENSUS OF (5)

Theorem 2. *Let the following conditions be satisfied:*

1. *The function $f : \mathbb{Z}_+ \times \mathbb{R} \to \mathbb{R}$ and there exists $\varepsilon > 0$ such that $|f(n,u) - f(n,v)| \leq L|u - v|$ uniformly in $n \in \mathbb{Z}_+$ for any $u, v \in \mathbb{R}:\ |u - v| \leq \varepsilon$.*

2. *The inequality $L + M < 1$ holds where $M = |\gamma|\sqrt{|\lambda_{max}|}$, λ_{max} is the eigenvalue of matrix $C\ C^T$ with the maximal modulus, $C = \mathcal{L} + D - \frac{1}{\gamma} I$, I is the unit n dimensional matrix.*

3. *There exists a positive constants $\xi :\ \xi < M + L$ such that for all $k = 1, 2, \dots$ the inequality $L + \max_{i=1,2,\dots,N} |1 - c_i| + \max_{i=1,2,\dots,N} |B_{i,k}| \leq \xi$ holds.*

Then under control law (3) multi-agent system (1) and (2) achieves the local leader-following consensus.

Proof. Choose the initial values $x_i^0 \in \mathbb{R}$ such that $|x_i^0 - x_0^0| \leq \varepsilon$, $i = 0, 1, 2, \dots, N$ with ε defined in condition 1 of Theorem 2. Therefore, $|e_i(0)| \leq \varepsilon,\ i = 1, 2, \dots, N$

and $||e(0)|| = \max_{i=1,2,\dots,N} |e_i(0)| \leq \varepsilon$. We will prove that the corresponding solution to (1), (2), (3) satisfies

$$|x_i(n) - x_0(n)| \leq \varepsilon \quad \text{for all} \quad n \in \mathbb{Z}_+. \tag{6}$$

According to (5) using that $x_i^0 \in S(x_0^0, \varepsilon)$ and condition 1 of Theorem 2 we get

$$\begin{aligned} |e_i(1)| \leq& |f(1, x_i(0)) - f(1, x_0(0))| \\ &+ \gamma \sum_{j=1}^{N} a_{ij} |e_j(0) - e_i(0)| + |1 - \gamma d_i| \; |e_i(0)| \\ \leq& L|x_i(0) - x_0(0)| \\ &+ \gamma \sum_{j=1}^{N} a_{ij} |e_j(0) - e_i(0)| + |1 - \gamma d_i| \; |e_i(0)| \\ \leq& (L + M)\varepsilon \leq \varepsilon. \end{aligned} \tag{7}$$

Using induction we prove inequality (6) holds for $n \in I_0$. Similarly to inequality (7) it follows that

$$||e(n)|| = \max_{i=1,2,\dots,N} |x_i(n) - x_0(n)| \leq (L + M)||e(n-1)||, \quad n \in I_0$$

or

$$||e(n)|| \leq (L + M)^n ||e(0)||, \quad n \in I_0. \tag{8}$$

Also, we obtain

$$\begin{aligned} |e_i(n_1 + 1)| &\leq (L + |1 - c_i| + |B_{i,1}|)|e_i(n_1)| \\ &\leq (L + \max_{i=1,2,\dots,N} |1 - c_i| + \max_{i=1,2,\dots,N} |B_{i,1}|)|e_i(n_1)| \end{aligned} \tag{9}$$

or

$$||e(n_1 + 1)|| \leq \Xi_1 ||e(n_1)|| \leq (L + M)^{n_1} \Xi_1 ||e(0)|| \tag{10}$$

where

$$\Xi_k = L + \max_{i=1,2,\dots,N} |1 - c_i| + \max_{i=1,2,\dots,N} |B_{i,k}|, \quad k = 1, 2 \dots.$$

Therefore, for $n \in J_1$ using (8) for $n = n_1$ we get

$$||e(n)|| \leq \Xi_1^{n-n_1} ||e(n_1)|| \leq (L + M)^{n_1} \Xi_1^{n-n_1} ||e(0)||, \quad n \in J_1. \tag{11}$$

As in the proof of inequality (8) using the inequalities (11) for $n = n_1 + m_1$, $M + L < 1$ and $m_1 < n_2 - n_1$ we get

$$\begin{aligned} ||e(n)|| &\leq (L + M)^{n-n_1-m_1} ||e(n_1 + m_1)|| \\ &\leq (L + M)^{n-m_1} \Xi_1^{m_1} ||e(0)||, \quad n \in I_1. \end{aligned} \tag{12}$$

Let $n \in J_2$. Then using (12) for $n = n_2$ and using the inequalities $M + L < 1$ and $m_1 < n_2 - n_1$ we obtain

$$\begin{aligned} ||e(n)|| &\le \Xi_2^{n-n_2}||e(n_2)|| \le (L+M)^{n_2-m_1}\Xi_1^{m_1}\Xi_2^{n-n_2}||e(0)|| \\ &\le (L+M)^{n_1}\Xi_1^{m_1}\Xi_2^{n-n_2}||e(0)||, \quad n \in J_2. \end{aligned} \tag{13}$$

Let $n \in I_2$. Then using the inequalities $M + L < 1$ and $m_2 < n_3 - n_2$

$$\begin{aligned} ||e(n)|| &\le (L+M)^{n-n_2-m_2}||e(n_2+m_2)|| \\ &\le (L+M)^{n+n_1-n_2-m_2}\Xi_1^{m_1}\Xi_2^{m_2}||e(0)||, \quad n \in I_2. \end{aligned} \tag{14}$$

Let $n \in J_3$. Then

$$\begin{aligned} ||e(n)|| &\le \Xi_3^{n-n_3}||e(n_3)|| \\ &\le \Xi_3^{n-n_3}(L+M)^{n_3-m_1-m_2}\Xi_1^{m_1}\Xi_2^{m_2}||e(0)||, \quad n \in J_3. \end{aligned} \tag{15}$$

Continue the induction process we obtain

$$||e(n)|| \le \begin{cases} ||e(0)||(M+L)^{n-\sum\limits_{i=1}^{k} m_i}\left(\prod\limits_{i=1}^{k}\Xi_i^{m_i}\right) \\ \qquad \text{for } \ n \in I_k,\ k = 0,1,2,\dots, \\ ||e(0)||(M+L)^{n_k-\sum\limits_{i=1}^{k-1} m_i}\left(\prod\limits_{i=1}^{k}\Xi_i^{m_i}\right) \\ \qquad \text{for } \ n \in J_k,\ k = 1,2,3,\dots. \end{cases}$$

Use the conditions 2 and 3 we obtain

$$||e(n)|| \le \begin{cases} ||e(0)||(M+L)^{n}\prod\limits_{i=1}^{k}\left(\frac{\xi}{M+L}\right)^{m_i} \\ \qquad \text{for } \ n \in I_k,\ \ k = 0,1,2,\dots, \\ ||e(0)||(M+L)^{n_k+m_k}\prod\limits_{i=1}^{k}\left(\frac{\xi}{M+L}\right)^{m_i} \\ \qquad \text{for } \ n \in J_k,\ \ k = 1,2,3,\dots. \end{cases}$$

Applying conditions 2 and 3 of Theorem 2 it follows the validity of $\lim\limits_{n\to\infty}||e(n)|| = 0$.

□

Theorem 3. *Let the conditions 2 and 3 of Theorem 2 be satisfied and the function* $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ *is Lipshitz with a constant* L *w.r.t. its second argument in* $\mathbb{R}$.

Then under control law (3) multi-agent system (1) and (2) achieves the leader-following consensus.

The proof of Theorem 3 is similar to the one of Theorem 2 and we omit it.

3. EXAMPLES

Let $n_k = 10k$ and $m_k = 3$, $k = 1, 2, \ldots$. Then $m_k < n_{k+1} - n_k = 10$. Therefore, $I_0 = 1, 2, \ldots 10$, $I_k = 10k + 4, 10k + 5, \ldots, 10k + 10$, $J_k = 10k + 1, 10k + 2, 10k + 3$ for $k = 1, 2, \ldots$.

Now we will study a group of 4 followers and the leader with two interacting topologies. The first one $\mathcal{G}$ is determining the interactions of the agents and the leader on each intervals I_k, $k = 1, 2, \ldots$, and the second one $\mathcal{F}$ is determining the switching interactions with the leader on the impulsive interval J_k, $k = 1, 2, \ldots$. Let the weighted adjacency matrix A, the diagonal matrix D, giving the leader adjacency matrix associated with $\mathcal{G}$ and the diagonal matrix B_k, giving the leader adjacency switching matrix associated with $\mathcal{F}$, are given by

$$A = \begin{bmatrix} 0 & 1.5 & 1 & 0 \\ 1.5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$B_k = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix}, \quad k = 1, 2, \ldots,$$

the intrinsic dynamics is described by $f(n, x) = 0.01 \ln(1 + x^2)$ and the constant $\gamma = 0.4$. Then $L = 0.01$ and $M \approx 0.95$, i.e. conditions 1 and 2 of Theorem 1 are satisfied.

We will study different case for this system illustrating the above theory.

Example 1. (Non-instantaneous changes of the behavior of the followers with small jumps). Let the initial values be: $x_0^0 = 10, x_1^0 = 3, x_2^0 = 15, x_3^0 = 5, x_4^0 = 20$.

Consider

$$\begin{aligned} x_0(n) &= x_0(n-1) + 0.01 \ln(1 + x_0(n-1)^2) \\ &\quad \text{for} \quad n = 1, 2, \ldots, 30 \\ x_i(n) &= x_i(n-1) + 0.01 \ln(1 + x_i(n-1)^2) \\ &\quad + 0.4 \sum_{j=1}^{4} a_{ij} \big(x_j(n-1) - x_i(n-1)\big) \\ &\quad + 0.4 d_i (x_0(n-1) - x_i(n-1)) \\ &\quad \text{for} \quad n \in \bigcup_{k=0}^{3} I_k, \ i = 1, 2, 3, 4, \end{aligned} \tag{16}$$

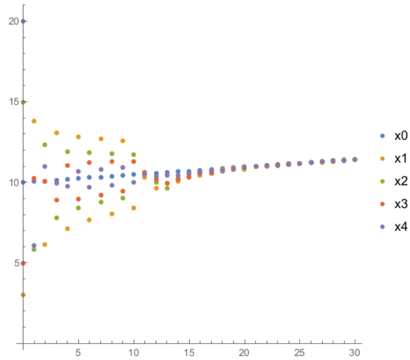

Figure 1. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \dots, 30$ (discretely).

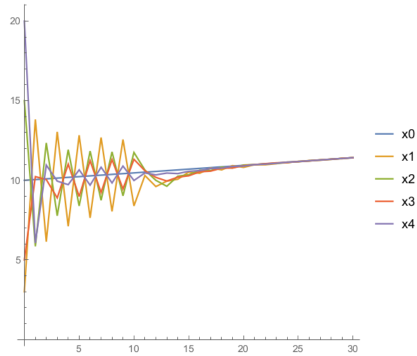

Figure 2. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \dots, 30$ (continuously).

$$\begin{aligned} x_i(n) &= x_i(n-1) + 0.01\ln(1 + x_i(n-1)^2) \\ &\qquad + c_i(x_0(n-1) - x_i(n-1)) + B_{i,k}(x_0(n_k) - x_i(n_k)), \\ &\qquad \text{for} \quad n \in J_k, \ k = 1,2,3, \ i = 1,2,3,4, \\ x_i(0) &= x_i^0, \quad i = 0,1,2,3,4, \end{aligned}$$

where $B_{i,k} = -0.5$, $c_i = 1.4$, $i = 1,4$ and $B_{i,k} = 0.5$, $c_i = 0.4$, $i = 2,3$, $k = 1,2,3,\dots$.

Then $L + \max\limits_{i=1,2,3,4} |1 - c_i| + \max\limits_{i=1,2,3,4} |B_{i,k}| = 0.01 + 0.4 + 0.5 = 0.91 < M + L = 0.96 < 1$, i.e. conditions 2 and 3 of Theorem 2 are satisfied. According to Theorem 3 the leader-following consensus is achieved. The state trajectories $x_i(n)$, $i = 0,1,2,3,4$ and $n = 0,1,2,\dots,30$ are shown in Figure 1 (discretely) and Figure 2 (continuously) and its values for $n = 1,2,\dots,12$ are shown in Table 1. From Table 1 and Figures 1 and 2 it could be seen that the state trajectory $x_i(n)$ of any agent approaches the state trajectory $x_0(n)$ of the leader.

Let's change the initial conditions, i.e. $x_0^0 = 20$, $x_1^0 = 13$, $x_2^0 = 18$, $x_3^0 = 15$, $x_4^0 = 27$. The state trajectories $x_i(n), i = 0,1,2,3,4$ and $n = 0,1,2,\dots,30$ are shown in Figures 3 and 4. Again the state trajectory $x_i(n)$ of any agent approaches the state trajectory $x_0(n)$ of the leader.

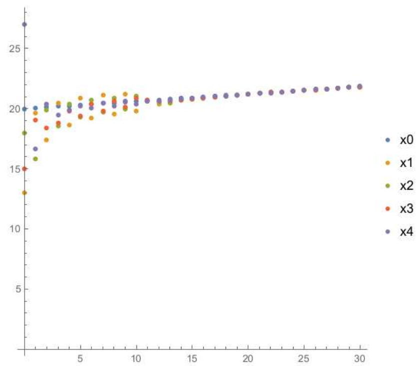

Figure 3. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (discretely).

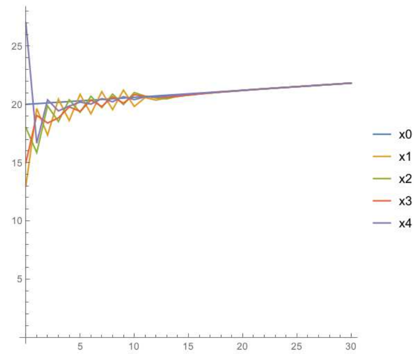

Figure 4. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (continuously).

n	x_0	x_1	x_2	x_3	x_4
0	10.0000	3.0000	15.0000	5.0000	20.0000
1	10.0462	13.8230	5.8542	10.2326	6.0599
2	10.0924	6.1474	12.3479	10.0463	10.9543
3	10.1387	13.0419	7.7757	8.8962	9.9495
4	10.1852	7.1141	11.9218	11.0196	9.7256
5	10.2317	12.8288	8.3921	8.9879	10.6565
6	10.2783	7.6427	11.8326	11.2357	9.6966
7	10.3250	12.6889	8.7464	9.2313	10.8231
8	10.3717	8.0457	11.7869	11.2957	9.8356
9	10.4186	12.5627	9.0255	9.4602	10.8944
10	10.4656	8.3924	11.7492	11.3199	9.9880
11	10.5126	10.3010	10.6433	10.5996	10.4640
12	10.5598	9.6075	9.9966	10.1850	10.3403

Table 1. Values of $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 12$.

n	x_0	x_1	x_2	x_3	x_4
0	20.0000	13.0000	18.0000	15.0000	27.0000
1	20.0599	19.6514	15.8578	19.0542	16.6659
2	20.1199	17.3594	19.8701	18.3967	20.3928
3	20.1800	20.4421	18.5234	18.8385	19.4364
4	20.2401	18.6050	20.3957	19.7778	19.8515
5	20.3003	20.8611	19.3194	19.3979	20.1927
6	20.3605	19.1872	20.6960	20.3604	20.0210
7	20.4208	21.0902	19.7172	19.8157	20.4884
8	20.4812	19.5498	20.8821	20.6543	20.2257
9	20.5416	21.2230	19.9832	20.1016	20.6617
10	20.6021	19.8191	21.0104	20.8343	20.4022
11	20.6626	20.5835	20.7038	20.6860	20.6424
12	20.7232	20.3633	20.5438	20.6212	20.6313

Table 2. Values of $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 12$.

Example 2. (Non-instantaneous changes of the behavior of the followers with at least one large jump). Now we will consider the case when the followers at same times change their behavior instantaneously, but at least one of them has a large jump.

Consider (16) with the same initial values as in Example 1 but changed both leader adjacency matrices B_k and C to

$$B_k = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Then the condition 3 of Theorem 1 is not satisfied because the leader adjacency coefficients are too large.

The state trajectories $x_i(n), i = 0, 1, 2, 3, 4$ and $n = 0, 1, \ldots, 100$ are shown in Figure 5 (discretely) and Figure 6 (continuously) and its values for $n = 1, 2, \ldots, 12$ are shown in Table 3. From Table 3 and Figures 5 and 6 it could be seen that the state trajectory $x_4(n)$ of the agent with a large jumps does not approach the state trajectory $x_0(n)$ of the leader. Therefore, the condition 3 of Theorem 2 is necessary condition to achieve leader-following consensus.

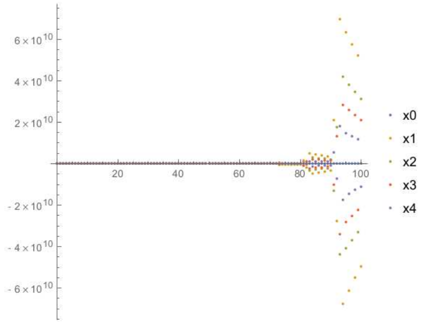

Figure 5. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 100$ (discretely).

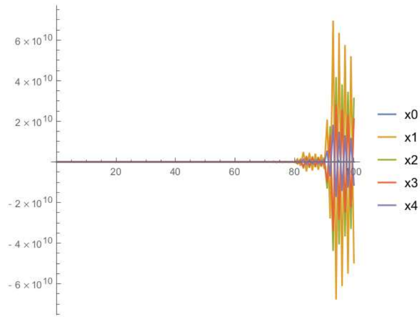

Figure 6. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 100$ (continuously).

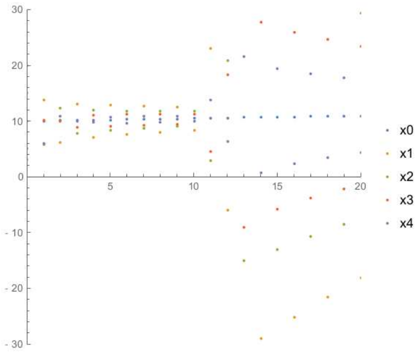

Figure 7. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 20$ (discretely).

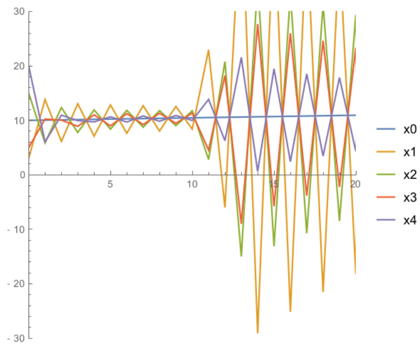

Figure 8. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 20$ (continuously).

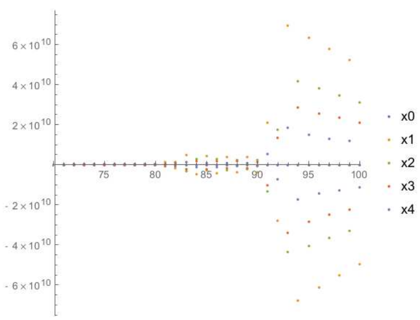

Figure 9. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 70, 71, 72, \ldots, 100$ (discretely).

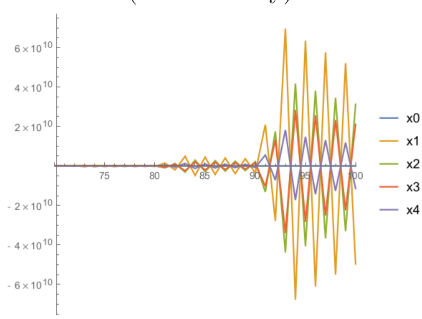

Figure 10. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 70, 71, 72, \ldots, 100$ (continuously).

n	x_0	x_1	x_2	x_3	x_4
0	10.0000	3.0000	15.0000	5.0000	20.0000
1	10.0462	13.8230	5.8542	10.2326	6.0599
2	10.0924	6.1474	12.3479	10.0463	10.9543
3	10.1387	13.0419	7.7757	8.8962	9.9495
4	10.1852	7.1141	11.9218	11.0196	9.7256
5	10.2317	12.8288	8.3921	8.9879	10.6565
6	10.2783	7.6427	11.8326	11.2357	9.6966
7	10.3250	12.6889	8.7464	9.2313	10.8231
8	10.3717	8.0457	11.7869	11.2957	9.8356
9	10.4186	12.5627	9.0255	9.4602	10.8944
10	10.4656	8.3924	11.7492	11.3199	9.9880
11	10.5126	22.9472	2.8134	4.5340	13.8552
12	10.5598	-6.0012	20.7986	18.2291	6.2684

Table 3. Values of $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 12$.

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