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ON THE APPROXIMATION OF THE GENERALIZED CUT FUNCTION OF DEGREE $p+1$ BY SMOOTH HYPER–LOG–LOGISTIC FUNCTION

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ABSTRACT: We introduce a modification of the familiar cut function by replacing the linear part in its definition by a polynomial of degree $p+1$ obtaining thus a sigmoid function called *generalized cut function of degree* $p+1$ *(GCFP)*. We then study the uniform approximation of the (GCFP) by smooth sigmoid functions such as the hyper–log–logistic and the shifted hyper–log–logistic functions. The limiting case of the interval-valued Heaviside step function is also discussed which imposes the use of Hausdorff metric. Numerical examples are presented using *CAS MATHEMATICA*.

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Key Words: sigmoid functions, cut function, generalized cut function of degree $p+1$, step function, hyper–log–logistic function, shifted hyper–log–logistic function, uniform and Hausdorff approximation

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1. INTRODUCTION

In this paper we introduce a modification of the familiar cut function by replacing the linear part in its definition by a polynomial of degree $p+1$ obtaining thus a differentiable sigmoid function called *generalized cut function of degree* $p+1$ *(GCFP)*. We then discuss some computational, modelling and approximation issues related to several classes of sigmoid functions. Sigmoid functions find numerous applications in various fields related to life sciences, chemistry, physics, artificial intelligence, etc. In fields such as signal processes, pattern recognition, machine learning, artificial neural networks, sigmoid functions are also known as "activation functions". A practically important class of sigmoid functions is the class of cut functions including the Heaviside step function as a limiting case. Cut functions are continuous but they are not differentiable at the two endpoints of the interval where they increase; step functions are not continuous but they are Hausdorff continuous (H-continuous). Section 2 contains preliminary definitions and motivations. In Section 3 we study the uniform and Hausdorff approximation [10] of the (GCFP) by hyper–log–logistic functions. We find an expression for the error of the best uniform approximation. Numerical examples are presented throughout the paper using the computer algebra system *MATHEMATICA*.

2. PRELIMINARIES

2.1. SIGMOID FUNCTIONS

In this work we consider *sigmoid functions* of a single variable defined on the real line, that is functions of the form $\mathbb{R} \longrightarrow \mathbb{R}$. Sigmoid functions can be defined as bounded monotone non-decreasing functions on $\mathbb{R}$. One usually makes use of normalized sigmoid functions defined as monotone non-decreasing functions $s(t), t \in \mathbb{R}$, such that $\lim s(t)_{t\to-\infty} = 0$ and $\lim s(t)_{t\to\infty} = 1$ (in some applications the left asymptote is assumed to be -1: $\lim s(t)_{t\to-\infty} = -1$).

In the fields of neural networks and machine learning sigmoid-like functions of many variables are used, familiar under the name *activation functions*.

2.2. THE CUT AND THE STEP FUNCTIONS

The cut function is the simplest piece-wise linear sigmoid function. Let $\Delta = [\gamma - \delta, \gamma + \delta]$ be an interval on the real line $\mathbb{R}$ with centre $\gamma \in \mathbb{R}$ and radius $\delta \in \mathbb{R}$. A cut function is defined as follows:

Definition 1. *The cut function $c_{\gamma,\delta}$ is defined for $t \in \mathbb{R}$ by*

$$c_{\gamma,\delta}(t) = \begin{cases} 0, & \text{if} \quad t < \gamma - \delta, \\ \dfrac{t - \gamma + \delta}{2\delta}, & \text{if} \quad |t - \gamma| < \delta, \\ 1, & \text{if} \quad t > \gamma + \delta. \end{cases} \tag{1}$$

Note that the slope of function $c_{\gamma,\delta}(t)$ on the interval Δ is $1/(2\delta)$ (the slope is constant in the whole interval Δ).

Two special cases and a limiting case are of interest for our discussion in the sequel.

Special case 1. For $\gamma = 0$ we obtain the special cut function on the interval $\Delta = [-\delta, \delta]$:

$$c_{0,\delta}(t) = \begin{cases} 0, & \text{if} \quad t < -\delta, \\ \dfrac{t+\delta}{2\delta}, & \text{if} \quad -\delta \leq t \leq \delta, \\ 1, & \text{if} \quad \delta < t. \end{cases} \tag{2}$$

Special case 2. For $\gamma = \delta$ we obtain the special cut function on the interval $\Delta = [0, 2\delta]$:

$$c_{\delta,\delta}(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ \dfrac{t}{2\delta}, & \text{if} \quad 0 \leq t \leq 2\delta, \\ 1, & \text{if} \quad 2\delta < t. \end{cases} \tag{3}$$

A limiting case. If $\delta \to 0$, then $c_{\delta,\delta}$ tends (in Hausdorff sense) to the Heaviside step function

$$c_0 = c_{0,0}(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ [0,1], & \text{if} \quad t = 0, \\ 1, & \text{if} \quad t > 0, \end{cases} \tag{4}$$

which is an interval-valued function [2], [3], [7], [11].

To prove that (3) tends to (4) let h be the H-distance using a square (box) unit ball between the step function (4) and the cut function (3).

By the definition of H-distance h is the side of the unit square, hence we have $1 - c_{\delta,\delta}(h) = h$, that is $1 - h/(2\delta) = h$, implying

$$h = \frac{2\delta}{1+2\delta} = 2\delta + O(\delta^2).$$

For the sake of simplicity throughout the paper we shall work with the special cut function (3) instead of the more general (arbitrary shifted) cut function (1); this special choice will not lead to any loss of generality concerning the results obtained.

2.3. THE GENERALIZED CUT SIGMOID FUNCTION OF DEGREE $P + 1$.

The *generalized cut function of degree* $p+1$ (GCFP) is obtained by substituting the linear function in the definition of the cut function by a polynomial of degree $p+1$.

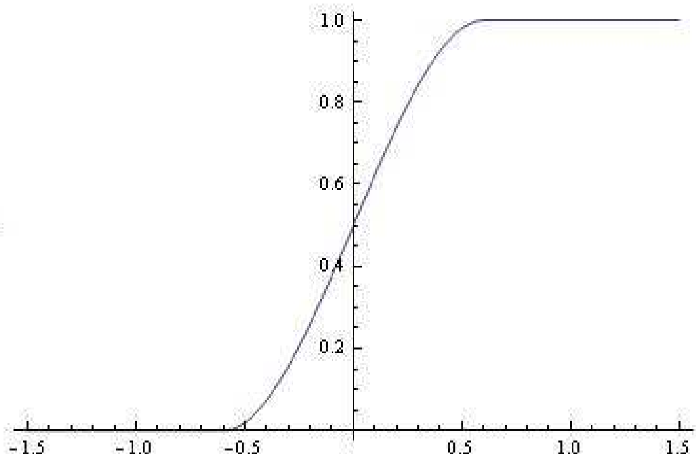

Figure 1: The (GCFP) function (7) with $\delta = 0.6$ and $p = 2$.

Let us define first a special case of the (GCFP). Consider the function

$$\overline{C}^*_{0,\delta}(t) = \begin{cases} -1, & \text{if} \quad t < -\delta, \\ kt\left((p+1)\delta^p - t^p\right), & \text{if} \quad -\delta \le t \le \delta, \\ 1, & \text{if} \quad \delta < t, \end{cases} \tag{5}$$

for some $k, \delta > 0, p$ when p is an even number. From $\overline{C}^{*\,\prime}_{0,\delta}(t) = k(p+1)(\delta^p - t^p)$ we obtain $\overline{C}^{*\,\prime}_{0,\delta}(t) \ge 0$, for $-\delta \le t \le \delta$, as well as $\overline{C}^{*\,\prime}_{0,\delta}(\pm\delta) = 0$.

Let us choose k so that $\overline{C}^*_{0,\delta}(\delta) = 1$. We have $\overline{C}^*_{0,\delta}(\delta) = kp\delta^{p+1} = 1$, hence $k = \dfrac{1}{p\delta^{p+1}}$.

Substituting k in (5) we obtain

$$\overline{C}^*_{0,\delta}(t) = \begin{cases} -1, & \text{if} \quad t < -\delta, \\ \dfrac{1}{p\delta^{p+1}} t\left((p+1)\delta^p - t^p\right), & \text{if} \quad -\delta \le t \le \delta, \\ 1, & \text{if} \quad \delta < t, \end{cases} \tag{6}$$

noticing that the slope of (6) at $t = 0$ is $\kappa = \dfrac{p+1}{p\delta}$.

Besides we have $\overline{C}^*_{0,\delta}(-\delta) = -1$ and (6) is differentiable at the points $\pm\delta$.

From presentation (6) we can pass to the normalized (GCFP) having as left asymp-

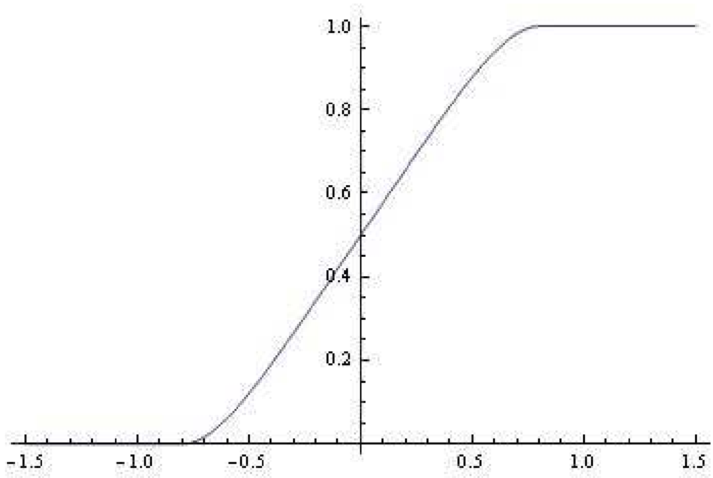

Figure 2: The (GCFP) function (7) with $\delta = 0.8$ and $p = 4$.

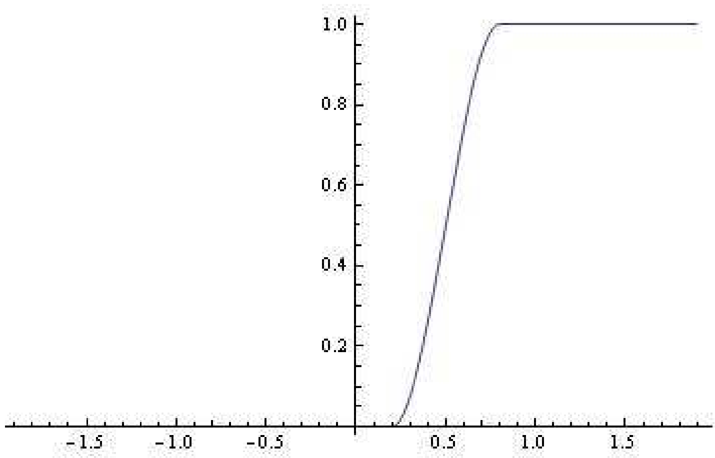

Figure 3: The (GCFP) function (8) with $\delta = 0.3$, $\gamma = 0.5$ and $p = 2$.

tote 0 instead of -1:

$$C^*_{0,\delta}(t) = \begin{cases} 0, & \text{if} \quad t < -\delta, \\ \dfrac{1}{2p\delta^{p+1}} t\left((p+1)\delta^p - t^p\right) + \frac{1}{2}, & \text{if} \quad -\delta \le t \le \delta, \\ 1, & \text{if} \quad \delta < t. \end{cases} \tag{7}$$

Note that the (steepest) slope of (7) at $t = 0$ is now $\kappa = \dfrac{p+1}{2p\delta}$.

Our last step is to generalize the function (7) up to a function $c_{\gamma,\delta}(t)$ shifted by γ.

This can be acheved by substituting t by $t - \gamma$ in (7) as follows:

$$C^*_{0,\delta}(t) = \begin{cases} 0, \text{ if } t < \gamma - \delta, \\ \dfrac{1}{2p\delta^{p+1}} (t-\gamma)\left((p+1)\delta^p - (t-\gamma)^p\right) + \frac{1}{2}, \\ \text{if } \gamma - \delta \le t \le \gamma + \delta, \\ 1, \text{ if } \gamma + \delta < t. \end{cases} \tag{8}$$

Definition 2. Define the logistic (Verhulst) function v on $\mathbb{R}$ as [12]

$$v_{\gamma,k}(t) = \frac{1}{1 + e^{-4k(t-\gamma)}}. \tag{9}$$

Note that the logistic function (9) has an inflection at its "centre" $(\gamma, 1/2)$ and its slope at γ is equal to k.

In [1] we prove the following proposition

Proposition 3. *The function $v_k(t)$ defined by (9) with $k = \frac{p+1}{2p\delta}$: i) is the logistic function of best uniform one-sided approximation to function $C^*_{0,\delta}(t)$ defined by (7); ii) approximates the (GSFP) function $C^*_{0,\delta}(t)$ in uniform metric with an error*

$$\rho = \rho(C^*, v) = \frac{1}{1 + e^{\frac{2(p+1)}{p}}}. \tag{10}$$

3. APPROXIMATION OF THE (GCFP) SIGMOID FUNCTION BY HYPER–LOG–LOGISTIC FUNCTION

Definition 4. Define the hyper-log–logistic function N on $\mathbb{R}$ as:

$$N_{\gamma,\beta,k}(t) = 1 - \frac{1}{1 + \left(1 + \frac{4k(t-\gamma)}{\beta}\right)^{\beta}}. \tag{11}$$

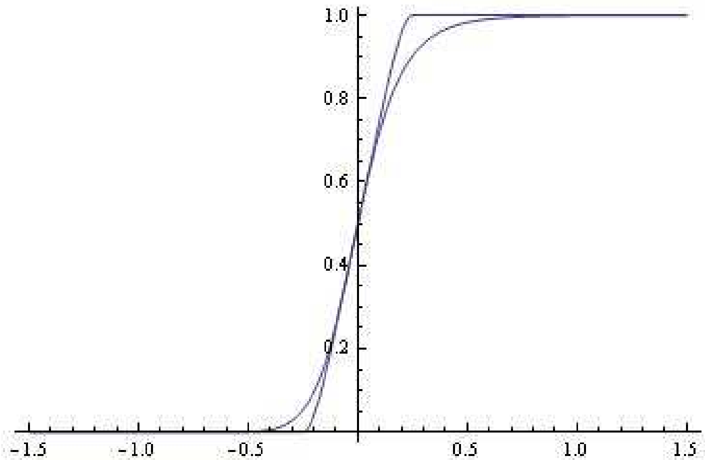

Figure 4: The approximation of the (GCFP) function (7) by hyper–log–logistic function with $\delta = 0.25$, $\beta = 10$ and $p = 4$.

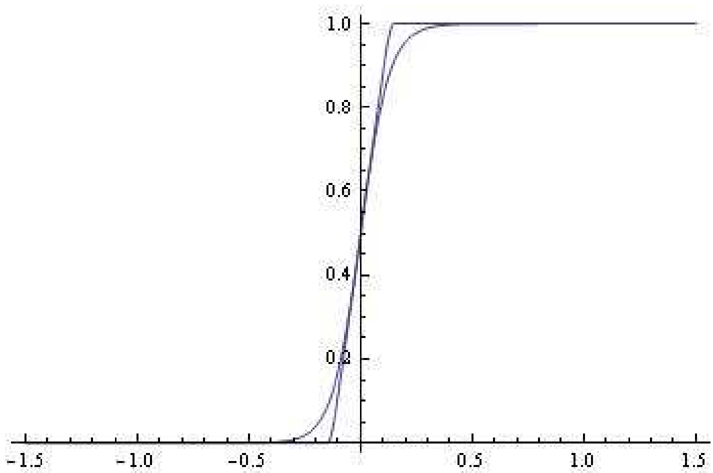

Figure 5: The approximation of the (GCFP) function (7) by hyper–log–logistic function with $\delta = 0.15$, $\beta = 100$ and $p = 6$.

Note that the logistic function (11) has an inflection at its "centre" $(\gamma, 1/2)$ and its slope at γ is equal to k.

Proposition 5. *The function $N_{\gamma,\beta,k}(t)$ defined by (11) with $k = \frac{p+1}{2p\delta}$: i) is the hyper–log–logistic function of best uniform one-sided approximation to function $C^*_{0,\delta}(t)$ defined by (7); ii) approximates the (GSFP) function $C^*_{0,\delta}(t)$ in uniform metric with an error*

$$\rho_1 = \rho_1(C^*, N) = 1 - \frac{1}{1 + \left(1 - \frac{2(p+1)}{p\beta}\right)^{\beta}}. \tag{12}$$

Proof. Let us choose k so that the slope of (11) is $k = \frac{p+1}{2p\delta}$.

Then, noticing that the largest uniform distance between the (GCFP) and hyper–log–logistic functions is achieved at the endpoints of the underlying interval $[-\delta, \delta]$, we have:

$$\rho = N_{0,\beta,k}(-\delta) - C^*_{0,\delta}(-\delta) = \tag{13}$$

$$1 - \frac{1}{1 + \left(1 - \frac{4k\delta}{\beta}\right)^{\beta}} = 1 - \frac{1}{1 + \left(1 - \frac{2(p+1)}{p\beta}\right)^{\beta}} = B(p, \beta).$$

This completes the proof of the proposition.

Some computational examples using relation (13) for various p and β are presented in Table 1.

p	β	$B(p,\beta)$ from (13)
6	10	0.0655569
6	100	0.0861965
6	10000	0.0883777
6	100000	0.0883975
6	1000000	0.0883995
10	100000	0.0997483
10	1000000	0.0997503

Table 1: The function $B(p, \beta)$ computed by (13) for various p and β.

Evidently

$$\lim_{\beta\to\infty} B(p, \beta) = 1 - \frac{1}{1 + e^{-\frac{2(p+1)}{p}}} = \frac{1}{1 + e^{\frac{2(p+1)}{p}}}$$

and we have the result from first Proposition.

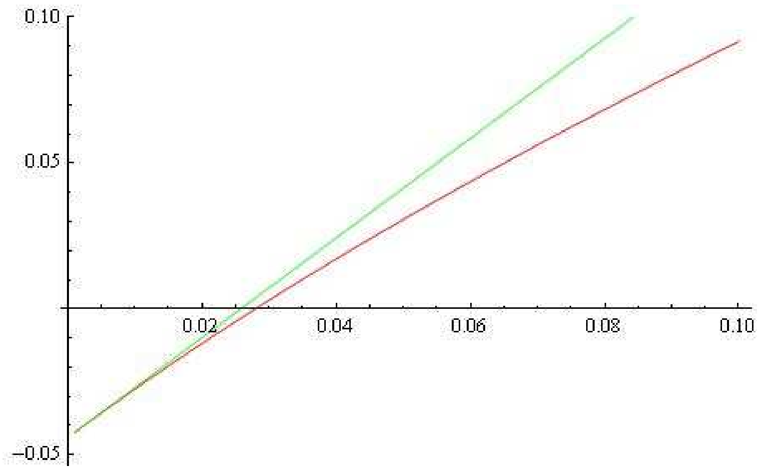

Figure 6: The functions $F(h)$ and $G(h)$ for $k = 4$, $\beta = 60$, $p = 2$.

Using $\delta = \frac{p+1}{2pk}$ we can write $\delta + h = \frac{p+1+2pkh}{2pk}$, resp.:

$$N(-\delta - h) = 1 - \frac{1}{1 + \left(1 - \frac{2}{p\beta}(p + 1 + 2pkh)\right)^{\beta}}. \tag{14}$$

The H-distance h using square unit ball (with a side h) satisfies the relation

$$N(-\delta - h) = h, \tag{15}$$

Let

$$p_1 = -1 + \frac{1}{1 + \left(1 - \frac{2}{\beta} - \frac{2}{p\beta}\right)^{\beta}};$$

$$q_1 = 1 + \frac{4k\left(1 - \frac{2}{\beta} - \frac{2}{p\beta}\right)^{\beta-1}}{\left(1 + \left(1 - \frac{2}{\beta} - \frac{2}{p\beta}\right)^{\beta}\right)^{2}};$$

$$r = -2.1\frac{q_1}{p_1};\ \ (p_1 < 0; q_1 > 0; r > 0).$$

Proposition 6. *For the H-distance $h = h(k, p, \beta)$ between the generalized cut and the hyper–log–logistic functions the following holds for $r > e^{2.1}$:*

$$h_1 = \frac{1}{r} < h(k, p, \beta) < \frac{\ln r}{r} = h_2. \tag{16}$$

Proof. From (15) we have

$$1 - h = \frac{1}{1 + \left(1 - \frac{2}{p\beta}(p + 1 + 2pkh)\right)^{\beta}}$$

Let us examine the function

$$F(h) = \frac{1}{1 + \left(1 - \frac{2}{p\beta}(p + 1 + 2pkh)\right)^{\beta}} - 1 + h.$$

From $F'(h) > 0$ we conclude that function F is strictly monotone increasing.

Consider function

$$G(h) = p_1 + q_1 h.$$

using the Taylor expansion $G(h) - F(h) = O(h^2)$.

Hence $G(h)$ approximates $F(h)$ with $h \to 0$ as $O(h^2)$ (see, Fig. 6).

In addition $G'(h) > 0$, hence function G is monotone increasing.

Further, for $r > e^{2.1}$

$$G\left(\frac{1}{r}\right) < 0, \quad G\left(\frac{\ln r}{r}\right) > 0.$$

This completes the proof of the proposition.

Example. For $k = 4$, $\beta = 60$, $p = 2$ from nonlinear equation (15) we find for the H–distance $h = 0.0279277$. From the two–sided bounds (16) we have $h_1 = 0.0122709$ and $h_2 = 0.0539983$.

For other results, see [9], [8], [4], [6], [5], [13]-[44].

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