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CAPUTO FRACTIONAL DIFFERENTIAL EQUATION WITH STATE DEPENDENT DELAY AND PRACTICAL STABILITY

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ABSTRACT: Practical stability properties of Caputo fractional delay differential equations is studied and, in particular, the case with state dependent delays is considered. These type of delays is a generalization of several types of delays such as constant delays, time variable delays, or distributed delays. In connection with the presence of a delay in a fractional differential equation and the application of the fractional generalization of the Razumikhin method, we give a brief overview of the most popular fractional order derivatives of Lyapunov functions among Caputo fractional delay differential equations. Three types of derivatives for Lyapunov functions, the Caputo fractional derivative, the Dini fractional derivative, and the Caputo fractional Dini derivative, are applied to obtain several sufficient conditions for practical stability. An appropriate Razumikhin condition is applied. These derivatives allow the application of non-quadratic Lyapunov function for studying stability properties. We illustrate our theory on several nonlinear Caputo fractional differential equations with different types of delays.

Key Words: functional-differential equations with fractional derivatives, stability, Lyapunov functions, state dependent delay

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1. INTRODUCTION

Fractional differential equations have been studied extensively in the literature because of their applications in various fields of engineering and science (see, for example, the monographs [4, 13], and the cited therein references). Also, various types of delays arise in differential equations due to more adequate modeling of real world problems. State-dependent delays in differential equations with ordinary derivative are applied in modeling milling [10] and in control theory [18].

In the qualitative study for nonlinear systems stability properties are important. For example, several stability results for fractional order systems with delay were obtained in [5]. An appropriate method for this study is the Lyapunov second method and its modification of the Razumikhin method (see, for example, [6] for stability of fractional delay differential equations, and [9], for Lyapunov-Razumikhin techniques for state-dependent delay differential equations). LaSalle and Lefschetz [12] introduced the so called practical stability which do not provide stability of the equilibrium point but it is connected with its boundedness. This type of stability is studied for various types of differential equations (for the Caputo fractional differential equations see [2]).

The main goal of this paper is the study practical stability properties of Caputo fractional delay differential equations. We study the general case of state dependent delays which includes for example the cases of time dependent variable delay, with a constant delay, and without delay. To the best of our knowledge, this is the first paper studying practical stability properties of Caputo fractional differential equation with state dependent delay. The investigation is based on the fractional modification of the Razumikhin method. It is worth pointing out that the Lyapunov functional method cannot be easily generalized to fractional order systems. Taking these factors into consideration, we present a brief overview of the literature, with various definitions of fractional order derivatives of Lyapunov functions among Caputo fractional delay differential equations. Three types of fractional derivatives of Lyapunov functions are applied: Caputo fractional derivative, Dini fractional derivative, and Caputo fractional Dini derivative. Although the most popular of them is the Caputo fractional derivative, it leads to some restrictions in applications, such as applying differentiable Lyapunov function and a Razumikhin condition over the whole past time interval.

When the other two types of derivatives of Lyapunov functions, the Dini and Caputo fractional Dini derivatives, are applied, then in the sufficient conditions we obtain, we use less restrictive conditions (similar to the Razumikhin condition and continuous Lyapunov functions without the restriction of differentiability). Several examples with various types of delays are provided to illustrate the application of the sufficient conditions we obtain.

2. NOTES ON FRACTIONAL CALCULUS

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [14]. In many applications in science and engineering, the fractional order q is often less than 1, so we restrict $q \in (0,1)$ everywhere in the paper. There are several definitions of fractional derivatives and fractional integrals, and three of them are presented next ([14]):

1. The Riemann–Liouville (RL) fractional derivative of order q of function m is given by

$$ {}_{t_0}^{RL}D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \ t \geq t_0 $$

where $\Gamma(.)$ denotes the Gamma function.

2. The Caputo fractional derivative of order q is defined by

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \geq t_0. $$

The Caputo and Riemann-Liouville formulations coincide when the initial conditions are zero. Note that the RL derivative is meaningful under weaker smoothness requirements.

3. The Grünwald-Letnikov fractional derivative is given by

$$ {}_{t_0}\widetilde{D}_t^q m(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \qquad t \geq t_0 $$

and the Grünwald-Letnikov fractional Dini derivative by

$$ {}_{t_0}\tilde{D}_+^q m(t) = \limsup_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \qquad t \geq t_0, \tag{1} $$

where $\binom{q}{r} = \frac{q(q-1)\dots(q-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

From the relation between the Caputo fractional derivative and the Grünwald-Letnikov fractional derivative, using (1), we define the Caputo fractional Dini derivative as

$$ {}_{t_0}^{c}D_+^q m(t) = {}_{t_0}\tilde{D}_+^q [m(t) - m(t_0)], $$

i.e.

$$ {}_{t_0}^{c}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q}\Big[m(t) - m(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\Big(m(t-rh) - m(t_0)\Big)\Big]. \quad (2) $$

3. STATEMENT OF THE PROBLEM

Let $\mathbb{R}_+ = [0,\infty)$ and $r > 0$ be a given number. Consider the space C_0 of all functions $y : [-r, 0] \to \mathbb{R}^n$ which are continuous endowed with the norm

$$ \|y\|_0 = \sup_{t\in[-r,0]} \{\|y(t)\| : \ y \in C_0\}, $$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$.

Consider the initial value problem (IVP) for a nonlinear system of fractional differential equations with finite state dependent delay (FrDDE) with $q \in (0,1)$:

$$ \begin{aligned} {}_{t_0}^{C}D_t^q x(t) &= f(t, x(t), x_{\rho(t,x_t)}), \quad \text{for } t > t_0, \\ x(t + t_0) &= \phi(t), \quad \text{for } t \in [-r, 0], \end{aligned} \quad (3) $$

where $x \in \mathbb{R}^n$, $t_0 \geq 0$ is the initial time, and ${}_{t_0}^{C}D_t^q y(t)$ denotes the Caputo fractional derivative for the state x. Also, $f : [t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\rho : [t_0,\infty) \times C_0 \to \mathbb{R}_+$, and $\phi \in C_0$, are three given functions, where $r > 0$. Here, $x_t(s) = x(t+s)$, $s \in [-r, 0]$ represents the history of the state from time $t-r$ up to the present time t. Note that, for any $t \geq 0$, we let $x_{\rho(t,x_t)} = x(\rho(t, x(t + s)))$, $s \in [-r, 0]$.

We introduce the following assumptions:

A1 The function f belong to $C([t_0,\infty) \times C_0 \times C_0, \mathbb{R}^n)$.

A2 There exists a set $\Omega \subset C_0$ such that the function $\rho \in C([t_0,\infty) \times C_0, \mathbb{R})$ and $t - r \leq \rho(t, u) \leq t$, for $u \in \Omega$.

A3 The function $f(t, 0) = 0$ for $t \geq t_0$.

Remark 1. Condition (A2) guarantees the delay in the argument of the unknown function in (3), i.e., the function ρ determines the state-dependent delay. Also, this condition guarantees the boundedness of the delay in (3).

Remark 2. The delay in (3) is in a very general form and it includes the time dependent variable delay (with $\rho(t,u) \equiv t-\tau(t)$, for $t \geq 0, u \in C_0$, and $\tau \in C(\mathbb{R}_+, R_+)$), the constant delay (with $\rho(t,u) = t - C$, for $t \geq 0, u \in C_0$, and $C = const > 0$), and without delay (with $\rho(t,u) \equiv t$, for $t \geq 0$, and $u \in C_0$).

Remark 3. Note that condition (A3) guarantees the existence of the zero solution of IVP for FrDDE (3) with the zero initial function $\varphi \equiv 0$.

Remark 4. The function $\rho(t,u) = t - \sin^2(u)$ satisfies the condition (A2) with $r = 1$, i.e. $t - 1 \leq t - \sin^2(u) \leq t$

Remark 5. Let x be a solution of (3). Then, $x_t \in C_0$ for any fixed $t \geq 0$. Define the function $\psi(s) = x(t+s),\ s \in [-r, 0]$. Then, $\psi_0 = x_t \in C_0$ and

$$x_{\rho(t,x_t)} = x(\rho(t, x(t+s))) = x(t+(\rho(t, x(t+s))-t)) = \psi((\rho(t, \psi(s))-t)) = \psi_{(\rho(t,\psi_0)-t)}.$$

If the condition (A2) is satisfied, then $\psi \in C_0$ and $\rho(t, \psi_0) - t \in [-r, 0]$.

Now we will define practical stability for the nonlinear Caputo FrDDE following the ideas for practical stability for ordinary differential equations ([12]).

Definition 1. The zero solution of FrDDE (3) with zero initial function is called

(S1) *practically stable* w.r.t. (λ, A), if there exits an initial time $t_0 \geq 0$ such that, for any initial function $\phi \in C_0 : \|\phi\|_0 < \lambda$, the inequality $\|x(t; t_0, \phi)\| < A$, for $t \geq t_0$, holds, where the real numbers (λ, A) with $0 < \lambda < A$ are given;

(S2) *uniformly practically stable* w.r.t. (λ, A), if (S1) is satisfied for all $t_0 \geq 0$;

(S3) *practically quasi stable with respect to* (λ, B, T) if there exists $t \geq t_0$ such that, for any $\phi \in C_0$, the inequality $\|\phi\|_0 < \lambda$ implies $\|x(t; t_0, \phi)\| < B$, for $t \geq t_0 + T$, where the positive constants λ, B, T with $0 < \lambda < B$ are given;

(S4) *uniformly practically quasi stable with respect to* (λ, B, T) if (S3) holds for all $t_0 \geq 0$.

Here, $x(t; t_0, \phi)$ is a solution of (3).

Remark 6. We note that in (S2) and (S4) of Definition 1, the change of the initial time t_0 leads to a change of the differential equation and not only on the initial condition (different than the case of ordinary differential equations).

Remark 7. Note that, from stability properties of the zero solution of (3), we have the practical stability but the opposite is not true.

Remark 8. Similar to Definition 1, various types of practical stability of a fixed nonzero solution could be defined. For convenience, we state all definitions and theorems for the case when the equilibrium point is the origin of $\mathbb{R}^n$ because any equilibrium point can be shifted to the origin via an appropriate change of variables.

Define the following sets:

$$\begin{aligned} \mathcal{K} &= \{a \in C(\mathbb{R}_+,\mathbb{R}_+) : a \text{ is strictly increasing and } a(0) = 0\}; \\ S_A &= \{x \in \mathbb{R}^n : \ \|x\| \leq A\}, \quad A > 0. \end{aligned}$$

We will use comparison results for the scalar fractional differential equation without any delay

$$ {}_{t_0}^{c}D^q u(t) = g\left(t, u\right), \qquad \text{for } t > t_0, \quad \text{s.t. } u(t_0) = v_0, \tag{4}$$

where $u, v_0 \in \mathbb{R}$ and $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$.

We denote the solution of the IVP for the scalar FrDE (4) by $u(t; t_0, v_0)$. In the case of non-uniqueness of the solution we will assume the existence of a maximal one.

We introduce the assumption:

A4 The function $g \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $g(t, 0) \equiv 0$, and for any $v_0 \in \mathbb{R}$, the IVP for the scalar FrDE (4) has a solution $u(t; t_0, v_0)$.

Remark 9. Practical stability properties of the scalar IVP (4) is defined similar to Definition 1.

Remark 10. We will study the practical stability of (3) or (4) in the case when the right side part depends on the unknown function. In the case $f(t, x) \equiv F(t)$ or $g(t, x) \equiv G(t)$, then the equation has no zero solution. The nonzero solution could be bounded by a bound depending on the initial condition.

Example 1. Consider the IVP for the scalar FrDE

$$\begin{aligned} {}_{0}^{C}D^{0.4}u(t) &= \frac{-t^{1.6}}{\Gamma(2.6)}\ {}_1F_2\Big(\{1\}, \{1.3, 1.8\}, -\frac{t^2}{4}\Big), \quad \text{for } t > 0, \\ u(0) &= u_0, \end{aligned} \tag{5}$$

where ${}_1F_2(1, \{1.3, 1.8\}, -\frac{t^2}{4})$ is the regularized generalized hypergeometric function. The IVP (5) has a solution ([15])

$$u(t) = u_0 + \cos(t) - 1, \ t \geq 0.$$

For any (λ, A), with $A = 2 + \lambda$, and for any initial value $|u_0| < \lambda$, the corresponding solution satisfies $|u(t)| < A, \ t \geq 0$. This is similar to (S1) in Definition 1 but we are not able to say that the zero solution is practically stable because there is no zero solution. The solution is bounded by a constant $2 + |u_0|$.

4. LYAPUNOV FUNCTIONS AND THEIR FRACTIONAL DERIVATIVES

We study the connection between the practical stability properties of the zero solution of the system of FrDDE (3) and the practical stability of the zero solution of the scalar FrDE (4) by applying an appropriate modification of Razumikhin method and Lyapunov functions. In connection with the application of fractional derivatives, we need an appropriate definitions of the derivative of Lyapunov functions among the studied fractional equations. We introduce the class Λ of Lyapunov-like functions which will be used to investigate the practical stability of the system FrDDE (3).

Definition 2. Let $I \subset \mathbb{R}_+$ and $\mathcal{D} \subset \mathbb{R}^n$. We say that the function $V : I \times \mathcal{D} \to \mathbb{R}_+$ belongs to the class $\Lambda(I, \mathcal{D})$ if V is continuous and locally Lipschitzian with respect to its second argument in $I \times \mathcal{D}$.

In connection with the Caputo fractional derivative, it is necessary to define in an appropriate way the derivative of the Lyapunov functions among the studied equation. We will give a brief overview of the three main types derivatives of Lyapunov functions $V \in \Lambda([t_0 - r, T), \mathcal{D})$ among solutions of fractional differential equations in the literature:

- *Caputo fractional derivative* - Let $x(t) \in \mathcal{D},\ t \in [t_0 - r, T)$, be a solution of the IVP for the FrDDE (3) and $V \in \Lambda([t_0 - r, T), \mathcal{D})$. We define

$$ {}_{t_0}^{c}D^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \frac{d}{ds}\Big(V(s, x(s))\Big) ds, \quad t \in (t_0, T). \tag{6}$$

  This type of derivative is applicable for continuously differentiable Lyapunov functions.

- *Dini fractional derivative* - Let $\psi \in C([-\tau, 0], \mathcal{D})$ and $V \in \Lambda([t_0 - r, T), \mathcal{D})$, $t_0 \geq 0$ is a given initial point. Then, for any $t \in (t_0, T)$, we define the Dini fractional derivative of V by

$$ {}_{t_0}D^q_{(3)} V(t, \psi(0), \psi) = \limsup_{h \to 0} \frac{1}{h^q} \Big[V(t, \psi(0)) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{q}{r} V(t - rh, \psi(0) - h^q f(t, \psi(0), \psi_{(\rho(t,\psi_0)-t)})) \Big], \tag{7}$$

  where $\psi_0(s) = \psi(s)$ and $\psi_{(\rho(t,\psi_0)-t)} = \psi(\rho(t, \psi(s)) - t)$, for $s \in [-r, 0]$.

  Note that, if condition (A2) is satisfied, then $\rho(t, \psi(s)) - t \in [-r, 0]$ and $\psi_{\rho(t,\psi(s))-t}$ is well defined.

- Caputo fractional Dini derivative - Let the initial data $(t_0,\phi)\in\mathbb{R}_+\times C([-\tau,0],\mathcal{D}))$ and $\psi\in C([-\tau,0],\mathcal{D})$ be given. Then, for the Lyapunov function $V\in\Lambda([t_0-r,T),\mathcal{D})$, we define the Caputo fractional Dini derivative by

$$\begin{aligned}
&{}_{t_0}^{\,c}D^q_{(3)}V(t,\psi(0),\psi;t_0,\phi(0))\\
&=\limsup_{h\to 0^+}\frac{1}{h^q}\Big\{V(t,\psi(0))-V(t_0,\phi(0))\\
&-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{q}{r}\Big(V(t-rh,\psi(0)-h^qf(t,\psi(0),\psi_{\rho(t,\psi_0)-t}))-V(t_0,\phi(0))\Big)\Big\},
\end{aligned}\tag{8}$$

for $t\in(t_0,T)$, or its equivalent form,

$$\begin{aligned}
&{}_{t_0}^{\,c}D^q_{(3)}V(t,\psi(0),\psi;t_0,\phi(0))\\
&=\limsup_{h\to 0^+}\frac{1}{h^q}\Big\{V(t,\psi(0))+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r}\binom{q}{r}V(t-rh,\psi(0)-h^qf(t,\psi(0),\psi_{\rho(t,\psi_0)-t})\Big\}\\
&\qquad-\frac{V(t_0,\phi(0))}{(t-t_0)^q\Gamma(1-q)},
\end{aligned}\tag{9}$$

for $t\in(t_0,T)$.

Remark 11. For any initial data $(t_0,\phi)\in\mathbb{R}_+\times C([-\tau,0],\mathcal{D})$ of the IVP for FrDDE (3) and $\psi\in C([-\tau,0],\mathcal{D})$, the relation between the Dini fractional derivative defined by (7) and the Caputo fractional Dini derivative defined by (9) is given by

$${}_{t_0}^{\,c}D^q_{(3)}V(t,\psi(0),\psi;t_0,\phi(0))={}_{t_0}D^q_{(3)}V(t,\psi(0),\psi)-\frac{V(t_0,\phi(0))}{(t-t_0)^q\Gamma(1-q)},$$

or by

$${}_{t_0}^{\,c}D^q_{(3)}V(t,\psi(0),\psi;t_0,\phi(0))={}_{t_0}D^q_{(3)}V(t,\psi(0),\psi)-{}_{t_0}^{RL}D^q\Big(V(t_0,\phi(0))\Big).$$

Remark 12. In the particular case of time variable delays $\rho(t,x_t)\equiv\tau(t)\le t$, some authors use the following definition for the derivative of the Lyapunov function among the fractional delay differential equations (see, for example, [16])

$$D^+V(t,\psi(0))=\limsup_{h\to 0}\frac{1}{h^q}\Big[V(t,\psi(0))-V(t-h,\psi(0)-h^qf(t,\psi_0))\Big],\tag{10}$$

with $\psi_0(s)=\psi(s)$.

This operator does not depend on the order q of the fractional derivative nor on the initial time t_0, which is typical for the Caputo fractional derivative. Also, it has no memory and if x is a solution of (3), then $D^+V(t,x(t))\neq{}_{t_0}^{\,c}D^qV(t,x(t))$.

We will give an example illustrating the above defined types of fractional derivatives of Lyapunov functions. To simplify the calculations and to emphasize the derivatives and their properties, we will consider the scalar case, i.e., $n = 1$.

Example 2. Let $V(t,x) = m(t)\, x^2$, where $m \in C^1(\mathbb{R}_+, \mathbb{R}_+)$.

Case 1. Caputo fractional derivative. Let $x(t) = x(t; t_0, \phi)$ be a solution of the IVP for FrDDE (3). The fractional derivative

$$ {}_{t_0}^{c}D^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{m'(s)x^2(s) + 2m(s)x(s)x'(s)}{(t-s)^q} ds $$

is difficult to obtain in the general case for any solution of (3). In the particular case $m(t) \equiv 1$, there is an upper bound ([8])

$$ {}_{t_0}^{c}D^q V(t, x(t)) = {}_{t_0}^{c}D^q (x(t))^2 \leq 2x(t) {}_{t_0}^{c}D^q x(t) = 2x(t) f(t, x(t), x_{\rho(t,x_t)}). $$

Case 2. Dini fractional derivative. Let $\psi \in C([-\tau, 0], \mathcal{D})$ be given. Applying (7), we obtain

$$ {}_{t_0}D^q_{(3)} V(t, \psi(0), \psi) = 2\psi(0)\; m(t) f(t, \psi(0), \psi_{\rho(t,\psi_0)-t}) + (\psi(0))^2 \; {}_{t_0}^{RL}D^q m(t). \quad (11) $$

Case 3. Caputo fractional Dini derivative. Let the initial data $(t_0, \phi) \in \mathbb{R}_+ \times C([-\tau, 0], \mathcal{D})$ and $\psi \in C([-\tau, 0], \mathcal{D})$ be given. Use (9) and we obtain

$$ \begin{aligned} & {}_{t_0}^{c}D^q_{(3)} V(t, \psi(0), \psi; t_0, \phi(0)) \\ & = 2\psi(0) m(t) f(t, \psi(0), \psi_{\rho(t,\psi_0)-t}) + (\psi(0))^2 \; {}_{t_0}^{RL}D^q m(t) - \frac{(\phi(0))^2 m(t_0)}{(t-t_0)^q \Gamma(1-q)}. \end{aligned} \quad (12) $$

5. COMPARISON RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAY

First we recall the following result for Caputo fractional Dini derivatives of continuous functions.

Lemma 1. *([1]) Let $m \in C([t_0, t_0 + \theta], \mathbb{R})$, $\theta > 0$, and suppose that there exist $t^* \in (t_0, t_0 + \theta]$ such that $m(t^*) = 0$, and $m(t) < 0$, for $t_0 \leq t < t^*$. Then, if the Caputo fractional Dini derivative* (2) *exists $t = t^*$, then the inequality ${}_{t_0}^{c}D^q_{+} m(t^*) > 0$ holds.*

Now we will obtain some comparison results for the case of state dependent delays.

Lemma 2. *(Comparison result by the Caputo fractional Dini derivative) Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t; t_0, x_0) \in \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^n$, *is a solution of IVP for FrDDE* (3) *defined for* $t \in [t_0, t_0 + \theta]$, $\theta > 0$.

2. *There exists a function* $G \in C([t_0, t_0 + \theta] \times \mathbb{R}, \mathbb{R})$ *and a real* $H > 0$ *such that, for any* $\epsilon \in [0, H]$ *and* $v_0 \in \mathbb{R}$, *the scalar FrDE*

$$ {}_{t_0}^{c}D^q u = G(t, u) + \epsilon \quad \text{for} \quad t \in [t_0, t_0 + \theta], \qquad \text{with } u(t_0) = v_0 \tag{13} $$

has a solution $u(t; t_0, v_0, \epsilon) \in C^q([t_0, t_0 + \theta], \mathbb{R})$.

3. *The function* $V \in \Lambda([t_0 - r, t_0 + \theta], \mathcal{D})$ *and, for any point* $t \in [t_0, t_0 + \theta]$ *such that*

$$ V(t, x^*(t)) = \sup_{s \in [t-r, t]} V(s, x^*(s)), $$

the inequality

$$ {}_{t_0}^{c}D^q_{(3)} V(t, \psi(0), \psi; t_0, \phi(0)) \le G(t, V(t, x^*(t))) \tag{14} $$

holds, where $\psi(s) = x^*(t + s)$ *for* $s \in [-r, 0]$ *and* $\psi_{(\rho(t,\psi_0)-t)} = x_{\rho(t,x_t)}$ *in Eq.* (8) *(see Remark 5).*

Then,

$$ \sup_{s \in [-r, 0]} V(t_0 - s, \phi(s)) \le u_0 $$

implies

$$ V(t, x^*(t)) \le u^*(t), \qquad \text{for } t \in [t_0, t_0 + \theta], $$

where $u^*(t) = u(t; t_0, u_0, 0)$ *is the maximal solution of IVP for scalar FrDE* (13) *for* $v_0 = u_0$ *and* $\varepsilon = 0$.

Proof. Assume that the condition

$$ \sup_{s \in [-r, 0]} V(t_0 - s, \phi(s)) \le u_0, $$

for a given u_0, holds. Let $\varepsilon \in (0, H]$ be an arbitrary fixed number and consider the initial value problem for the scalar FrDE (13) with $v_0 = u_0 + \varepsilon$. According to Condition 2, the IVP for the scalar FrDE (13) has a solution $u_\varepsilon(t) = u(t; t_0, u_0 + \varepsilon, \varepsilon)$. Then, it satisfies the Volterra fractional integral equation ([7, Lemma 6.2])

$$ u_\varepsilon(t) = u_0 + \varepsilon + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} \Big(G(s, u_\varepsilon(s)) + \varepsilon \Big) ds, \qquad \text{for} \quad t \in [t_0, t_0 + \theta]. \tag{15} $$

Let the function $m \in C([t_0, t_0 + \theta], \mathbb{R}_+)$ be defined by $m(t) = V(t, x^*(t))$. We now prove that

$$ m(t) < u_\varepsilon(t), \qquad \text{for} \quad t \in [t_0, t_0 + \theta]. \tag{16} $$

Note that the inequality (16) holds for $t = t_0$ since

$$m(t_0) = V(t_0, x^*(t_0)) = V(t_0, \phi(0)) \leq \sup_{s\in[-r,0]} V(t_0 - s, \phi(s)) \leq u_0 < u_\varepsilon(t_0).$$

Assume that the inequality (16) is not true for some $t \in (t_0, t_0 + \theta]$. Then, there exists a point $t^* \in (t_0, t_0 + \theta]$ such that $m(t^*) = u_\varepsilon(t^*)$ and $m(t) < u_\varepsilon(t)$, for $t \in [t_0, t^*)$. Now, Lemma 1 (applied to $m(t) - u_\varepsilon(t)$) yields

$${}_{t_0}^{c}D_+^q(m(t^*) - u_\varepsilon(t^*) > 0,$$

i.e.

$${}_{t_0}^{c}D_+^q m(t^*) > G(t^*, u_\varepsilon(t^*)) + \varepsilon > G(t^*, m(t^*)). \tag{17}$$

Then, from Eq. (2), we obtain for $t \in (t_0, t_0 + \theta]$ the equality

$$\limsup_{h\to 0+} \frac{1}{h^q}\Big[x^*(t) - \phi(0) - S(x^*(t), h)\Big] = f(t, x^*(t), x^*_{\rho(t,x^*_t)}),$$

where

$$S\left(x^*(t), h\right) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}[x^*(t - rh) - \phi(0)]. \tag{18}$$

Therefore,

$$x^*(t) - \phi(0) - S\left(x^*(t), h\right) = h^q f(t, x^*(t), x^*_{\rho(t,x^*_t)}) + \Xi(h^q),$$

or

$$x^*(t) - h^q f(t, x^*(t), x^*_{\rho(t,x^*_t)}) = S\left(x^*(t), h\right) + \phi(0) + \Xi(h^q),$$

with $\frac{\|\Xi(h^q)\|}{h^q} \to 0$ as $h \to 0$.

Then, for any $t \in (t_0, t_0 + \theta]$, we obtain

$$\begin{aligned}
&m(t) - m(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\Big[m(t - rh) - m(t_0)\Big] \\
&= \Bigg\{V(t, x^*(t)) - V(t_0, \phi(0)) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\Big[V\Big(t - rh, x^*(t) - h^q f(t, x^*(t), x^*_{\rho(t,x^*_t)})\Big) - V(t_0, \phi(0))\Big]\Bigg\} \\
&\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r} V\Big(t - rh, S\left(x^*(t), h\right) + \phi(0) + \Xi(h^q)\Big) \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r} V\Big(t - rh, x^*(t - rh)\Big).
\end{aligned} \tag{19}$$

Since V is locally Lipschitzian in its second argument with a Lipschitz constant $L > 0$, applying (18), we obtain

$$\begin{aligned}
&\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\Big\{V\Big(t-rh, S\left(x^*(t),h\right)+\phi(0)+\Xi(h^q)\Big) \\
&\qquad - V\Big(t-rh, x^*(t-rh)\Big)\Big\} \\
&\leq \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r} L\|S\left(x^*(t),h\right)+\phi(0)-x^*(t-rh)\| + L\|\Xi(h^q)\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}\binom{q}{r} \\
&\leq L\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\| \sum_{j=1}^{[\frac{t-t_0}{h}]} (-1)^{j+1}\binom{q}{j}[x^*(t-jh)-\phi(0)]+\phi(0)-x^*(t-rh)\| \\
&\qquad + L\|\Xi(h^q)\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}\binom{q}{r} \\
&\leq L\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\|x^*(t-rh)-\phi(0)\| \\
&\quad + L\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\binom{q}{r}\| \sum_{j=1}^{[\frac{t-t_0}{h}]} (-1)^{j+1}\binom{q}{j}[x^*(t-jh)-\phi(0)]\| \\
&\qquad + L\|\Xi(h^q)\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}\binom{q}{r}.
\end{aligned} \tag{20}$$

Note that

$$\limsup_{h\to 0+} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}\binom{q}{r} = -1. \tag{21}$$

Substitute (20) in (19), divide both sides by h^q, take the limit as $h \to 0^+$, use (21), and we obtain that, for any $t \in (t_0, t_0+T]$,

$$
\begin{aligned}
{}_{t_0}^{c}D_+^q m(t) \leq\ & {}_{t_0}^{c}D_{(3)}^q V(t,\psi(0),\psi;t_0,\phi(0)) + L \lim_{h\to 0+} \frac{\Xi(h^q)}{h^q} \lim_{h\to 0+} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} \\
& + L \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{q}{r} \|x^*(t-rh) - \phi(0)\| \\
& + L \lim_{h\to 0+} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{q}{r} \limsup_{h\to 0+} \frac{1}{h^q} \sum_{j=1}^{[\frac{t-t_0}{h}]} (-1)^{j+1} \binom{q}{j} \|x^*(t-jh) - \phi(0)\| \\
=\ & {}_{t_0}^{c}D_{(3)}^q V(t,\psi(0),\psi;t_0,\phi(0)) \leq G(t, V(t, x^*(t))) = G(t, m(t)),
\end{aligned}
\tag{22}
$$

where $\psi_0 = x_t^*$ and $\psi_{(\rho(t,\psi_0)-t)} = x^*_{\rho(t,x_t)}$ in Eq. (8) (see Remark 5). Now, (22) with $t = t^*$ contradicts (17). Therefore, (16) holds on $[t_0, t_0+\theta]$. Since $\varepsilon \in (0, H]$ is an arbitrary number, it follows that (16) holds on $[t_0, t_0+\theta]$ and any $\varepsilon \in (0, H]$.

We now show that, if $0 < \varepsilon_2 < \varepsilon_1 \leq H$, then

$$
u_{\varepsilon_2}(t) < u_{\varepsilon_1}(t) \quad \text{ for } \ t \in [t_0, t_0+\theta]. \tag{23}
$$

Note that the inequality (23) holds for $t = t_0$. Assume that inequality (23) is not true. Then, there exists a point $t^* \in (t_0, t_0+\theta]$ such that $u_{\varepsilon_2}(t^*) = u_{\varepsilon_1}(t^*)$, and $u_{\varepsilon_2}(t) < u_{\varepsilon_1}(t)$, for $t \in [t_0, t^*)$. Now, Lemma 1 (applied to $u_{\varepsilon_2}(t) - u_{\varepsilon_1}(t)$) yields

$$
{}_{t_0}^{c}D_+^q (u_{\varepsilon_2}(t^*) - u_{\varepsilon_1}(t^*)) > 0.
$$

However,

$$
{}_{t_0}^{c}D_+^q (u_{\varepsilon_2}(t^*) - u_{\varepsilon_1}(t^*)) = G(t^*, u_{\varepsilon_2}(t^*)) + \varepsilon_2 - [G(t^*, u_{\varepsilon_1}(t^*)) + \varepsilon_1] = \varepsilon_2 - \varepsilon_1 < 0,
$$

obtaining a contradiction. Thus, (23) is true.

Now, given $0 < \varepsilon \leq H$, (16) together with (23), guarantee that the family of solutions $\{u_\varepsilon(t) : t \in [t_0, t_0+\theta]\}$ of (13) is uniformly bounded, i.e., there exists $K > 0$ with

$$
|u_\varepsilon(t)| \leq K, \quad \text{ for } (t,\varepsilon) \in [t_0, t_0+\theta] \times [0, H].
$$

Let

$$
M = \sup\{|G(t,x)| : (t,x) \in [t_0, t_0+\theta] \times [-K, K]\}.
$$

Consider a decreasing sequence of positive numbers $\{\varepsilon_j\}_{j=0}^{\infty}$, $0 < \varepsilon_0 \leq H$, such that $\lim_{j\to\infty} \varepsilon_j = 0$, and consider the sequence of solutions $u_{\varepsilon_j}(t)$. Now, for $t_1, t_2 \in [t_0, t_0+\theta]$, with $t_1 < t_2$, using the inequalities

$$
\begin{aligned}
a^q - b^q &\leq 2(a-b)^q, \quad \text{ for } a \geq b \geq 0, \\
(t_1 - s)^q &\leq (t_2 - s)^q, \quad \text{ for } s \in [t_0, t_1],
\end{aligned}
$$

and

$$\int_{t_0}^{t_1}\Big((t_2-s)^{q-1}-(t_1-s)^{q-1}\Big)ds=\frac{1}{q}\Big((t_2-t_0)^q-(t_1-t_0)^q-(t_2-t_1)^q\Big)\le\frac{(t_2-t_1)^q}{q},$$

we get

$$\begin{aligned}|u_{\varepsilon_j}(t_2)-u_{\varepsilon_j}(t_1)|&\le\frac{1}{\Gamma(q)}\Big|\int_{t_0}^{t_1}\Big((t_2-s)^{q-1}-(t_1-s)^{q-1}\Big)\Big(G(s,u_{\varepsilon_j}(s))+\varepsilon_j\Big)ds\Big|\\&\quad+\Big|\int_{t_1}^{t_2}((t_2-s)^{q-1})(G(s,u_{\varepsilon_j}(s))+\varepsilon_j)ds\Big|\\&\le\frac{M+H}{\Gamma(q)}\Big\{\frac{(t_2-t_1)^q}{q}+\frac{(t_2-t_1)^q}{q}\Big\}=2\frac{M+1}{q\Gamma(q)}(t_2-t_1)^q.\end{aligned}\tag{24}$$

Thus, the family $\{u_{\varepsilon_j}(t)\}$ is equicontinuous on $[t_0,t_0+\theta]$. The Arzela-Ascoli Theorem guarantees that there exists a subsequence $\{u_{\varepsilon_{j_k}}(t)\}$ that is uniformly convergent in the interval $[t_0,t_0+\theta]$. Let

$$w(t)=\lim_{k\to\infty}u_{\varepsilon_{j_k}}(t).$$

Take the limit in (15) as $k\to\infty$ and we see that w satisfies the initial value problem (4) for $t\in[t_0,t_0+\theta]$, i.e., it is a solution of IVP (13) for $v_0=u_0$ and $\varepsilon=0$. Now, take the limit in (16) for $\varepsilon=\varepsilon_{j_k}$ as $k\to\infty$ and we have $m(t)\le w(t)\le u^*(t)$ on $[t_0,t_0+\theta]$. □

In the case when the Dini fractional derivative is applied instead of the Caputo fractional Dini derivative, the following result is obtained.

Lemma 3. *(Comparison result by the Dini fractional derivative). Assume the conditions of Lemma 2 are satisfied where inequality* (14) *is replaced by*

$$ {}_{t_0}^{c}D^q_{(3)}V(t,\psi(0),\psi)\le G(t,V(t,x^*(t))),\tag{25}$$

where $\psi(s)=x^*(t+s)$ *for* $s\in[-r,0]$ *and* $\psi_{(\rho(t,\psi_0)-t)}=x_{\rho(t,x_t)}$ *in Eq.* (7).

Then,

$$\sup_{s\in[-r,0]}V(t_0-s,\phi(s))\le u_0$$

implies

$$V(t,x^*(t))\le u^*(t),\qquad for\ t\in[t_0,t_0+\theta],$$

where $u^*(t)=u(t;t_0,u_0,0)$ *is the maximal solution of IVP for scalar FrDE* (13) *for* $v_0=u_0$ *and* $\varepsilon=0$.

The proof of Lemma 3 follows from Remark 11 and Lemma 2.

Corollary 1. *Let Condition 1 of Lemma 2 be satisfied and the function* $V \in \Lambda([t_0 - r, t_0 + \theta], \Delta)$ *be such that, for any point* $t \in [t_0, t_0 + \theta]$ *such that*

$$V(t, x^*(t)) = \sup_{s \in [t-r, t]} V(s, x^*(s)),$$

the inequality

$$ {}_{t_0}^{c}\mathcal{D}_{(3)}^{q} V(t, \psi(0)) \leq 0 \tag{26}$$

holds, where ${}_{t_0}^{c}\mathcal{D}_{(3)}^{q} V(t, \psi(0))$ *denotes one of: the Caputo fractional Dini derivative* ${}_{t_0}^{c}D_{(3)}^{q} V(t, \psi(0), \psi; t_0, \phi(0))$ *or the Dini fractional derivative* ${}_{t_0}^{c}D_{(3)}^{q} V(t, \psi(0), \psi)$ *with* $\psi_{(\rho(t,\psi_0)-t)} = x_{\rho(t,x_t)}$ *in Eq.* (8) *or* (7), *respectively.*

Then, for $t \in [t_0, t_0 + \theta]$, *the inequality*

$$V(t, x^*(t)) \leq \sup_{s \in [-r, 0]} V(t_0 - s, \phi(s))$$

holds.

Proof. The proof of Corollary 2 follows from the fact that the corresponding IVP for the scalar FrDE (13) with $G(t, u) = 0$, $\varepsilon = 0$, and $v_0 = \sup_{s \in [-r,0]} V(t_0 - s, \phi(s))$, i.e., the equation ${}_{t_0}^{c}D^q u = 0$, has a unique solution

$$u(t) = \sup_{s \in [-r, 0]} V(t_0 - s, \phi(s)),$$

for $t \in [t_0, t_0 + \theta]$. □

The result of Lemma 2 is also true on the half line.

6. PRACTICAL STABILITY RESULTS.

We will use various types of fractional derivatives of Lyapunov functions to obtain sufficient conditions for different types of practical stability. The base of the study will be the comparison results applying scalar fractional differential equation without any type of delay. We will use in our method a fractional extension of the Razumikhin method. Note that in [6] some stability results for delay FrDDE are obtained applying the Caputo fractional derivative of the Lyapunov function and the generalized Razumikhin condition

$$\sup_{\Theta \in [-r, t]} V(\Theta, x(\Theta)) = V(t, x(t)). \tag{27}$$

Remark 13. Note that condition (27) is not very similar to the idea of Razumikhin condition and it is very restrictive, but it is necessarily because of the application of Caputo fractional derivative of the Lyapunov function.

We will give sufficient conditions for practical stability of zero solution of FrDDE (3) applying the Caputo fractional derivative of the Lyapunov function.

Theorem 1. *(*Practical stability by the Caputo fractional derivative*). Let conditions (A1)-(A3) be satisfied for a given number $t_0 \geq 0$ and for $\Omega \equiv C_0$. Suppose also that here exists a continuously differentiable Lyapunov function $V \in \Lambda([t_0-r,\infty),\mathbb{R}^n)$ with $V(t,0)=0$, such that*

(i) the inequalities

$$\alpha_1(\|x\|) \leq V(t,x), \ \textit{for} \ \ t \geq t_0, x \in \mathbb{R}^n$$
$$V(t,x) \leq \alpha_2(\|x\|), \ \textit{for} \ \ t \geq t_0, x \in S_\lambda,$$

hold, where $\alpha_i \in \mathcal{K}$, $i=1,2$, and $\lambda > 0$ is a given number.

(ii) for any $t > t_0$ and for any solution $x(t) = x(t;t_0,\phi)$ of (3) with $\phi \in C_0$ such that

$$\sup_{\Theta \in [t_0-r,t]} V(\Theta, x(\Theta)) = V(t,x(t)),$$

the inequality

$${}_{t_0}^{C}D_t^q V(t,x(t)) \leq 0$$

holds.

Then, the zero solution of (3) is practically stable w.r.t. $(\lambda, \alpha_1^{-1}(\alpha_2(\lambda)))$.

Proof. Let $x(t) = x(t;t_0,\phi)$ be a solution of FrDDE (3) with $\|\phi\|_0 < \lambda$. Define the function

$$v(t) = \sup_{s \in [t_0-r,t]} V(s,x(s)), \ t \geq t_0.$$

Obviously, the function v is nondecreasing. We will prove that

$$v(t) = v(t_0), \quad \text{for } t \geq t_0. \tag{28}$$

Assume that (28) is not true, i.e., there exists a point $T > t_0$ such that $v(t) = v(t_0)$, for $t \in [t_0,T]$, but $v(t) > v(t_0)$ and v is strictly increasing for $t \in (T, T+\varepsilon]$, $\varepsilon > 0$ is a small enough number. Then $v(s) = v(t_0) \geq V(s,x(s))$, for $s \in [t_0,T]$, and $v(t) = V(t,x(t))$, for $t \in (T,T+\varepsilon]$. Therefore, for $t \in (T,T+\varepsilon]$, the inequality

$$\sup_{\Theta \in [t_0-r,t]} V(\Theta, x(\Theta)) = v(t) = V(t,x(t))$$

holds, and according to condition (ii), the inequality ${}_{t_0}^{C}D_t^q V(t,x(t)) \leq 0$ holds. Then, for any $t \in (T,T+\varepsilon]$, we obtain

$${}_{t_0}^{c}D^q v(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{v'(s)}{(t-s)^q} ds \leq \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{V'(s,x(s))}{(t-s)^q} ds = {}_{t_0}^{C} D_t^q V(t,x(t)) \leq 0,$$

because $v(s) \geq V(s, x(s))$ for $s \in [t_0, T+\varepsilon]$, and

$$\int_{t_0}^{t} \frac{v'(s)}{(t-s)^q} ds = \int_{t_0}^{t} \frac{v'(s) - V'(s, x(s))}{(t-s)^q} ds + \int_{t_0}^{t} \frac{V'(s, x(s))}{(t-s)^q} ds$$

$$= -\frac{v(t_0) - V(t_0, x(t_0))}{(t-t_0)^q} - q \int_{t_0}^{t} \frac{v(s) - V(s, x(s))}{(t-s)^{q+1}} ds + \int_{t_0}^{t} \frac{V'(s, x(s))}{(t-s)^q} ds.$$

According to the assumption, we get $v'(t) = 0$, for $t \in [t_0, T]$, and $v'(t) > 0$, for $t \in (T, T+\varepsilon]$. Then, for any $t \in (T, T+\varepsilon]$, we obtain

$${}_{t_0}^{c}D^q v(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{v'(s)}{(t-s)^q} ds = \frac{1}{\Gamma(1-q)} \int_{T}^{t} \frac{v'(s)}{(t-s)^q} ds > 0, \quad t \in (T, T+\varepsilon].$$

The contradiction proves the inequality (28).

From condition (i), we get

$$\alpha_1(\|x(t)\|) \leq V(t, x(t)) \leq v(t) = v(t_0) = \sup_{s \in [t_0 - r, t_0]} V(s, \phi(t_0 + s))$$

$$\leq \sup_{s \in [t_0 - r, t_0]} \alpha_2(\|\phi(t_0 + s)\|) \leq \alpha_2(\lambda).$$

□

Remark 14. Note that Theorem 1 is similar to [6, Theorem 3.1], except for condition (i). This condition leads to practical stability and it gives the bounds of both the initial function and the solution.

Theorem 2. *(*Uniform practical stability by the Caputo fractional derivative*). Let conditions (A1)-(A3) be satisfied for $t_0 = 0$ and $\Omega \equiv C_0$. Suppose that there exists a continuously differentiable Lyapunov function $V \in \Lambda([-r, \infty), \mathbb{R}^n)$, with $V(t, 0) = 0$, $t \geq 0$, such that condition (i) of Theorem 1 is satisfied for $t \geq 0$ and*

(ii) for any $t_0 \geq 0$ and any $t > t_0$ such that

$$\sup_{\Theta \in [t_0 - r, t]} V(\Theta, x(\Theta; t_0, \phi)) = V(t, x(t; t_0, \phi)),$$

the inequality

$${}_{t_0}^{C}D_t^q V(t, x(t; t_0, \phi)) \leq 0$$

holds, where $x(t; t_0, \phi)$ a solution of (3) *with initial time t_0 and initial function $\phi \in C_0$.*

Then, the zero solution of (3) *is uniformly practically stable w.r.t. $(\lambda, \alpha_1^{-1}(\alpha_2(\lambda)))$.*

Now we will apply the Dini fractional derivative and the Caputo fractional Dini derivative of Lyapunov functions to obtain sufficient conditions for practical stability.

In this case, we will use the Razumikhin condition

$$\sup_{\Theta\in[-r,0]} V(t+\Theta,\psi(\Theta)) = V(t,\psi(0)) \tag{29}$$

for a function $\psi \in C_0$.

Remark 15. Note that condition (29) is less restrictive than condition (27) and it is closer to the idea of the original Razumikhin condition.

We will study the connection between the practical stability properties of the system FrDDE (3) and the practical stability properties of the scalar FrDE (4) .

Theorem 3. *(*Practical stability by the Caputo fractional Dini derivative*). Let the following conditions be satisfied:*

1. *The conditions (A1)-(A4) are satisfied for a given t_0 and $\Omega \equiv C_0$.*
2. *There exists a function $V \in \Lambda([t_0-r,\infty),\mathbb{R}^n)$ with $V(t,0)=0$ such that*
   - *(i) the inequalities*

     $$\alpha_1(\|x\|) \le V(t,x) \text{ for } t \ge t_0,\ x\in\mathbb{R}^n,$$
     $$V(t,x) \le \alpha_2(\|x\|) \text{ for } t \ge t_0,\ x\in S_\lambda,$$

     hold, where $\alpha_i \in \mathcal{K},\ i=1,2,$ and $\lambda>0$ is a given number;
   - *(ii) For any functions $\psi \in C_0$ and $\phi \in C_0$, with $\|\phi\|_0 \le \lambda$, and for any point $t \ge t_0$ such that*

     $$\sup_{s\in[-r,0]} V(t+s,\psi(s)) = V(t,\psi(0)),$$

     the inequality

     $${}_{t_0}^{c}D^q_{(3)}V(t,\psi(0),\psi;t_0,\phi) \le g(t,V(t,\psi(0))) \tag{30}$$

     holds.
3. *The zero solution of the scalar FrDE (4) is practically stable w.r.t. $(\alpha_2(\lambda),A)$, where $A \ge \alpha_2(\lambda)$ is a given number.*

Then, the zero solution of FrDDE (3) is practically stable w.r.t. $(\lambda,\alpha_1^{-1}(A))$.

Proof. From condition 3 of Theorem 3, for $u_0 \in \mathbb{R}$ with $|u_0| < \alpha_2(\lambda)$, we have

$$|u(t;t_0,u_0)| < A, \text{ for } t \ge t_0, \tag{31}$$

where $u(t; t_0, u_0)$ is a solution of FrDE (4). Let $x(t) = x(t; t_0, \phi)$ be any solution of FrDDE (3) with $\|\phi\|_0 < \lambda$, and let

$$\tilde{u}_0 = \max_{s\in[-r,0]} V(t_0 + s, \phi(s)).$$

From Condition 2 (i), it follows that

$$\tilde{u}_0 \leq \max_{s\in[-r,0]} \alpha_2(\|\phi(s)\|) \leq \alpha_2(\lambda).$$

Therefore, the maximal solution $u^*(t) = u(t; t_0, \tilde{u}_0)$ of FrDE (4) satisfies inequality (31). The conditions of Lemma 2 are satisfied for $\mathcal{D} = \mathbb{R}^n$, $\theta = \infty$, and $G(t, u) = g(t, u)$. According to Lemma 2, the inequality

$$V(t, x^*(t)) \leq u^*(t), \text{for } t \geq t_0, \tag{32}$$

holds. Then, from condition 2 (i) and inequalities (31), (32) we obtain

$$\alpha_1(\|x^*(t)\|) \leq V(t, x^*(t)) \leq u^*(t) < A,$$

which implies that

$$\|x^*(t)\| < a_1^{-1}(A), \quad \text{for } t \geq t_0.$$

Note that $A \geq \alpha_2(\lambda) \geq \alpha_1(\lambda)$ or $\lambda \leq \alpha_1^{-1}(A)$. Therefore, according to Definition 1, the zero solution is practically stable w.r.t. $(\lambda, a_1^{-1}(A))$. □

In the case when the Caputo fractional derivative ${}_{t_0}^{c}D^q_{(3)}V(t, \psi(0), \psi; t_0, \phi)$ in Theorem 3 is replaced by the Dini fractional derivative ${}_{t_0}^{c}D^q_{(3)}V(t, \psi(0), \psi)$, we obtain the following result:

Theorem 4. *(*Practical stability by the Dini fractional derivative*). Let conditions 1 and 3 of Theorem 3 be satisfied, and*

2. There exists a function $V \in \Lambda([t_0 - r, \infty), \mathbb{R}^n)$ with $V(t, 0) = 0$, such that

(i) the inequalities

$$\alpha_1(\|x\|) \leq V(t, x) \textit{ for } t \geq t_0,\ x \in \mathbb{R}^n,$$
$$V(t, x) \leq \alpha_2(\|x\|) \textit{ for } t \geq t_0,\ x \in S_\lambda,$$

hold, where $\alpha_i \in \mathcal{K}$, $i = 1, 2$, and $\lambda > 0$ is a given number;

(ii) for any function $\psi \in C_0$ and for any point $t \geq t_0$ such that

$$\sup_{s\in[-r,0]} V(t + s, \psi(s)) = V(t, \psi(0)),$$

the inequality

$${}_{t_0}^{c}D^q_{(3)}V(t, \psi(0), \psi) \leq g(t, V(t, \psi(0))) \tag{33}$$

holds.

Then, the zero solution of FrDDE (3) *is practically stable w.r.t.* $(\lambda, \alpha_1^{-1}(A))$.

The proof of Theorem 4 is similar to the one of Theorem 3 where Lemma 3 is used instead of Lemma 2.

In the case of the zero function on the right side part of the FrDE (4), we obtain:

Corollary 2. *Let the conditions of Theorem 3/ Theorem 4 be satisfied with* $g(t,u) \equiv 0$. *Then, the zero solution of FrDDE* (3) *is practically stable w.r.t.* $(\lambda, \alpha_1^{-1}(\alpha_2(\lambda)))$.

Theorem 5. *(*Uniform practical stability by the Caputo fractional Dini derivative*). Let conditions (A1)-(A3) be satisfied for* $t_0 = 0$ *and* $\Omega \equiv C_0$. *Suppose that there exists a continuously differentiable Lyapunov function* $V \in \Lambda([-r,\infty), \mathbb{R}^n)$ *with* $V(t,0) = 0,\ t \geq 0$, *such that condition (i) of Theorem 3 is satisfied for* $t \geq 0$, *and*

(ii) for any $t_0 \geq 0$, *for any functions* $\psi \in C_0$ *and* $\phi \in C_0$ *with* $\|\phi\|_0 \leq \lambda$, *and for any point* $t \geq t_0$ *such that*

$$\sup_{s\in[-r,0]} V(t+s, \psi(s)) = V(t, \psi(0)),$$

the inequality

$$ {}_{t_0}^{c}D_{(3)}^{q} V(t, \psi(0), \psi; t_0, \phi) \leq 0 \tag{34}$$

holds. Then, the zero solution of (3) *is uniformly practically stable w.r.t.* $(\lambda, \alpha_1^{-1}(\lambda))$.

The proof of Theorem 5 follows from Theorem 3 and the fact that the solution of FrDE (4) with $g(t,u) \equiv 0$ is $u(t; t_0, v_0) = v_0$, for any $t_0 \geq 0$ and $v_0 \in \mathbb{R}$, i.e., it is uniformly practically stable w.r.t. (λ, λ).

Theorem 6. *(*Uniform practical stability by the Dini fractional derivative*). Let conditions (A1)-(A3) be satisfied for* $t_0 = 0$ *and* $\Omega \equiv C_0$. *Suppose that there exists a continuously differentiable Lyapunov function* $V \in \Lambda([-r,\infty), \mathbb{R}^n)$ *with* $V(t,0) = 0,\ t \geq 0$, *such that condition (i) of Theorem 3 is satisfied for* $t \geq 0$ *and*

(ii) for any $t_0 \geq 0$, *any functions* $\psi \in C_0$ *and any point* $t \geq t_0$ *such that*

$$\sup_{s\in[-r,0]} V(t+s, \psi(s)) = V(t, \psi(0)),$$

the inequality

$$ {}_{t_0}^{c}D_{(3)}^{q} V(t, \psi(0), \psi) \leq 0 \tag{35}$$

holds. Then, the zero solution of (3) *is uniformly practically stable w.r.t.* $(\lambda, \alpha_1^{-1}(\lambda))$.

Remark 16. Note that when the Dini fractional derivative or the Caputo fractional Dini derivative is applied in the obtained sufficient conditions

- the less restrictive condition (29) is used instead of the condition (27);

- the Lyapunov function need only be continuous without the restriction of differentiability.

7. APPLICATIONS

Example 3. (*Constant delay and the Caputo fractional derivative*). Consider the IVP for the scalar linear FrDDE

$$\begin{aligned} & {}_0^C D_t^{0.4} x(t) = -(|\cos(t)| + 1.5)x(t) + |\sin(t)|x(t-\pi) \text{ for } t > 0, \\ & x(s) = \phi(s) \text{ for } s \in [-\pi, 0]. \end{aligned} \tag{36}$$

Consider the quadratic Lyapunov function, i.e., $V(t,x) = x^2$. Let $\alpha_1(x) = 0.5x^2$, $\alpha_2(x) = x$, and $\lambda = 1$. Then, condition (i) of Theorem 1 is satisfied.

Let x be a solution of IVP for FrDDE (37), and $t > 0$ be such that

$$V(t, x(t)) = x^2(t) \geq x^2(s) = V(s, x(s)),$$

for all $s \in [-\pi, t)$. Then, according to [8], we get

$$\begin{aligned} {}_0^C D_t^{0.4} V(t, x(t)) \leq 2x(t)\, {}_0^C D_t^{0.4} x(t) &= -2(|\cos(t)| + 1.5)(x(t))^2 + 2|\sin(t)|x(t)x(t-\pi) \\ &\leq -2(|\cos(t)| - |\sin(t)| + 1.5)(x(t))^2 \leq 0. \end{aligned}$$

According to Theorem 1, the zero solution of (36) is practically stable w.r.t. (1,1), i.e. for any initial function $\phi \in S_1$, the solution is also in the ball S_1. The graphs of the absolute values of the solutions x_1, x_2, x_3 of the IVP for FDDE (36), with initial functions $\phi_1(s) = \sin(s), \phi_2(s) = \cos(s)$ and $\phi_3(s) = 0.5\frac{s}{s-1}$, for $s \in [-\pi, 0]$, are given in Figure 1.

Example 4. (*Constant delay*). Consider the IVP for the scalar linear FrDDE

$$\begin{aligned} & {}_0^C D_t^{0.4} x(t) = g(t)x(t) + 0.1x(t-\pi) \text{ for } t > 0, \\ & x(s) = \phi(s) \text{ for } s \in [-\pi, 0], \end{aligned} \tag{37}$$

where

$$g(t) = -0.5 \frac{{}_0^{RL} D^{0.4} \Big(\sin^2(t) + 1 \Big)}{\sin^2(t) + 1} - 0.1, \quad t > 0,$$

$x_t(s) = x(t - \pi)$ for $s \in [-\pi, 0]$, and $\rho(t, u) \equiv t - \pi$. Therefore, $x_{\rho(t,x_t)} = x(\rho(t, x_t)) = x(t - \pi)$.

Denote $f(t, x, y) = g(t)x + 0.1y$.

Case 1. Consider the quadratic Lyapunov function, i.e. $V(t,x) = x^2$. Let x be a solution of IVP for FrDDE (37), and $t > 0$ be such that

$$V(t, x(t)) = x^2(t) \geq x^2(s) = V(s, x(s)),$$

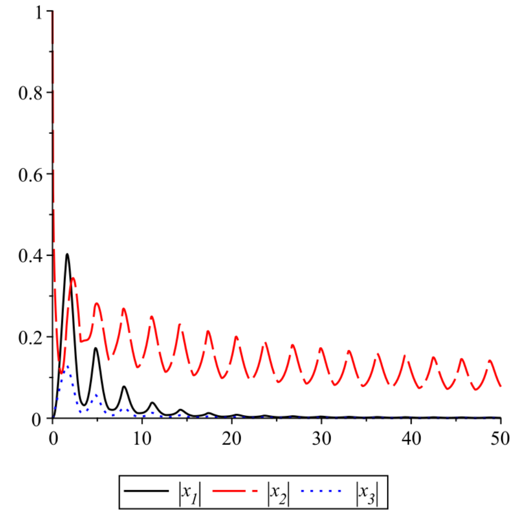

Figure 1: Graph of the the absolute values of the solutions x_1, x_2, x_3.

for all $s \in [-\pi, t)$. Then, according to [8], we get

$$\begin{aligned} {}_{0}^{C}D_t^{0.4}V(t,x(t)) &\leq 2x(t)\, {}_{0}^{C}D_t^{0.4}x(t) = 2(x(t))^2 g(t) + 0.2x(t)x(t-\pi) \\ &\leq 2(x(t))^2 g(t) + 0.1x^2(t) + 0.1x^2(t-\pi) \leq 2x^2(t)(0.1+g(t)). \end{aligned} \tag{38}$$

The sign of ${}_{0}^{C}D_t^{0.4}V(t,x(t))$ is changeable (the graph of the function $g+0.1$ is given in Figure 2).

Case 2. Consider the function

$$V(t,x) = (\sin^2(t)+1)x^2,$$

for $t \geq -\pi$ and $x \in \mathbb{R}$. Then, condition 2(i) of Theorem 3 is satisfied with $t_0 = 0$, $\alpha_1(u) = u^2$, $\alpha_2(u) = 2u$, and $\lambda = 1$, because

$$\sin^2(t) + 1 \in [1,2], \quad \forall t \geq 0.$$

Case 2.1: Caputo fractional derivative. Let x be a solution of IVP for FrDDE (37). Then, we obtain

$${}_{0}^{C}D_t^{0.4}V(t,x(t)) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-0.4} \Big(2(\sin^2(s)+1)x(s)x'(s) + \sin(2s)x^2(s)\Big)ds.$$

The fractional derivative of this function V is difficult to obtain so it is difficult to discuss its sign.

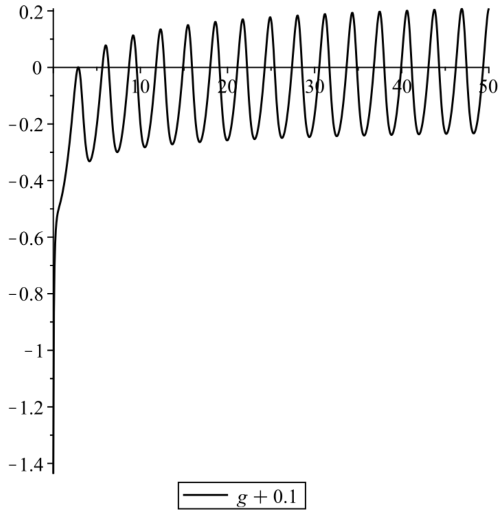

Figure 2: Graph of the function $g+0.1$.

Case 2.2: Dini fractional derivative. Let $\psi \in C([-\pi,0],\mathbb{R})$ be a function and $t>0$ be a point such that

$$V(t,\psi(0)) = (\sin^2(t)+1)\psi^2(0) \geq (\sin^2(t+s)+1)\psi^2(s), \quad s\in[-\pi,0).$$

Therefore,

$$(\sin^2(t)+1)\psi^2(-\pi) = (\sin^2(t-\pi)+1)\psi^2(-\pi) \leq (\sin^2(t)+1)\psi^2(0).$$

Note that, in this case, $\psi_{\rho(t,\psi_0)-t} = \psi(-\pi)$. Then, according to Example 2 and Eq. (11), we obtain

$$\begin{aligned}
&{}_0D^{0.4}_{(37)}V(t,\psi(0),\psi(-\pi))\\
&= 2\psi(0)(\sin^2(t)+1)f(t,\psi(0),\psi(-\pi)) + (\psi(0))^2\, {}_0^{RL}D^{0.4}\Big((\sin^2(t)+1)\Big)\\
&= 2\psi(0)(\sin^2(t)+1)\Big(g(t)\psi(0)+0.1\psi(-\pi)\Big) + (\psi(0))^2\, {}_0^{RL}D^{0.4}\Big(\sin^2(t)+1\Big)\\
&= \psi^2(0)\Big(2(\sin^2(t)+1)g(t) + {}_0^{RL}D^{0.4}\Big(\sin^2(t)+1\Big)\Big)\\
&\quad + 0.2\psi(0)(\sin^2(t)+1)\psi(-\pi)\\
&\leq -0.2\psi^2(0)(\sin^2(t)+1) + 0.1(\sin^2(t)+1)(\psi^2(0)+\psi^2(-\pi)) = 0.
\end{aligned}$$

Therefore, the conditions of Theorem 4 are satisfied, with $g(t,u)\equiv 0$, and according to Corollary 2, the zero solution of (37) is practically stable w.r.t. $(1,\sqrt{11})$ because $\alpha_1^{-1}(u)=\sqrt{u}$ and $\alpha_1^{-1}(\alpha_2(1))=\sqrt{2}$.

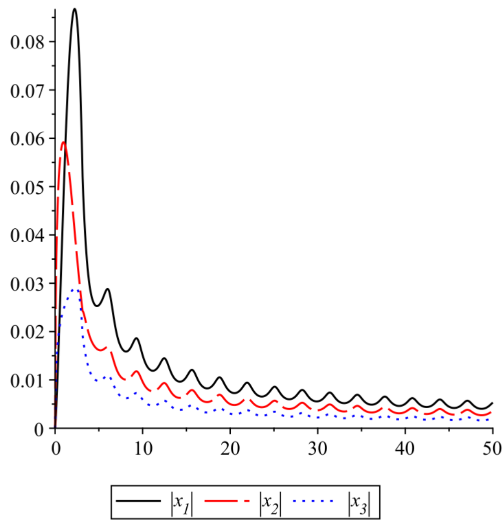

Figure 3: Graph of the the absolute values of the solutions x_1, x_2, x_3.

Case 2.3: Caputo fractional Dini derivative . According to Remark 11 and Case 2.2, the inequality

$$ {}_0^cD_{(37)}^{0.4}V(t,\psi(0),\psi(-h);0,\phi(0)) = {}_0D_{(37)}^{0.4}V(t,\psi(0),\psi(-h)) - \frac{0.1\phi^2(0)}{t^{0.4}\Gamma(0.6)} \leq 0 $$

holds. Therefore, the conditions of Theorem 3 are satisfied, with $g(t,u) \equiv 0$, and according to Corollary 2, the zero solution of (37) is practically stable w.r.t. $(1, \sqrt{2})$.

The graphs of the absolute values of the solutions x_1, x_2, x_3 of the IVP for FrDDE (37), with initial functions $\phi_1(s) = \sin(s), \phi_2(s) = 0.5 - 0.5\cos(s)$ and $\phi_3(s) = 0.5\frac{s}{s-1}$, for $s \in [-\pi, 0]$, are given in Figure 3.

Example 5. (*Time variable delay*). Consider the IVP for the scalar linear FrDDE

$$ \begin{aligned} &{}_0^CD_t^{0.4}x(t) = g(t)x(t) + 0.1x(t - \sin^2(t) + 1 - \pi) \text{ for } t > 0, \\ &x(s) = \phi(s) \text{ for } s \in [-\pi, 0], \end{aligned} \tag{39} $$

where

$$ g(t) = -0.5\frac{{}_0^{RL}D^{0.4}\Big(m(t)\Big)}{m(t)} - 0.1, \quad m(t) = \frac{0.1t + \pi}{t + 2\pi}, \quad t > 0, $$

and $\rho(t,u) = t - \sin^2(t) + 1 - \pi$. Note that $t - \pi \leq \rho(t,u) \leq t$, for $t \geq 0$, i.e., condition (A2) is satisfied. Consider the function $V(t,x) = m(t)x^2$, for $t \geq -\pi$, $x \in \mathbb{R}$. Then, condition 2(i) of Theorem 3 is satisfied, with $t_0 = 0$, $\alpha_1(u) = 0.5u^2$, $\alpha_2(u) = 0.9u$, and $\lambda = 1$, because

$$ \frac{0.1t + \pi}{t + 2\pi} \in [0.5, 0.9], \quad \forall t \geq 0. $$

Let $\psi \in C([-\pi,0],\mathbb{R})$ be a function and $t \geq 0$ be a point such that

$$V(t,\psi(0)) = m(t)\psi^2(0) \geq m(t+s)\psi^2(s) = V(t+s,\psi(s)), \quad s \in [-\pi,0)$$

Then, since $\xi = -\sin^2(t)+1-\pi \in [-\pi,0]$, we have

$$\begin{aligned} m(t)\phi^2(0) \geq m(t+\xi)\psi^2(\xi) &= m(t-\sin^2(t)+1-\pi)\psi^2(-\sin^2(t)+1-\pi) \\ &\geq m(t)\psi^2(-\sin^2(t)+1-\pi). \end{aligned}$$

Note that, in this case, $\psi_{\rho(t,\psi_0)-t} = \psi(-\sin^2(t)+1-\pi)$. According to Example 2 and Eq. (11), we obtain

$$\begin{aligned} &{}_0D^{0.4}_{(37)}V(t,\psi(0),\psi) \\ &= 2\psi(0)m(t)f(t,\psi(0),\psi_{\rho(t,\psi_0)-t}) + (\psi(0))^2\ {}_0^{RL}D^{0.4}\Big(m(t)\Big) \\ &= 2\psi(0)m(t)\Big(g(t)\psi(0)+0.1\psi(-\sin^2(t)+1-\pi)\Big) + (\psi(0))^2\ {}_0^{RL}D^{0.4}\Big(m(t)\Big) \\ &= \psi^2(0)\Big(2m(t)g(t) + {}_0^{RL}D^{0.4}m(t)\Big) + 0.2\psi(0)m(t)\psi(-\sin^2(t)+1-\pi) \\ &\leq -0.2\psi^2(0)m(t) + 0.1m(t)(\psi^2(0)+\psi^2(-\sin^2(t)+1-\pi)) = 0, \end{aligned}$$

and

$${}_0^cD^{0.4}_{(39)}V(t,\psi(0),\psi;0,\phi(0)) = {}_0D^{0.4}_{(39)}V(t,\psi(0),\psi) - \frac{0.5\phi^2(0)}{t^{0.4}\Gamma(0.6)} \leq 0.$$

Therefore, the conditions of Theorem 3/ Theorem 4 are satisfied with $g(t,u) \equiv 0$, and according to Corollary 2, the zero solution of (39) is practically stable w.r.t. $(\lambda, \alpha_1^{-1}(\alpha_2(1))) = (1,\sqrt{1.8})$.

The graphs of the absolute values of the solutions x_1, x_2, x_3 of the IVP for FDDE (39), with initial functions $\phi_1(s) = \sin(s), \phi_2(s) = \cos(s)$, and $\phi_3(s) = 0.5\frac{s}{s-1}$, for $s \in [-\pi,0]$, are given in Figure 4.

Example 6. (*State dependent delay*). Consider the IVP for the scalar linear FrDDE with state dependent delay

$$\begin{aligned} &{}_0^CD_t^{0.4}x(t) = g(t)x(t) + 0.1x(t-\sin^2(x(t-\pi))+1-\pi) \text{ for } t>0, \\ &x(s) = \phi(s) \text{ for } s \in [-\pi,0], \end{aligned} \tag{40}$$

where $f(t,x,u) = g(t)x + 0.1u$,

$$g(t) = -0.5\frac{t+1}{0.1t+1.1}\ {}_0^{RL}D^{0.4}\frac{0.1t+1.1}{t+1} - 0.1, \quad t>0,$$

$x_t(s) \equiv x(t-\pi)$, and $\rho(t,u) = t-\sin^2(u)+1-\pi \in [t-\pi,t]$, for $t \geq 0$, i.e., the condition (A2) is satisfied.

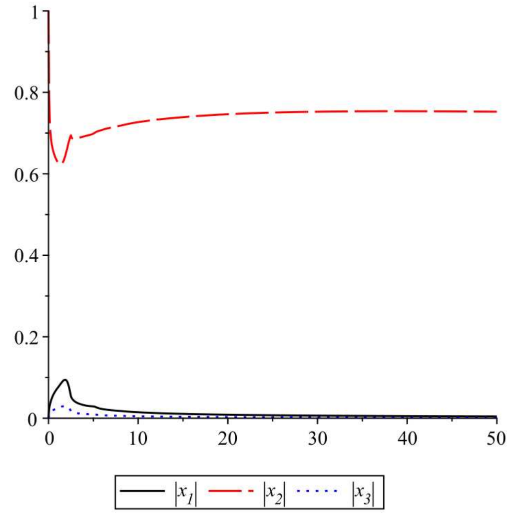

Figure 4: Graph of the the absolute values of the solutions x_1, x_2, x_3.

Consider the Lyapunov function

$$V(t,x) = \frac{0.1t+1.1}{t+1}x^2.$$

Then, condition 2(i) of Theorem 3 is satisfied, with $t_0 = 0$, $\alpha_1(u) = 0.1u^2$, $\alpha_2(u) = 1.1u$, and $\lambda = 1$, because $\frac{0.1t+1.1}{t+1} \in [0.1, 1.1]$.

Let $\psi \in C([-\pi, 0], \mathbb{R})$ be such that, for any $t > 0$, the inequality

$$V(t,\psi(0)) = \frac{0.1t+1.1}{t+1}\psi^2(0) \geq \frac{0.1(t+s)+1.1}{t+s+1}\psi^2(s) = V(t+s,\psi(s))$$

holds, for $s \in [-\pi, 0)$. Similar to Example 2 and Eq. (11), applying $\rho(t,\psi_0) - t \in [-\pi, 0]$ and

$$\frac{0.1(t+s)+1.1}{t+s+1} \geq \frac{0.1t+1.1}{t+1}, \ s \in [-r, 0],$$

we obtain

$$\begin{aligned}
&{}_0D^{0.4}_{(40)}V(t,\psi(0),\psi) = 2\psi(0)\ \frac{0.1t+1.1}{t+1}\Big(g(t)\psi(0) + 0.1\psi_{\rho(t,\psi_0)-t}\Big)\\
&= 2\psi^2(0)\ \frac{0.1t+1.1}{t+1}g(t) + 0.2\psi(0)\ \frac{0.1t+1.1}{t+1}\psi_{\rho(t,\psi_0)-t}\\
&\quad + (\psi(0))^2\ {}^{RL}_{0}D^{0.4}\frac{0.1t+1.1}{t+1}\\
&= 0.2\frac{0.1t+1.1}{t+1}\psi(0)\ \psi_{\rho(t,\psi_0)-t} - 0.2\psi^2(0)\ \frac{0.1t+1.1}{t+1}\\
&\leq 0.1\frac{0.1t+1.1}{t+1}\psi^2(0) + 0.1\frac{0.1t+1.1}{t+1}\psi^2_{\rho(t,\psi_0)-t} - 0.2\psi^2(0)\ \frac{1.1t+0.01}{t+0.1}\\
&\leq 0.1\frac{1}{t+0.1}\psi^2(0) + 0.1\frac{0.1(t+s)+1.1}{t+s+1}\psi^2_{\rho(t,\psi_0)-t} - 0.2\psi^2(0)\ \frac{1}{t+0.1} \leq 0.
\end{aligned} \tag{41}$$

Therefore, conditions of Theorem 4 are satisfied and, according to Corollary 2, the zero solution of (39) is practically stable w.r.t. $(1, \sqrt{11})$.

Remark 17. The above examples illustrate the importance of practically applicable sufficient conditions to study stability properties of Caputo fractional derivatives with delays, especially in the case of state dependent delays.

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