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S^2_γ-WPAA FUNCTIONS AND APPLICATIONS TO STOCHASTIC NEUTRAL FUNCTIONAL EQUATIONS WITH INFINITE DELAY

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ABSTRACT: In this paper, we introduce the concept of S^2_γ-weighted pseudo almost automorphy. And then, we investigate some basic properties such as completeness of spaces, ergodic and composition theorems of such stochastic processes. Finally, by virtue of theories of evolution systems, fading phase spaces for infinite delay and the stochastic analysis techniques, we apply the results obtained to consider the existence and uniqueness results of weighted pseudo almost automorphic solutions in distribution to a class of nonautonomous stochastic neutral functional equations with infinite delay under S^2_γ-weighted pseudo almost automorphic coefficients in real separable Hilbert spaces.

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1. INTRODUCTION

The concept of almost automorphy is an important generalization of the classical almost periodicity. It was introduced by S. Bochner [4, 5], for more details about this topics we refer the reader to [9, 22, 23]. In recent years, the existence of almost periodic and almost automorphic solutions on different kinds of deterministic differential equations have been considerably investigated in lots of publications [1, 3, 11, 12, 21, 20, 24, 34, 35, 36] because of their significance and applications in physics, mechanics and mathematical biology. Recently, Diagana [10] presented the notion of S^p_γ-pseudo almost automorphy(or generalized Stepanov-like pseudo almost automorphy), which generalizes the well-known concept of S^p-pseudo almost automorphy.

Stochastic differential equations arise in the mathematical modeling of many phenomena in real world problems (see [2, 18, 19, 25, 26, 27, 28, 30, 31, 33, 37]). More recently, Fu and Liu [14] generalized the almost automorphic theory from the deterministic version to the stochastic one and studied the square-mean almost automorphic mild solution for the stochastic differential equations. Chen and Lin [7, 8] investigated the existence, uniqueness and stability of the square-mean (weighted) pseudo almost automorphic solutions for some stochastic differential equations via the Banach fixed point theorem and the stochastic analysis techniques. Chang et al [6, 32] investigated the existence of S^2-almost automorphic and S^2-weighted pseudo almost automorphic mild solutions for a stochastic differential equation in a real separable Hilbert space with the help of composition theorems together with fixed point theorems. As far as we know, however, there have been very few applicable results on S^2_γ-weighted pseudo almost automorphy for stochastic processes compared with the concept in [10] in deterministic sense.

From above mentioned works, in this paper we introduce a concept of S^2_γ-weighted pseudo almost automorphy for stochastic processes and investigate some basic properties such as completeness of spaces, ergodic and composition theorems of S^2_γ-weighted pseudo almost automorphic stochastic processes. Finally, by virtue of theories of evolution systems, fading phase spaces for infinite delay and the stochastic analysis techniques, we establish the existence and uniqueness results of weighted pseudo almost automorphic solutions in distribution to a class of non-autonomous stochastic neutral differential equations with infinite delay in the abstract form

$$d[u(t) + f(t, u_t)] = [A(t)u(t) + g(t, u_t)]dt + \sigma(t, u_t)dW(t), t \in \mathbb{R}, \tag{1.1}$$

where $A(t) : D(A(t)) \subset L^2(P, \mathbb{H}) \to L^2(P, \mathbb{H})$ is a family of densely defined closed linear operators satisfying the Acquistapace-Terrani conditions, the coefficients $f, g, \sigma : \mathbb{R} \times \mathcal{B} \to L^2(P, \mathbb{H})$ are appropriate functions, $\mathcal{B}$ is a uniform fading memory phase space defined in the next section. Also, the history $u_t : (-\infty, 0] \to L^2(P, \mathbb{H})$, defined by $u_t(\theta) = u(t + \theta)$ for each $\theta \in (-\infty, 0]$. Further, $W(t)$ is a two-sided standard one dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ where $\mathcal{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$.

The rest of this paper is organized as follows. In Section 2, we present some basic definitions, lemmas and preliminary facts. In Section 3, we introduce the concept of S^2_γ-weighted pseudo almost automorphy for stochastic processes and prove some fundamental properties of such stochastic processes. In Section 4, we investigate the existence and uniqueness results of weighted pseudo almost automorphic solutions in distribution to the Eq (1.1) with S^2_γ-weighted pseudo almost automorphic coefficients.

2. PRELIMINARIES

In this section, we fix some basic definitions, lemmas and preliminary facts which will be used in the sequel. Throughout the paper, we assume that $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ are two real and separable Hilbert spaces, $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is supposed to be a probability space and $L^2(P,\mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that $E\|x\|^2 = \int_\Omega \|x\|^2 dP < \infty$. Note that $L^2(P,\mathbb{H})$ is a Hilbert space when it is equipped with the norm $\|x\|_2 = (E\|x\|^2)^{\frac{1}{2}}$. Let $\mathcal{L}(\mathbb{K},\mathbb{H})$ be the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$ and this is denoted by $\mathcal{L}(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$.

We let $C(\mathbb{R}, L^2(P,\mathbb{H}))$ (respectively, $C(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(P,\mathbb{H})))$ denote the collection of all continuous stochastic processes from $\mathbb{R}$ into $L^2(P,\mathbb{H})$ (respectively, the collection of all jointly continuous stochastic processes from $\mathbb{R}\times L^2(P,\mathbb{H})$ into $L^2(P,\mathbb{H})$). Furthermore, $BC(\mathbb{R}, L^2(P,\mathbb{H}))$ (respectively, $BC(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ stands for the class of all bounded continuous stochastic processes from $\mathbb{R}$ into $L^2(P,\mathbb{H})$ (respectively, the class of all jointly bounded continuous stochastic processes from $\mathbb{R}\times L^2(P,\mathbb{H})$ into $L^2(P,\mathbb{H})$. Note that $BC(\mathbb{R}, L^2(P,\mathbb{H}))$ is a Banach space with the sup norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{\frac{1}{2}}.$$

A Brownian motion plays a key role in the construction of stochastic integrals.

Definition 2.1. A (standard one-dimensional) *Brownian motion* is a continuous adapted real-valued process $(W(t),\ t \geq 0)$ such that

(i) $W(0) = 0$;

(ii) $W(t) - W(s)$ is independent of $\mathscr{F}_s$ for all $0 \leq s < t$;

(iii) $W(t) - W(s)$ is $N(0, t-s)$-distributed for all $0 \leq s \leq t$.

Note that the Brownian motion W has the following properties:

(a) W has independent increments, that is, for $t_1 < t_2 < \cdots < t_n$, $W(t_1) - W(0)$, $W(t_2) - W(t_1), \ldots W(t_n) - W(t_{n-1})$ are independent random variables;

(b) W has stationary increments, that is, $W(t+s)-W(t)$ has the same distribution as $W(s)-W(0)$.

Definition 2.2. [25] Let $\mathcal{V}=\mathcal{V}(S,T)$ be the class of functions $f(t,\omega):[0,\infty)\times\Omega\to\mathbb{R}$ such that

(i) $(t,\omega)\to f(t,\omega)$ is $\mathscr{B}\times\mathscr{F}$-measurable, where $\mathscr{B}$ denotes the Borel σ-algebra on $[0,\infty)$.

(ii) $f(t,\omega)$ is $\mathscr{F}_t$-adapted.

(iii) $E\left[\int_s^T f(t,\omega)^2dt\right]<\infty$.

Definition 2.3. [25] Let $f\in\mathcal{V}(S,T)$. Then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t,\omega)dW_t(\omega)=\lim_{n\to\infty}\int_S^T f_n(t,\omega)dW_t(\omega)\quad(\text{limit in } L^2(\mathbb{P})),$$

where f_n is a sequence of elementary functions such that

$$E\left[\int_S^T (f(t,\omega)-f_n(t,\omega))^2dt\right]\to 0\ \text{ as } n\to\infty,$$

and W_t is one-dimensional Brownian motion. Moreover, we have the following Itô isometry, for all $f\in\mathcal{V}(S,T)$,

$$E\left[\left(\int_S^T f(t,\omega)dW_t(\omega)\right)^2\right]=E\left[\int_S^T f^2(t,\omega)dt\right].$$

Definition 2.4. [14] A stochastic process $x:\mathbb{R}\to L^2(P,\mathbb{H})$ is said to be stochastically continuous if

$$\lim_{t\to s}E\|x(t)-x(s)\|^2=0.$$

Definition 2.5. [14] A stochastically continuous stochastic process $x:\mathbb{R}\to L^2(P,\mathbb{H})$ is said to be square-mean almost automorphic if for every sequence of real numbers there exists a subsequence $\{s_n\}$ and a stochastic process $y:\mathbb{R}\to L^2(P,\mathbb{H})$ such that

$$\lim_{n\to\infty}E\|x(t+s_n)-y(t)\|^2=0\ \text{ and }\ \lim_{n\to\infty}E\|y(t-s_n)-x(t)\|^2=0,$$

hold for each $t\in\mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x:\mathbb{R}\to L^2(P,\mathbb{H})$ is denoted by $AA(\mathbb{R},L^2(P,\mathbb{H}))$.

Definition 2.6. [14] A stochastic process $f:\mathbb{R}\times L^2(P,\mathbb{H})\to L^2(P,\mathbb{H})$, $(t,x)\to f(t,x)$, which is jointly continuous, is said to be square-mean almost automorphic in

$t \in \mathbb{R}$ for each $x \in L^2(P, \mathbb{H})$ if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and a stochastic process $\widetilde{f} : \mathbb{R} \times L^2(P, \mathbb{H}) \to L^2(P, \mathbb{H})$ such that

$$\lim_{n\to\infty} E\|f(t+s_n, x) - \widetilde{f}(t, x)\|^2 = 0, \lim_{n\to\infty} E\|\widetilde{f}(t-s_n, x) - f(t, x)\|^2 = 0,$$

for each $t \in \mathbb{R}$ and each $x \in L^2(P, \mathbb{H})$. The collection of such process is denoted by $AA(\mathbb{R} \times L^2(P, \mathbb{H}), L^2(P, \mathbb{H}))$.

Lemma 2.1. [14] $(AA(\mathbb{R}, L^2(P, \mathbb{H})), \|\cdot\|_\infty)$ is a Banach space when it is equipped with the norm $\|\cdot\|_\infty$, for $x \in AA(\mathbb{R}, L^2(P, \mathbb{H}))$.

Let $\mathbb{V}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $q > 0$ and for each $\rho \in \mathbb{V}$, we set $m(q, \rho) := \int_{-q}^{q} \rho(t)dt$.

Thus the space of weights $\mathbb{V}_\infty$ is defined by

$$\mathbb{V}_\infty := \{\rho \in \mathbb{V} : \lim_{r\to\infty} m(q, \rho) = \infty\}.$$

Now for $\rho \in \mathbb{V}_\infty$, we define a class of stochastic process

$$\begin{aligned} &PAA_0(\mathbb{R}, \rho) \\ &:= \left\{ f \in BC(\mathbb{R}, L^2(P, \mathbb{H})) : \lim_{q\to\infty} \frac{1}{m(q, \rho)} \int_{-q}^{q} E\|f(t)\|^2 \rho(t)dt = 0 \right\}, \end{aligned}$$

$$\begin{aligned} &PAA_0(\mathbb{R} \times L^2(P, \mathbb{H}), \rho) \\ := \quad & \Big\{ f \in C(\mathbb{R} \times L^2(P, \mathbb{H}), L^2(P, \mathbb{H})) : f(\cdot, x) \text{ is bounded for each} \\ & x \in L^2(P, \mathbb{H}) \text{ and } \lim_{q\to\infty} \frac{1}{m(q, \rho)} \int_{-q}^{q} E\|f(t, x)\|^2 \rho(t)dt = 0 \Big\}. \end{aligned}$$

To study issues related to delay terms, we consider the new space of functions defined for each $p > 0$ by

$$\begin{aligned} &PAA_0(\mathbb{R}, \rho, p) \\ := \quad & \left\{ f \in PAA_0(\mathbb{R}, \rho) : \lim_{q\to\infty} \frac{1}{m(q, \rho)} \int_{-q}^{q} \sup_{\theta\in[t-p,t]} (E\|f(\theta)\|^2) \rho(t)dt = 0 \right\}, \end{aligned}$$

$$\begin{aligned} &PAA_0(\mathbb{R} \times L^2(P, \mathbb{H}), \rho, p) \\ := \quad & \Big\{ f \in PAA_0(\mathbb{R} \times L^2(P, \mathbb{H}), \rho) : f(\cdot, x) \text{ is bounded for each} \\ & x \in L^2(P, \mathbb{H}) \text{ and } \lim_{q\to\infty} \frac{1}{m(q, \rho)} \int_{-q}^{q} \sup_{\theta\in[t-p,t]} (E\|f(\theta, x)\|^2) \rho(t)dt = 0 \Big\}. \end{aligned}$$

In view of the previous definitions it is clear that $PAA_0(\mathbb{R}, \rho, p)$ and $PAA_0(\mathbb{R} \times L^2(P, \mathbb{H}), \rho, p)$ are continuously embedded and closed in the $PAA_0(\mathbb{R}, \rho)$ and $PAA_0(\mathbb{R} \times L^2(P, \mathbb{H}), \rho)$, respectively.

Definition 2.7. [8] Let $\rho \in \mathbb{V}_\infty$. A stochastically continuous process $f : \mathbb{R} \to L^2(P, \mathbb{H})$ is said to be square-mean weighted pseudo almost automorphic provided that it can be decomposed as $f = h + \varphi$, where $h \in AA(\mathbb{R}, L^2(P, \mathbb{H}))$ and $\varphi \in PAA_0(\mathbb{R}, \rho)$. The collection of all such processes is denoted by $WPAA(\mathbb{R}, L^2(P, \mathbb{H}))$.

Definition 2.8. [8] Let $\rho \in \mathbb{V}_\infty$. A stochastically continuous process $f : \mathbb{R} \times L^2(P, \mathbb{H}) \to L^2(P, \mathbb{H})$ is said to be square-mean weighted pseudo almost automorphic in t for any $x \in L^2(P, \mathbb{H})$ provided that it is decomposed as $f = h + \varphi$, where $h \in AA(\mathbb{R} \times L^2(P, \mathbb{H}), L^2(P, \mathbb{H}))$ and $\varphi \in PAA_0(\mathbb{R} \times L^2(P, \mathbb{H}), \rho)$. Also denote by $WPAA(\mathbb{R} \times L^2(P, \mathbb{H}), L^2(P, \mathbb{H}))$ the set of all such stochastically continuous processes.

Lemma 2.2. [8] For some $\rho \in \mathbb{V}_\infty$, $PAA_0(\mathbb{R}, \rho)$ may be translation invariant. In fact, if ρ satisfies conditions:

$$\lim_{t\to\infty} \sup \frac{\rho(t+\tau)}{\rho(t)} < \infty \quad \text{and} \quad \lim_{r\to\infty} \sup \frac{m(r+\tau, \rho)}{m(r, \rho)} < \infty,$$

for every $\tau \in \mathbb{R}$, one can validate that $PAA_0(\mathbb{R}, \rho)$ may be translation invariant.

Lemma 2.3. [8] $WPAA(\mathbb{R}, L^2(P, \mathbb{H}))$ equipped with the norm $\|\cdot\|_\infty$ becomes a Banach space if $PAA_0(\mathbb{R}, \rho)$ is translation invariant.

Lemma 2.4. [8] Suppose ρ satisfies the conditions in Lemma 2.2, $f(t, x) \in WPAA(\mathbb{R} \times L^2(P, \mathbb{H}), L^2(P, \mathbb{H}))$ and there exists a number $L > 0$ such that for any $x, y \in L^2(P, \mathbb{H})$,

$$E\|f(t, x) - f(t, y)\|^2 \le LE\|x - y\|^2, \quad t \in \mathbb{R}.$$

Then for any $x(\cdot) \in WPAA(\mathbb{R}, L^2(P, \mathbb{H}))$, then $f(\cdot, x(\cdot)) \in WPAA(\mathbb{R}, L^2(P, \mathbb{H}))$.

From [13] and [19], we define $\mathcal{P}(\mathbb{H})$ the space of all Borel probability measures on $\mathbb{H}$ with the β metric:

$$\beta(\mu, \eta) := \sup\left\{\left|\int f d\mu - \int f d\eta\right| : \|f\|_{BL} \le 1\right\}, \mu, \eta \in \mathcal{P}(\mathbb{H}),$$

where f are Lipschitz continuous real-valued functions on $\mathbb{H}$ with

$$\|f\|_{BL} = \|f\|_L + \|f\|_\infty, \|f\|_L = \sup_{x\neq y} \frac{|f(x) - f(y)|}{\|x - y\|}, \|f\|_\infty = \sup_{x\in\mathbb{H}} |f(x)|.$$

Definition 2.9. [19] An $\mathbb{H}$-valued stochastic process $u(t)$ is said to be almost automorphic in distribution if its law $\mu(t)$ is a $\mathcal{P}(\mathbb{H})$- valued almost automorphic mapping, i.e. for every sequence of real numbers $\{s_n'\}$, there exist a subsequence $\{s_n\}$ and a $\mathcal{P}(\mathbb{H})$- valued mapping $\widetilde{\mu}(t)$ such that

$$\lim_{n\to\infty} \beta(\mu(t + s_n), \widetilde{\mu}(t))) = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta(\widetilde{\mu}(t - s_n), \mu(t))) = 0,$$

hold for each $t \in \mathbb{R}$.

Definition 2.10. [17] An $\mathbb{H}$-valued stochastic process $f(t)$ is said to be weighted pseudo almost automorphic in distribution with respect to $\rho \in \mathbb{V}_\infty$, provided that it can be decomposed as $f = h + \varphi$, where h is almost automorphic in distribution and $\varphi \in PAA_0(\mathbb{R}, \rho)$.

We now introduce positively bi-almost automorphic functions. For that, let $\mathbb{T}$ be the set defined by:

$$\mathbb{T} := \{(t, s) \in \mathbb{R} \times \mathbb{R} : t \geq s)\}.$$

Definition 2.11. [10] A stochastically process $L : \mathbb{T} \to L^2(P, \mathbb{H})$ is called positively square-mean bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, we can extract a subsequence $(s_n)_{n\in\mathbb{N}}$ such that for some stochastic processes $H : \mathbb{T} \to L^2(P, \mathbb{H})$

$$\lim_{n\to\infty} E\|L(t + s_n, s + s_n) - H(t, s)\|^2 = 0$$

$$\lim_{n\to\infty} E\|H(t - s_n, s - s_n) - L(t, s)\|^2 = 0$$

for each $(t, s) \in \mathbb{T}$. The collection of such processes will be denoted by $bAA(\mathbb{T}, L^2(P, \mathbb{H}))$.

Definition 2.12. [2] The Bochner transform $x^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$ of a stochastic process $x : \mathbb{R} \to L^2(P, \mathbb{H})$ is defined by

$$x^b(t, s) := x(t + s).$$

Remark 2.1. (i) A function $\varphi(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, is the Bochner transform of a certain function f, $\varphi(t, s) = f^b(t, s)$, if and only if $\varphi(t + \tau, s - \tau) = \varphi(t, s)$ for all $t \in \mathbb{R}$, $s \in [0, 1]$ and $\tau \in [s - 1, s]$.

(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.13. [2] The Bochner transform $f^b(t, s, u)$, $t \in \mathbb{R}$, $s \in [0, 1]$, $u \in L^2(P, \mathbb{H})$ of a stochastic process $f : \mathbb{R} \times L^2(P, \mathbb{H}) \to L^2(P, \mathbb{H})$ is defined by

$$f^b(t, s, u) := f(t + s, u)$$

for each $u \in L^2(P, \mathbb{H})$.

Now, we recall the definition of fading memory space (phase space) $\mathcal{B}$ axiomatically presented in [15, 21]. Let $\mathcal{B}$ denote the vector space of function $x_t : (-\infty, 0] \to L^2(P, \mathbb{H})$, defined as $x_t(s) = x(t + s)$ for $s \in \mathbb{R}^-$, endowed with a seminorm denoted by $\|\cdot\|_{\mathcal{B}}$. A Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ which consists of such functions $\psi : (-\infty, 0] \to L^2(P, \mathbb{H})$, is called a fading memory space, if it satisfies the following axioms.
(ai) If $x : (-\infty, r + a) \to L^2(P, \mathbb{H})$ with $a > 0$, $r \in \mathbb{R}$, is continuous on $[r, r + a)$ and $x_r \in \mathcal{B}$, then for each $t \in [r, r + a)$ the following conditions hold:

(i) $x_t \in \mathcal{B}$,

(ii) $\|x(t)\| \leq L\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq \sup\{G(t-r)\|x(s)\| : r \leq s \leq t\} + N(t-r)\|x_r\|_{\mathcal{B}}$,

where $L > 0$ is a constant, and $G, N : [0, \infty) \to [1, \infty)$ are functions such that $G(\cdot)$ and $N(\cdot)$ are respectively continuous and locally bounded, and L, G, N are independent of $x(\cdot)$.

(aii) If $x(\cdot)$ is a function as in (ai), then x_t is a $\mathcal{B}$ valued continuous function on $[r, r+a)$.

(aiii) The space $\mathcal{B}$ is complete.

(aiv) If $(\phi^n)_{n\in\mathbb{N}}$ is a sequence of continuous functions with compact support defined from $(-\infty, 0]$ into $L^2(P, \mathbb{H})$, which converges to ϕ uniformly on compact subsets of $(-\infty, 0]$ and if $\{\phi^n\}$ is a cauchy sequence in $\mathcal{B}$, then $\phi \in \mathcal{B}$ and $\phi^n \to \phi$ in $\mathcal{B}$.

Definition 2.14. [21] Let $S(t) : \mathcal{B} \to \mathcal{B}$ be a C_0 semigroup defined by $S(t)\phi(\theta) = \phi(0)$ on $[-t, 0]$ and $S(t)\phi(\theta) = \phi(t+\theta)$ on $(-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory space if $\|S(t)\phi\|_{\mathcal{B}} \to 0$ as $t \to \infty$ for each $\phi \in \mathcal{B}$ with $\phi(0) = 0$.

Also, by axiom (aiv), there exists a constant $\Re > 0$ such that $\|\phi\|_{\mathcal{B}} \leq \Re \sup\limits_{\theta\leq 0}\|\phi(\theta)\|$ for every $\phi \in \mathcal{B}$ bounded continuous. Moreover, if $\mathcal{B}$ is a fading memory, we assume that $max\{G(t), N(t))\} \leq \mathcal{K}$ for $t \geq 0$. Further, it should be mentioned that the phase $\mathcal{B}$ is a uniform fading memory space if and only if axiom (aiv) holds, the function G is bounded and $\lim\limits_{t\to\infty} N(t) = 0$. For more details please refer the article [21].

Lemma 2.5. [18] Let $x : (-\infty, r+a) \to L^2(P, \mathbb{H})$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \phi \in L_0^2(\Omega, \mathcal{B})$, then

$$E\|x_s\|_{\mathcal{B}} \leq \mathcal{D}E\|\phi\|_{\mathcal{B}} + \Im \sup_{s\in\mathbb{R}} E\|x(s)\|,$$

where $\mathcal{D} = \sup\limits_{t\in\mathbb{R}}\{N(t)\}$ and $\Im = \sup\limits_{t\in\mathbb{R}}\{G(t)\}$.

3. GENERALIZED STEPANOV-LIKE WPAA STOCHASTIC PROCESSES

In this section, we shall introduce the concept of S^2_γ-weighted pseudo almost automorphy for stochastic processes and give some fundamental properties of such stochastic processes.

Let $\mathbb{U}$ denote the collection of all measurable (weighted) functions $\gamma : (0, \infty) \to (0, \infty)$ satisfying

$$\gamma_0 := \lim_{\varepsilon\to 0}\int_\varepsilon^1 \gamma(\sigma)d\sigma = \int_0^1 \gamma(\sigma)d\sigma < \infty. \tag{3.1}$$

Moreover, we let $\mathbb{U}_\infty$ be the collection of all functions $\gamma \in \mathbb{U}$, which are differentiable. Define the set of weights

$$\mathbb{U}_\infty^+ := \{\gamma \in \mathbb{U}_\infty : \frac{d\gamma}{dt} > 0 \text{ for all } t \in (0,\infty)\},$$

$$\mathbb{U}_\infty^- := \{\gamma \in \mathbb{U}_\infty : \frac{d\gamma}{dt} < 0 \text{ for all } t \in (0,\infty)\}.$$

In addition to the above, we define the set of weights

$$\mathbb{U}_B := \{\gamma \in \mathbb{U} : \sup_{t\in(0,\infty)} \gamma(t) < \infty\}.$$

Throughout the rest of the paper, we suppose that γ satisfies

$$\inf_{t\in(0,\infty)} \gamma(t) = m_0 > 0.$$

Definition 3.1. [10] Let $\mu, \nu \in \mathbb{U}_\infty$. One says that μ is equivalent to ν and denote it $\mu \prec \nu$, if $\frac{\mu}{\nu} \in \mathbb{U}_B$.

Remark 3.1. [10] Let $\mu, \nu, \gamma \in \mathbb{U}_\infty$. Note that $\mu \prec \mu$ (reflexivity). If $\mu \prec \nu$, then $\nu \prec \mu$ (symmetry). If $\mu \prec \nu$ and $\nu \prec \gamma$, then $\mu \prec \gamma$ (transitivity). Therefore, $\prec$ is a binary equivalence relation on $\mathbb{U}_\infty$.

We now introduce the space $BS^2_\gamma(L^2(P,\mathbb{H}))$ of all generalized Stepanov spaces as follows.

Definition 3.2. [10] Let $\gamma \in \mathbb{U}$. The space $BS^2_\gamma(L^2(P,\mathbb{H}))$ of all generalized Stepanov bounded stochastic processes consists of all γds-measurable stochastic processes $f : \mathbb{R} \to L^2(P,\mathbb{H})$ such that $f^b \in L^\infty(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds))$. This is a Banach space with the norm

$$\begin{aligned}\|f\|_{S^2_\gamma} : &= \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \gamma(s-t) E\|f(s)\|^2 ds\right)^{\frac{1}{2}} \\ &= \sup_{t\in\mathbb{R}} \left(\int_0^1 \gamma(s) E\|f(s+t)\|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

Definition 3.3. [10] Let $\gamma \in \mathbb{U}$. A stochastic process $f \in BS^2_\gamma(L^2(P,\mathbb{H}))$ is called S^2_γ-almost automorphic if $f^b \in AA(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds))$. In other words, a stochastic process $f \in L^2_{loc}(\mathbb{R}, \gamma ds)$ is said to be S^2_γ-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^2(0,1; L^2(P,\mathbb{H}), \gamma ds)$ is square-mean almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}$ there exists a subsequence $\{s_n\}$ and a stochastic process $g \in L^2_{loc}(\mathbb{R}, \gamma ds)$ such that

$$\int_t^{t+1} \gamma(s-t) E\|f(s+s_n) - g(s)\|^2 ds = \int_0^1 \gamma(s) E\|f(s+t+s_n) - g(s+t)\|^2 ds \to 0,$$

$$\int_t^{t+1} \gamma(s-t)E\|g(s-s_n)-f(s)\|^2 ds = \int_0^1 \gamma(s)E\|g(s+t-s_n)-f(s+t)\|^2 ds \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$. The collection of all such processes will be denoted by $AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$.

Definition 3.4. [10] Let $\gamma \in \mathbb{U}$. A process $F : \mathbb{R} \times L^2(P,\mathbb{H}) \to L^2(P,\mathbb{H})$, $(t,x) \to F(t,x)$ with $F(\cdot,x) \in L^2_{loc}(\mathbb{R},\gamma ds)$ for each $x \in L^2(P,\mathbb{H})$, is said to be S^2_γ-almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in L^2(P,\mathbb{H})$ if $t \to F(t,x)$ is S^2_γ-almost automorphic for each $x \in L^2(P,\mathbb{H})$. That means, for every sequence of real numbers $\{s'_n\}$ there exist a subsequence $\{s_n\}$ and a process $G(\cdot,x) \in L^2_{loc}(\mathbb{R},\gamma ds)$ such that

$$\int_t^{t+1} \gamma(s-t)E\|F(s+s_n,x)-G(s,x)\|^2 ds \to 0,$$

$$\int_t^{t+1} \gamma(s-t)E\|G(s-s_n,x)-F(s,x)\|^2 ds \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$ and for each $x \in L^2(P,\mathbb{H})$. We denote by $AS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,$
$\mathbb{H}))$ the set of all such processes.

Similarly, as in Ding et al [12], for each $K \subset L^2(P,\mathbb{H})$ compact subset, we denote by $AS^2_{\gamma,K}(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ the collection of all functions $f \in AS^2_{\gamma,K}(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ satisfying that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G : \mathbb{R} \times L^2(P,\mathbb{H}) \to L^2(P,\mathbb{H})$ with $G(\cdot,x) \in L^2_{loc}(\mathbb{R},\gamma ds)$ such that

$$\int_0^1 \gamma(s)\left(\sup_{x\in K} E\|F(s+t+s_n,x)-G(s+t,x)\|\right)^2 ds \to 0,$$

$$\int_0^1 \gamma(s)\left(\sup_{x\in K} E\|G(s+t-s_n,x)-F(s+t,x)\|\right)^2 ds \to 0,$$

as $n \to \infty$ for each $t \in \mathbb{R}$.

Lemma 3.1. [10] Let $f \in AS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ and suppose f is Lipschitz, that is, there exists $L > 0$ such that for all $x, y \in L^2(P,\mathbb{H})$ and $t \in \mathbb{R}$

$$E\|f(t,x)-f(t,y)\|^2 \le LE\|x-y\|^2 \tag{3.2}$$

Then for every $K \subset L^2(P,\mathbb{H})$ a compact subset, the process

$$f \in AS^2_{\gamma,K}(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H})).$$

Lemma 3.2. [10] Suppose $\varphi \in AS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ such that $K = \overline{\{\varphi(t) : t \in \mathbb{R}\}} \subset L^2(P,\mathbb{H}))$ is a compact subset. If the process

$$F \in AS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$$

and satisfies the Lipschitz condition (3.2), then it holds $t \to F(t,\varphi(t))$ belongs to $AS^2_\gamma(\mathbb{R}\times L^2(P,\mathbb{H}),L^2(P,\mathbb{H}))$.

Let $\gamma \in \mathbb{U}$ and $\rho \in \mathbb{V}_\infty$. We now introduce the new concept of S^2_γ-weighted pseudo almost automorphy for stochastic processes.

Definition 3.5. A stochastic process $f \in BS^2_\gamma(L^2(P,\mathbb{H}))$ is called S^2_γ-weighted pseudo almost automorphic if it can be expressed as $f = h+\varphi$, where $h \in AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$ and $\varphi^b \in PAA_0(\mathbb{R}, L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$. The collection of such processes will be denoted by $WPAAS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$.

Definition 3.6. A stochastic process $F : \mathbb{R}\times L^2(P,\mathbb{H}) \to L^2(P,\mathbb{H})$, $(t,x) \to F(t,x)$ with $F(\cdot,x) \in L^2_{loc}(\mathbb{R},\gamma ds)$ for each $x \in L^2(P,\mathbb{H})$, is said to be S^2_γ-weighted pseudo almost automorphic in $t\in\mathbb{R}$ uniformly in $x \in L^2(P,\mathbb{H})$ if it can be expressed as $F = H + \Phi$, where $H \in AS^2_\gamma(\mathbb{R}\times L^2(P,\mathbb{H}),L^2(P,\mathbb{H}))$ and $\Phi^b \in PAA_0(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$. The collection of such processes will be denoted by $WPAAS^2_\gamma(\mathbb{R}\times L^2(P,\mathbb{H}),L^2(P,\mathbb{H}))$.

Lemma 3.3. If $f \in WPAA(\mathbb{R},L^2(P,\mathbb{H}))$, then $f \in WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$. In other words, $WPAA(\mathbb{R},L^2(P,\mathbb{H})) \subset WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$.

Lemma 3.4. Let $\gamma,\nu \in \mathbb{U}$. If $\gamma \prec \nu$, then it holds $WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H})) = WPAAS^2_\nu(\mathbb{R},L^2(P,\mathbb{H}))$.

The proofs of the Lemmas 3.3 and 3.4 are straightforward and hence omitted.

Lemma 3.5. If $\phi^b(\cdot) \in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$ and ρ satisfies the conditions in Lemma 2.2, then for any $\tau\in\mathbb{R}$, it holds that

$$\phi^b(\cdot-\tau) \in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho).$$

Proof. Let $\phi^b(\cdot) \in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$, for the arbitrary $\tau \in \mathbb{R}$, one has

$$\begin{aligned}
&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_\theta^{\theta+1}\gamma(s-\theta)E\|\phi(s-\tau)\|^2ds\right)\right)\rho(t)dt\\
=\;&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_0^{1}\gamma(s)E\|\phi(s+\theta-\tau)\|^2ds\right)\right)\rho(t)dt\\
=\;&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p-\tau,t-\tau]}\left(\int_0^{1}\gamma(s)E\|\phi(s+\theta)\|^2ds\right)\right)\rho(t)dt\\
=\;&\frac{1}{m(q,\rho)}\int_{-q-\tau}^{q-\tau}\left(\sup_{\theta\in[t-p,t]}\left(\int_0^{1}\gamma(s)E\|\phi(s+\theta)\|^2ds\right)\right)\rho(t+\tau)dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{m(q,\rho)}\int_{-q+|\tau|}^{q-|\tau|}\left(\sup_{\theta\in[t-p,t]}\left(\int_0^1\gamma(s)E\|\phi(s+\theta)\|^2ds\right)\right)\rho(t+\tau)dt\\
&\leq \limsup_{t\to\infty}\frac{\rho(t+\tau)}{\rho(t)}\times\limsup_{r\to\infty}\frac{m(q+|\tau|,\rho)}{m(q,\rho)}\\
&\quad\times\frac{1}{m(q+|\tau|,\rho)}\int_{-q+|\tau|}^{q-|\tau|}\left(\sup_{\theta\in[t-p,t]}\left(\int_0^1\gamma(s)E\|\phi(s+\theta)\|^2ds\right)\right)\rho(t)dt,
\end{aligned}
$$

which implies that

$$
\lim_{q\to\infty}\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_\theta^{\theta+1}\gamma(s-\theta)E\|\phi(s-\tau)\|^2ds\right)\right)\rho(t)dt=0,
$$

that is $\phi^b(\cdot-\tau)\in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$. This completes the proof. □

Theorem 3.1. Let $\gamma\in\mathbb{U}$ and $PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$ be translation invariant. The space $WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$ equipped with the norm $\|\cdot\|_{S^2_\gamma}$ is a Banach space.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$. And let $(h_n)_{n\in\mathbb{N}}$, $(\varphi_n)_{n\in\mathbb{N}}$ be sequences such that $f_n=h_n+\varphi_n$, where $(h_n)_{n\in\mathbb{N}}\subset AS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$ and $(\varphi^b_n)_{n\in\mathbb{N}}\subset PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$. Using similar ideas as in the proof of [35, Lemma 3.1] it can be shown that the following holds

$$
\|h_n\|_{S^2_\gamma}\leq\|f_n\|_{S^2_\gamma},
$$

for all $n\in\mathbb{N}$. Thus there exists a function $h\in AS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$ such that $\|h_n-h\|_{S^2_\gamma}\to 0$ as $n\to\infty$. Using the previous fact, it easily follows that there exists a function $\varphi\in BS^2_\gamma(L^2(P,\mathbb{H}))$ such that $\|\varphi_n-\varphi\|_{S^2_\gamma}\to 0$ as $n\to\infty$. Now, for $q>0$, we obtain

$$
\begin{aligned}
&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\int_t^{t+1}\gamma(s-t)E\|\varphi(s)\|^2ds\right)\rho(t)dt\\
\leq\;&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\int_t^{t+1}\gamma(s-t)E\|\varphi_n(s)-\varphi(s)\|^2ds\right)\rho(t)dt\\
&+\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\int_t^{t+1}\gamma(s-t)E\|\varphi_n(s)\|^2ds\right)\rho(t)dt\\
\leq\;&\|\varphi_n-\varphi\|^2_{S^2_\gamma}+\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\int_t^{t+1}\gamma(s-t)E\|\varphi_n(s)\|^2ds\right)\rho(t)dt.
\end{aligned}
$$

Letting $q\to\infty$ and then $n\to\infty$ in the previous inequality, it follows that $\varphi^b\in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$. Above all we have $f=h+\varphi\in WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$, which completes the proof. □

Next, we prove a useful composition theorem.

Theorem 3.2. Let $F = H + \Phi \in WPAAS^2_\gamma(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$, $\phi = \phi_1 + \phi_2 \in WPAAS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$, where $H \in AS^2_\gamma(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ and $\Phi^b \in PAA_0(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$, $\phi_1 \in AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$, $\phi_2 \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$. If $K = \overline{\{\phi_1(t) : t\in\mathbb{R}\}} \subset L^2(P,\mathbb{H}))$ is a compact subset and there exists a continuous function $L_F(\cdot) : \mathbb{R} \to [0,\infty)$ such that for all $x, y \in L^2(P,\mathbb{H})$ and $t \in \mathbb{R}$

$$\int_t^{t+1} \gamma(s-t)E\|F(s,x) - F(s,y)\|^2 ds \le L_F(t)E\|x-y\|^2.$$

If for every $\chi^b \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$ such that

$$\limsup_{r\to\infty} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} L_F(\theta) \right) \rho(t)dt < \infty, \tag{3.3}$$

$$\lim_{r\to\infty} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} L_F(\theta) \right) \chi(t)\rho(t)dt = 0. \tag{3.4}$$

(I) $H(t,x)$ is uniformly continuous in any bounded subset $K' \subset L^2(P,\mathbb{H})$ uniformly for $t\in\mathbb{R}$, then $F(\cdot,\phi(\cdot)) \in WPAAS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$.

Proof. Since $F = H + \Phi$, $\phi = \phi_1 + \phi_2$, where $H \in AS^2_\gamma(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$, $\Phi^b \in PAA_0(\mathbb{R}\times L^2(P,\mathbb{H}), L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$, $\phi_1 \in AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$, $\phi_2 \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$ and $Q = \overline{\{\phi_1(t) : t\in\mathbb{R}\}}$ is compact. Now the function F can be decomposed as

$$F(t,\phi(t)) = H(t,\phi_1(t)) + F(t,\phi(t)) - F(t,\phi_1(t)) + \Phi(t,\phi_1(t)).$$

denote

$$\zeta(t) = H(t,\phi_1(t)), \quad \eta(t) = F(t,\phi(t)) - F(t,\phi_1(t)), \quad \varpi(t) = \Phi(t,\phi_1(t)).$$

Then $F(t,\phi(t)) = \zeta(t) + \eta(t) + \varpi(t)$. Since the function H satisfies the condition (I) and $Q = \overline{\{\phi_1(t) : t\in\mathbb{R}\}}$ is compact, we have $\zeta(t) \in AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$. So it is sufficient to prove $\eta^b(t) \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$ and $\varpi^b(t) \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$. For $p > 0$ and $q > 0$, by using the conditions in the theorem above, we obtain

$$\begin{aligned}
&\frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} \left(\int_\theta^{\theta+1} \gamma(s-\theta)E\|\eta(s)\|^2 ds \right) \right) \rho(t)dt \\
= \; &\frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} \left(\int_\theta^{\theta+1} \gamma(s-\theta) \right.\right. \\
&\left.\left. \times E\|F(t,\phi(t)) - F(t,\phi_1(t))\|^2 ds \right) \right) \rho(t)dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]} L_F(\theta)E\|\phi_2(\theta)\|^2\right)\rho(t)dt \\
&\leq \frac{m_0^{-1}}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]} L_F(\theta)\right)\cdot\left(\sup_{\theta\in[t-p,t]}\left(E\|\phi_2(\theta)\|^2\gamma(s-\theta)\right)\right)\rho(t)dt \\
&\leq \frac{m_0^{-1}}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]} L_F(\theta)\right) \\
&\quad\times\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta}^{\theta+1}\gamma(s-\theta)E\|\phi_2(s)\|^2ds\right)\right)\rho(t)dt.
\end{aligned}
$$

By using the (3.4), we obtain $\eta^b(t)\in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$.

Next we prove $\varpi^b(t)\in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$. Since the function H satisfies the condition (I), for the arbitrary $\epsilon>0$, there exists $\delta>0$ such that for each $t\in\mathbb{R}$, $E\|H(t,x)-H(t,\overline{x})\|^2<\epsilon$, whenever $E\|x-\overline{x}\|^2\leq\delta$, where $x,\overline{x}\in K'$. Furthermore,

$$
\int_t^{t+1}\gamma(s-t)E\|H(s,x)-H(s,\overline{x})\|^2ds<\epsilon,\quad t\in\mathbb{R}.
$$

Now we fix $x_1,x_2,\ldots,x_n\in\phi_1(\mathbb{R})$ such that $\phi_1(\mathbb{R})\subset\bigcup_{i=1}^n B_\delta(x_i,L^2(P,\mathbb{H}))$. Obviously, the set $E_i=\phi_1^{-1}(B_\delta(x_i))$ generates an open covering on $\mathbb{R}$, let $B_1=E_1$,$B_k=E_k\backslash(\bigcup_{i=1}^k E_i)(2\leq k\leq m)$. Then $\mathbb{R}=\bigcup_{k=1}^m B_k$ and $B_i\cap B_j=\varnothing, i\neq j, 1\leq i,j\leq m$. When $t\in\mathbb{R}$ and $\phi_1(t)\in B_\delta(x_i)$, it follows that

$$
\begin{aligned}
&\int_t^{t+1}\gamma(s-t)E\|\varpi(t)\|^2ds \\
=\ &\int_t^{t+1}\gamma(s-t)E\|\Phi(s,\phi_1(s))\|^2ds \\
\leq\ &\int_t^{t+1}\gamma(s-t)E\|F(s,\phi_1(s))-F(s,x_i)\|^2ds \\
&+\int_t^{t+1}\gamma(s-t)E\|\Phi(s,x_i)\|^2ds \\
&+\int_t^{t+1}\gamma(s-t)E\|H(s,\phi_1(s))-H(s,x_i)\|^2ds \\
\leq\ & L_F(t)E\|\phi_1(t)-x_i\|^2+\epsilon+\int_t^{t+1}\gamma(s-t)E\|\Phi(s,x_i)\|^2ds \\
\leq\ & (L_F(t)+1)\epsilon+\int_t^{t+1}\gamma(s-t)E\|\Phi(s,x_i)\|^2ds.
\end{aligned}
$$

For each $q>0$, we obtain

$$
\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_\theta^{\theta+1}\gamma(s-\theta)E\|\Phi(s,\phi_1(s))\|^2ds\right)\right)\rho(t)dt
$$

$$
\begin{aligned}
&\leq \frac{1}{m(q,\rho)}\sum_{i=1}^{n}\int_{B_i\cap[-q,q]}\Bigg(\sup_{j=1,\dots,n}\Bigg[\sup_{\theta\in[t-p,t]\cap B_j}\Bigg(\int_{\theta}^{\theta+1}\gamma(s-\theta)\\
&\quad\times E\|\Phi(s,\phi_1(s))\|^2ds\Bigg)\Bigg]\Bigg)\rho(t)dt\\
&\leq \frac{1}{m(q,\rho)}\sum_{i=1}^{n}\int_{B_i\cap[-q,q]}\Bigg(\sup_{j=1,\dots,n}\Bigg[\sup_{\theta\in[t-p,t]\cap B_j}\Bigg(\int_{\theta}^{\theta+1}\gamma(s-\theta)\\
&\quad\times E\|F(s,\phi_1(s))-F(s,x_j)\|^2ds\Bigg)\Bigg]\Bigg)\rho(t)dt\\
&\quad+\frac{1}{m(q,\rho)}\sum_{i=1}^{n}\int_{B_i\cap[-q,q]}\Bigg(\sup_{j=1,\dots,n}\Bigg[\sup_{\theta\in[t-p,t]\cap B_j}\Bigg(\int_{\theta}^{\theta+1}\gamma(s-\theta)\\
&\quad\times E\|H(s,\phi_1(s))-H(s,x_j)\|^2ds\Bigg)\Bigg]\Bigg)\rho(t)dt\\
&\quad+\frac{1}{m(q,\rho)}\sum_{i=1}^{n}\int_{B_i\cap[-q,q]}\Bigg(\sup_{j=1,\dots,n}\Bigg[\sup_{\theta\in[t-p,t]\cap B_j}\Bigg(\int_{\theta}^{\theta+1}\gamma(s-\theta)\\
&\quad\times E\|\Phi(s,x_j)\|^2ds\Bigg)\Bigg]\Bigg)\rho(t)dt\\
&\leq \sum_{j=1}^{n}\frac{1}{m(q,\rho)}\int_{-q}^{q}\Bigg(\sup_{\theta\in[t-p,t]}\Bigg(\int_{\theta}^{\theta+1}\gamma(s-\theta)E\|\Phi(s,x_j)\|^2ds\Bigg)\Bigg)\rho(t)dt\\
&\quad+\frac{1}{m(q,\rho)}\Bigg(\sup_{\theta\in[t-p,t]}L_F(\theta)\epsilon+\epsilon\Bigg)\rho(t)dt.
\end{aligned}
$$

From the arbitrary ϵ, it follows that

$$\lim_{q\to\infty}\frac{1}{m(q,\rho)}\int_{-q}^{q}\Bigg(\sup_{\theta\in[t-p,t]}\Bigg(\int_{\theta}^{\theta+1}\gamma(s-\theta)E\|\Phi(s,\phi_1(s)\|^2ds\Bigg)\Bigg)\rho(t)dt=0,$$

which implies that $F(\cdot,\phi(\cdot))\in WPAAS_{\gamma}^{2}(\mathbb{R},L^2(P,\mathbb{H}))$. $\square$

4. WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS IN DISTRIBUTION

Fix $\gamma\in\mathbb{U}$ and $\rho\in\mathbb{V}_{\infty}$. This section is devoted to the search of the existence and uniqueness results of weighted pseudo almost automorphic solutions in distribution to Eq. (1.1) with S_{γ}^{2}-weighted pseudo almost automorphic coefficients. For that, we assume that the following assumptions hold:

(H1) There exist constants $\lambda_0\geq 0$, $\theta\in(\frac{\pi}{2},\pi)$, $K_1,K_2\geq 0$ and $\beta_1,\beta_2\in(0,1]$ with $\beta_1+\beta_2>1$ such that

$$\Sigma_{\theta}\cup\{0\}\subset\rho(A(t)-\lambda_0),\qquad \|R(\lambda,A(t)-\lambda_0)\|\leq\frac{K_1}{1+|\lambda|},$$

and

$$\|[A(t)-\lambda_0]R(\lambda, A(t)-\lambda_0)[R(\lambda_0, A(t))-R(\lambda_0, A(s))]\| \leq K_2|t-s|^{\beta_1}|\lambda|^{-\beta_2}$$

for $t, s \in \mathbb{R}$ and $\lambda \in \Sigma_\theta = \{\lambda \in \mathcal{C}\ \{0\}|\ |\arg\lambda| \leq \theta\}$. Then there exists a unique evolution family $\{U(t,s)\}_{-\infty<s\leq t<\infty}$, which governs the following linear equation

$$x'(t) = A(t)x(t), \quad t \geq s, \quad x(s) = \varphi \in \mathbb{X},$$

where $\mathbb{X}$ is a Banach space.
(H2) $U(t,s)$ is an exponentially stable evolution family on $L^2(P, \mathbb{H})$, that is, there exist two numbers $M, \delta > 0$ such that $\|U(t,s)\| \leq Me^{-\delta(t-s)}$, for $t \geq s$.
(H3) The function $s \to A(s)U(t,s)$ defined from $(-\infty, t]$ into $\mathcal{L}(L^2(P, \mathbb{H}))$ is strongly measurable and there exists a non-increasing function $H : [0,\infty) \to [0,\infty)$ with $H \in L^1(0,\infty)$ and a constant $\omega > 0$ such that

$$\|A(s)U(t,s)\| \leq e^{-\omega(t-s)}H(t-s), \quad t > s.$$

(H4) The function $\mathbb{R}\times\mathbb{R} \to L^2(P, \mathbb{H})$, $(t,s) \to U(t,s)x \in bAA(\mathbb{T}, L^2(P, \mathbb{H}))$ uniformly for all x in any bounded subset of $L^2(P, \mathbb{H})$.
(H5) The function $\mathbb{R}\times\mathbb{R} \to L^2(P, \mathbb{H})$, $(t,s) \to A(s)U(t,s)x \in bAA(\mathbb{T}, L^2(P, \mathbb{H}))$ uniformly for all x in any bounded subset of $L^2(P, \mathbb{H})$.

Definition 4.1. A continuous stochastic function $u : \mathbb{R}\times\Omega \to L^2(P, \mathbb{H})$, $a \in \mathbb{R}$, is called a mild solution of (1.1), provided that $\sup_{t\in\mathbb{R}} E\|u(t)\|^2 < \infty$, the function $s \to A(s)U(t,s)f(s, u_s)$ is integrable on $\mathbb{R}$ and the following conditions hold:
(i) $u_s \in \mathcal{B}$ for every $s \in \mathbb{R}$.
(ii) for $t \geq a$, $a \in \mathbb{R}$, $u(t)$ satisfies the following integral equation

$$\begin{aligned} u(t) &= U(t,a)[\phi(a) + f(a,\phi)] - f(t,u_t) - \int_a^t A(s)U(t,s)f(s,u_s)ds \\ &\quad + \int_a^t U(t,s)g(s,u_s)ds + \int_a^t U(t,s)\sigma(s,u_s)dW(s). \end{aligned}$$

Under assumptions (H1)-(H3), it can be easily shown that the equation

$$\begin{aligned} u(t) &= -f(t,u_t) - \int_{-\infty}^t A(s)U(t,s)f(s,u_s)ds + \int_{-\infty}^t U(t,s)g(s,u_s)ds \\ &\quad + \int_{-\infty}^t U(t,s)\sigma(s,u_s)dW(s) \end{aligned}$$

for each $t \in \mathbb{R}$, is a mild solution of (1.1).

Lemma 4.1. [29] Let $u \in WPAA(\mathbb{R}, L^2(P, \mathbb{H}))$ and assume that $\mathcal{B}$ is a uniform fading memory space. Then the function $t \to u_t$ belongs to $WPAA(\mathbb{R}, L^2(P, \mathbb{H}))$.

Lemma 4.2. Let $u \in WPAAS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$ and assume that $\mathcal{B}$ is a uniform fading memory space. Then $t \to u_t$ belongs to $WPAAS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$.

Proof. Assume that $u = h + \varphi$ with $h \in AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$ and also

$$\varphi^b \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho).$$

Clearly, in the light of the translation property of $AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$, we can obtain that $h_t \in AS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$. So it is sufficient to prove that

$$t \to \varphi^b_t \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho).$$

Let $p > 0$ and $\epsilon > 0$. Since $\mathcal{B}$ is a uniform fading memory space, there exists $\tau_\epsilon > p$ such that $N(\tau) < \epsilon$ for every $\tau > \tau_\epsilon$ and consequently $\mathcal{D} = \sup N(\tau) < \epsilon$. Under these conditions, for $q > 0$ and $\tau > \tau_\epsilon$ we find that

$$\begin{aligned}
&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_\theta^{\theta+1}\gamma(s-\theta)E\|\varphi_\theta\|^2_{\mathcal{B}}ds\right)\right)\rho(t)dt\\
\le\;& \frac{1}{m(q,\rho)}\int_{-q}^{q}\sup_{\theta\in[t-p,t]}\left(\mathcal{D}\left(\int_\theta^{\theta+1}\gamma(s-\theta)E\|\varphi_{\theta-\tau}\|^2_{\mathcal{B}}ds\right)\right.\\
&\left.+\,\Im\sup_{s\in[\theta-\tau,\theta]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\right)\rho(t)dt\\
\le\;& \Re\|\varphi\|_{S^2_\gamma}\epsilon + \frac{\Im}{m(q,\rho)}\int_{-q}^{q}\sup_{s\in[t-2\tau,t]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\rho(t)dt.
\end{aligned}$$

Next, we consider the second term of the inequality above

$$\begin{aligned}
&\frac{\Im}{m(q,\rho)}\int_{-q}^{q}\sup_{s\in[t-2\tau,t]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\rho(t)dt\\
\le\;& \frac{\Im}{m(q,\rho)}\int_{-q-\tau}^{q-\tau}\left(\sup_{s\in[t-\tau,t]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\right.\\
&\left.+\sup_{s\in[t,t+\tau]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\right)\rho(t)dt\\
\le\;& \frac{\Im}{m(q,\rho)}\int_{-q-\tau}^{q-\tau}\left(\sup_{s\in[t-\tau,t]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\right)\rho(t)dt\\
&+\frac{\Im}{m(q,\rho)}\int_{-q-\tau}^{q-\tau}\left(\sup_{s\in[t,t+\tau]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\right)\rho(t)dt\\
\le\;& \frac{m(q+\tau,\rho)}{m(q,\rho)}\frac{\Im}{m(q+\tau,\rho)}\\
&\times\int_{-q-\tau}^{q+\tau}\left(\sup_{s\in[t-\tau,t]}\left(\int_s^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\right)\rho(t)dt
\end{aligned}$$

$$+\frac{\Im}{m(q,\rho)}\int_{-q}^{q}\sup_{s\in[t-\tau,t]}\left(\int_{s}^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\rho(t)dt.$$

Consequently,

$$\begin{aligned}
&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta}^{\theta+1}\gamma(s-\theta)E\|\varphi_\theta\|_{\mathcal{B}}^2ds\right)\right)\rho(t)dt\\
\leq\ &\Re\|\varphi\|_{S^2_\gamma}\epsilon+\frac{\Im}{m(q,\rho)}\int_{-q}^{q}\sup_{s\in[t-\tau,t]}\left(\int_{s}^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\rho(t)dt\\
&+\frac{m(q+\tau,\rho)}{m(q,\rho)}\frac{\Im}{m(q+\tau,\rho)}\\
&\times\int_{-q-\tau}^{q+\tau}\left(\sup_{s\in[t-\tau,t]}\left(\int_{s}^{s+1}\gamma(\xi-s)E\|\varphi(\xi)\|^2d\xi\right)\right)\rho(t)dt,
\end{aligned}$$

which enables us to complete the proof, since ϵ is arbitrary and the conditions in Lemma 2.2. □

Lemma 4.3. Let $u\in WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$. Under assumptions (H1), (H2), (H4), the function $v(\cdot)$ is defined by

$$v(t):=\int_{-\infty}^{t}U(t,s)u(s)ds$$

then $v\in WPAA(\mathbb{R},L^2(P,\mathbb{H}))$.

Proof. Since $u\in WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$, there exist $h\in AS^2_\gamma$ and

$$\varphi^b\in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$$

such that $u=h+\varphi$. Therefore,

$$v(t):=\int_{-\infty}^{t}U(t,s)h(s)ds+\int_{-\infty}^{t}U(t,s)\varphi(s)ds=X(t)+Y(t).$$

Let us show that $X(t)\in AA(\mathbb{R},L^2(P,\mathbb{H}))$. For $n=1,2,\cdots$, consider the integral

$$X_n(t)=\int_{t-n}^{t-n+1}U(t,r)h(r)dr.$$

Now, by using the Cauchy-Schwartz inequality and the exponential dichotomy, it follows that

$$\begin{aligned}
&E\|X_n(t)\|^2\\
\leq\ &E\left(\int_{t-n}^{t-n+1}\|U(t,r)h(r)\|dr\right)^2
\end{aligned}$$

$$\begin{aligned}
&\leq M^2 E\left(\int_{t-n}^{t-n+1} \gamma^{-\frac{1}{2}}(r-t+n)e^{-\delta(t-r)}\|h(r)\|\gamma^{\frac{1}{2}}(r-t+n)dr\right)^2 \\
&\leq M^2\left[\int_{t-n}^{t-n+1}\gamma^{-1}(r-t+n)e^{-2\delta(t-r)}dr\right] \\
&\quad\times\left[\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|h(r)\|^2dr\right] \\
&\leq M^2 m_0^{-1}\left(\int_{n-1}^{n} e^{-2\delta s}ds\right)\|h\|^2_{S^2_\gamma} \\
&\leq \left[m_0^{-1}e^{-2\delta n}\frac{M^2(1+e^{2\delta})}{2\delta}\right]\|h\|^2_{S^2_\gamma}.
\end{aligned}$$

Since $m_0^{-1}\frac{M^2(1+e^{2\delta})}{2\delta}\Sigma_{n=1}^{\infty}e^{-2\delta n} < \infty$, we deduce from the well-known Weierstrass theorem that the series $\Sigma_{n=1}^{\infty}X_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$X(t) := \int_{-\infty}^{t} U(t,r)h(r)dr = \sum_{n=1}^{\infty} X_n(t),$$

$X \in C(\mathbb{R}, L^2(P,\mathbb{H}))$ and $E\|X(t)\|^2 \leq \Sigma_{n=1}^{\infty}E\|X_n(t)\|^2 \leq L(M,m_0,\delta)\|h\|^2_{S^2_\gamma}$, where $L(M,m_0,\delta) > 0$ is a constant. Since $h \in AS^2_\gamma$, then for every sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_m\}_{m\in\mathbb{N}}$ and functions U_1 and $\widetilde{h} \in L^2_{loc}(\mathbb{R},\gamma ds)$ such that

$$\lim_{m\to\infty} U(t+s_m, s+s_m)x = U_1(t,s)x, \quad t,s\in\mathbb{R}, \quad x \in L^2_{loc}(\mathbb{R},\gamma ds) \tag{4.1}$$

$$\lim_{m\to\infty} U_1(t-s_m, s-s_m)x = U(t,s)x, \quad t,s\in\mathbb{R}, \quad x \in L^2_{loc}(\mathbb{R},\gamma ds) \tag{4.2}$$

$$\lim_{m\to\infty}\int_t^{t+1}\gamma(s-t)E\|h(s+s_m)-\widetilde{h}(s)\|^2ds = 0, \tag{4.3}$$

$$\lim_{m\to\infty}\int_t^{t+1}\gamma(s-t)E\|\widetilde{h}(s-s_m)-h(s)\|^2ds = 0. \tag{4.4}$$

Let $\widetilde{X}_n(t) = \int_{t-n}^{t-n+1}U_1(t,r)\widetilde{h}(r)dr$. By using the Cauchy-Schwartz inequality and the exponential dichotomy, one has

$$\begin{aligned}
&E\|X_n(t+s_m)-\widetilde{X}_n(t)\|^2 \\
&= E\left\|\int_{t+s_m-n}^{t+s_m-n+1}U(t+s_m,r)h(r)dr - \int_{t-n}^{t-n+1}U_1(t,r)\widetilde{h}(r)dr\right\|^2 \\
&= E\left\|\int_{t-n}^{t-n+1}[U(t+s_m,r+s_m)h(r+s_m)-U_1(t,r)\widetilde{h}(r)]dr\right\|^2 \\
&\leq 2E\left\|\int_{t-n}^{t-n+1}U(t+s_m,r+s_m)[h(r+s_m)-\widetilde{h}(r)]dr\right\|^2
\end{aligned}$$

$$+2E\left\|\int_{t-n}^{t-n+1}[U(t+s_m,r+s_m)-U_1(t,r)]\widetilde{h}(r)dr\right\|^2$$

For the first term of the inequality above

$$\begin{aligned}
& 2E\left\|\int_{t-n}^{t-n+1}U(t+s_m,r+s_m)[h(r+s_m)-\widetilde{h}(r)]dr\right\|^2 \\
\leq\ & 2M^2E\bigg(\int_{t-n}^{t-n+1}\gamma^{-\frac{1}{2}}(r-t+n)e^{-\delta(t-r)} \\
& \times\|h(r+s_m)-\widetilde{h}(r)\|\gamma^{\frac{1}{2}}(r-t+n)dr\bigg)^2 \\
\leq\ & 2M^2\left[\int_{t-n}^{t-n+1}\gamma^{-1}(r-t+n)e^{-2\delta(t-r)}dr\right] \\
& \times\left[\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|h(r+s_m)-\widetilde{h}(r)\|^2dr\right] \\
\leq\ & 2L(M,m_0,\delta)\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|h(r+s_m)-\widetilde{h}(r)\|^2dr.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& E\|X_n(t+s_m)-\widetilde{X}_n(t)\|^2 \\
\leq\ & 2L(M,m_0,\delta)\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|h(r+s_m)-\widetilde{h}(r)\|^2dr \\
& +2E\left\|\int_{t-n}^{t-n+1}[U(t+s_m,r+s_m)-U_1(t,r)]\widetilde{h}(r)dr\right\|^2.
\end{aligned}$$

Now by using (4.1), (4.3) and also the Lebesgue Dominated Convergence theorem, for each $t\in\mathbb{R}$, it follows that

$$\lim_{m\to\infty}E\|X_n(t+s_m)-\widetilde{X}_n(t)\|^2=0.$$

Similarly, by using (4.2) and (4.4) it can be shown that

$$\lim_{m\to\infty}E\|\widetilde{X}_n(t-s_m)-X_n(t)\|^2=0.$$

Therefore, for each $X_n\in AA(\mathbb{R},L^2(P,\mathbb{H}))$ for each n and their uniform limit $X(t)\in AA(\mathbb{R},L^2(P,\mathbb{H}))$.

Next we prove $Y(t)\in PAA_0(\mathbb{R},\rho)$. Similarly, for $n=1,2,\cdots$, consider the integral

$$Y_n(t)=\int_{t-n}^{t-n+1}U(t,r)\varphi(r)dr.$$

For $p>0$, we can obtain

$$\sup_{\theta\in[t-p,t]}E\|Y_n(\theta)\|^2$$

$$
\begin{aligned}
&\le \sup_{\theta\in[t-p,t]} E\left(\int_{\theta-n}^{\theta-n+1}\|U(\theta,r)\varphi(r)\|dr\right)^2\\
&\le M^2\sup_{\theta\in[t-p,t]} E\left(\int_{\theta-n}^{\theta-n+1}\gamma^{-\frac{1}{2}}(r-\theta+n)e^{-\delta(\theta-r)}\|\varphi(r)\|\gamma^{\frac{1}{2}}(r-\theta+n)dr\right)^2\\
&\le M^2m_0^{-1}e^{2\delta p}\left(\int_{t-n}^{t-n+1}e^{-2\delta(t-r)}dr\right)\\
&\quad\times\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta-n}^{\theta-n+1}\gamma(r-\theta+n)E\|\varphi(r)\|^2dr\right)\right)\\
&\le \left[m_0^{-1}e^{-2\delta(n-p)}\frac{M^2(1+e^{2\delta})}{2\delta}\right]\\
&\quad\times\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta-n}^{\theta-n+1}\gamma(r-\theta+n)E\|\varphi(r)\|^2dr\right)\right).
\end{aligned}
$$

For $q>0$, we have

$$
\begin{aligned}
&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}E\|Y_n(\theta)\|^2\right)\rho(t)dt\\
&\le m_0^{-1}e^{-2\delta(n-p)}\frac{M^2(1+e^{2\delta})}{2\delta}\\
&\quad\times\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta-n}^{\theta-n+1}\gamma(r-\theta+n)E\|\varphi(r)\|^2dr\right)\right)\rho(t)dt.
\end{aligned}
$$

Since $\varphi^b\in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$, according to the inequality above, it follows that

$$
E\|Y_n(t)\|^2\le m_0^{-1}e^{-2\delta(n-p)}\frac{M^2(1+e^{2\delta})}{2\delta}\|\varphi\|^2_{S^2_\gamma}.
$$

Note that $m_0^{-1}e^{2\delta p}\frac{M^2(1+e^{2\delta})}{2\delta}\Sigma_{n=1}^{\infty}e^{-2\delta n}<\infty$, therefore, we deduce from the well-known Weierstrass theorem that the series $\Sigma_{n=1}^{\infty}Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$
Y(t):=\int_{-\infty}^{t}U(t,r)\varphi(r)dr=\sum_{n=1}^{\infty}Y_n(t),
$$

$Y\in C(\mathbb{R},L^2(P,\mathbb{H}))$ and $E\|Y(t)\|^2\le\Sigma_{n=1}^{\infty}E\|Y_n(t)\|^2\le L(M,m_0,\delta,p)\|\varphi\|^2_{S^2_\gamma}$, where $L(M,m_0,\delta,p)>0$ is a constant. Since $Y_n\in PAA_0(\mathbb{R},\rho)$ and the following inequality

$$
\begin{aligned}
&\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}E\|Y(\theta)\|^2\right)\rho(t)dt\\
\le\ &\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}E\|Y(\theta)-\sum_{k=1}^{n}Y_k(\theta)\|^2\right)\rho(t)dt
\end{aligned}
$$

$$+\sum_{k=1}^{n}\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}E\|Y_k(\theta)\|^2\right)\rho(t)dt.$$

Consequently, the uniform limit $Y(t)=\Sigma_{n=1}^{\infty}Y_n(t)\in PAA_0(\mathbb{R},\rho)$. Therefore, $v(t)=X(t)+Y(t)\in WPAA(\mathbb{R},L^2(P,\mathbb{H}))$. □

Lemma 4.4. Let $u'\in WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$. Under assumptions (H1), (H2), (H4), the function $v'(\cdot)$ is defined by

$$v'(t):=\int_{-\infty}^{t}U(t,s)u'(s)dW(s)$$

then $v'\in WPAA(\mathbb{R},L^2(P,\mathbb{H}))$.

Proof. Since $u'\in WPAAS^2_\gamma(\mathbb{R},L^2(P,\mathbb{H}))$, there exist $\alpha\in AS^2_\gamma$ and $\beta^b\in PAA_0(\mathbb{R},L^2(0,1;L^2(P,\mathbb{H}),\gamma ds),\rho)$ such that $u'=\alpha+\beta$. Therefore,

$$v'(t):=\int_{-\infty}^{t}U(t,s)\alpha(s)dW(s)+\int_{-\infty}^{t}U(t,s)\beta(s)dW(s)=\Upsilon(t)+\Psi(t).$$

Let us show that $\Upsilon(t)\in AA(\mathbb{R},L^2(P,\mathbb{H}))$. For $n=1,2,\cdots$, consider the integral

$$\Upsilon_n(t)=\int_{t-n}^{t-n+1}U(t,r)\alpha(r)dW(r).$$

Now using the Itô's integral and Hölder's inequality, it follows that

$$\begin{aligned}
E\|\Upsilon_n(t)\|^2 &\leq E\left(\int_{t-n}^{t-n+1}\|U(t,r)\alpha(r)\|^2dr\right)\\
&\leq M^2\int_{t-n}^{t-n+1}e^{-2\delta(t-r)}E\|\alpha(r)\|^2dr\\
&= M^2\int_{t-n}^{t-n+1}\gamma^{-1}(r-t+n)e^{-2\delta(t-r)}E\|\alpha(r)\|^2\gamma(r-t+n)dr\\
&\leq M^2\left[\int_{t-n}^{t-n+1}\gamma^{-1}(r-t+n)e^{-2\delta(t-r)}dr\right]\\
&\quad\times\left[\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|\alpha(r)\|^2dr\right]\\
&\leq M^2m_0^{-1}\left(\int_{n-1}^{n}e^{-2\delta s}ds\right)\|\alpha\|^2_{S^2_\gamma}\\
&\leq \left[m_0^{-1}e^{-2\delta n}\frac{M^2(1+e^{2\delta})}{2\delta}\right]\|\alpha\|^2_{S^2_\gamma}.
\end{aligned}$$

Since $m_0^{-1}\frac{M^2(1+e^{2\delta})}{2\delta}\Sigma_{n=1}^{\infty}e^{-2\delta n}<\infty$, we deduce from the well-known Weierstrass theorem that the series $\Sigma_{n=1}^{\infty}\Upsilon_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$\Upsilon(t):=\int_{-\infty}^{t}U(t,r)\alpha(r)dr=\sum_{n=1}^{\infty}\Upsilon_n(t),$$

$\Upsilon \in C(\mathbb{R}, L^2(P,\mathbb{H}))$ and $E\|\Upsilon(t)\|^2 \leq \Sigma_{n=1}^{\infty} E\|\Upsilon_n(t)\|^2 \leq L(M, m_0, \delta)\|\alpha\|_{S_\gamma^2}^2$, where $L(M, m_0, \delta) > 0$ is a constant. Since $\alpha \in AS_\gamma^2$, then for every sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_m\}_{m\in\mathbb{N}}$ and functions U_2 and $\widetilde{\alpha} \in L^2_{loc}(\mathbb{R}, \gamma ds)$ such that

$$\lim_{m\to\infty} U(t+s_m, s+s_m)x = U_2(t,s)x, \quad t,s \in \mathbb{R}, \quad x \in L^2_{loc}(\mathbb{R}, \gamma ds) \tag{4.5}$$

$$\lim_{m\to\infty} U_2(t-s_m, s-s_m)x = U(t,s)x, \quad t,s \in \mathbb{R}, \quad x \in L^2_{loc}(\mathbb{R}, \gamma ds) \tag{4.6}$$

$$\lim_{m\to\infty} \int_t^{t+1} \gamma(s-t)E\|\alpha(s+s_m) - \widetilde{\alpha}(s)\|^2 ds = 0, \tag{4.7}$$

$$\lim_{m\to\infty} \int_t^{t+1} \gamma(s-t)E\|\widetilde{\alpha}(s-s_m) - \alpha(s)\|^2 ds = 0. \tag{4.8}$$

Let $\widetilde{\Upsilon}_n(t) = \int_{t-n}^{t-n+1} U_2(t,r)\widetilde{\alpha}(r)dr$. By using Hölder's inequality, one has

$$\begin{aligned}
& E\|\Upsilon_n(t+s_m) - \widetilde{\Upsilon}_n(t)\|^2 \\
= \;& E\left\|\int_{t+s_m-n}^{t+s_m-n+1} U(t+s_m, r)\alpha(r)dr - \int_{t-n}^{t-n+1} U_2(t,r)\widetilde{\alpha}(r)dW(r)\right\|^2 \\
= \;& E\left\|\int_{t-n}^{t-n+1} [U(t+s_m, r+s_m)\alpha(r+s_m) - U_2(t,r)\widetilde{\alpha}(r)]dW(r)\right\|^2 \\
\leq \;& 2E\left\|\int_{t-n}^{t-n+1} U(t+s_m, r+s_m)[\alpha(r+s_m) - \widetilde{\alpha}(r)]dW(r)\right\|^2 \\
& +2E\left\|\int_{t-n}^{t-n+1} [U(t+s_m, r+s_m) - U_2(t,r)]\widetilde{\alpha}(r)dW(r)\right\|^2 \\
\leq \;& 2E\left\|\int_{t-n}^{t-n+1} U(t+s_m, r+s_m)[\alpha(r+s_m) - \widetilde{\alpha}(r)]dW(r)\right\|^2 \\
& +2\int_{t-n}^{t-n+1} \|U(t+s_m, r+s_m) - U_2(t,r)\| E\|\widetilde{\alpha}(r)\|^2 dr
\end{aligned}$$

For the first term of the inequality above

$$\begin{aligned}
& 2E\left\|\int_{t-n}^{t-n+1} U(t+s_m, r+s_m)[\alpha(r+s_m) - \widetilde{\alpha}(r)]dW(r)\right\|^2 \\
\leq \;& 2M^2 \int_{t-n}^{t-n+1} \gamma^{-1}(r-t+n)e^{-2\delta(t-r)} \\
& \times E\|\alpha(r+s_m) - \widetilde{\alpha}(r)\|^2 \gamma(r-t+n)dr \\
\leq \;& 2M^2 \left[\int_{t-n}^{t-n+1} \gamma^{-1}(r-t+n)e^{-2\delta(t-r)}dr\right] \\
& \times \left[\int_{t-n}^{t-n+1} \gamma(r-t+n)E\|\alpha(r+s_m) - \widetilde{\alpha}(r)\|^2 dr\right]
\end{aligned}$$

$$\leq \quad 2L(M,m_0,\delta)\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|\alpha(r+s_m)-\widetilde{\alpha}(r)\|^2dr$$

Consequently,

$$\begin{aligned}
& E\|\Upsilon_n(t+s_m)-\widetilde{\Upsilon}_n(t)\|^2 \\
\leq \quad & 2L(M,m_0,\delta)\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|\alpha(r+s_m)-\widetilde{\alpha}(r)\|^2dr \\
& +2\int_{t-n}^{t-n+1}\|U(t+s_m,r+s_m)-U_2(t,r)\|E\|\widetilde{\alpha}(r)\|^2dr
\end{aligned}$$

Now by using (4.5), (4.7) and also the Lebesgue Dominated Convergence theorem, for each $t\in\mathbb{R}$, it follows that

$$\lim_{m\to\infty}E\|\Upsilon_n(t+s_m)-\widetilde{\Upsilon}_n(t)\|^2=0.$$

Similarly, by using (4.6) and (4.8) it can be shown that

$$\lim_{m\to\infty}E\|\widetilde{\Upsilon}_n(t-s_m)-\Upsilon_n(t)\|^2=0.$$

Therefore, for each $\Upsilon_n\in AA(\mathbb{R},L^2(P,\mathbb{H}))$ for each n and their uniform limit $\Upsilon(t)\in AA(\mathbb{R},L^2(P,\mathbb{H}))$.

Next we prove $\Psi(t)\in PAA_0(\mathbb{R},\rho)$. Similarly, for $n=1,2,\cdots$, consider the integral

$$\Psi_n(t)=\int_{t-n}^{t-n+1}U(t,r)\beta(r)dW(r).$$

For $p>0$, we can obtain

$$\begin{aligned}
& \sup_{\theta\in[t-p,t]}E\|\Psi_n(\theta)\|^2 \\
\leq \quad & \sup_{\theta\in[t-p,t]}E\left(\int_{\theta-n}^{\theta-n+1}\|U(\theta,r)\beta(r)\|^2dr\right) \\
\leq \quad & M^2\sup_{\theta\in[t-p,t]}\int_{\theta-n}^{\theta-n+1}e^{-2\delta(\theta-r)}E\|\beta(r)\|^2dr \\
\leq \quad & M^2\sup_{\theta\in[t-p,t]}e^{2\delta p}\int_{\theta-n}^{\theta-n+1}\gamma^{-1}(r-\theta+n)e^{-2\delta(t-r)}E\|\beta(r)\|^2\gamma(r-\theta+n)dr \\
\leq \quad & M^2m_0^{-1}e^{2\delta p}\left(\int_{t-n}^{t-n+1}e^{-2\delta(t-r)}dr\right) \\
& \times\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta-n}^{\theta-n+1}\gamma(r-\theta+n)E\|\beta(r)\|^2dr\right)\right) \\
\leq \quad & \left[m_0^{-1}e^{-2\delta(n-p)}\frac{M^2(1+e^{2\delta})}{2\delta}\right]
\end{aligned}$$

$$\times \left(\sup_{\theta \in [t-p,t]} \left(\int_{\theta-n}^{\theta-n+1} \gamma(r-\theta+n) E\|\beta(r)\|^2 dr \right) \right).$$

for $q > 0$, we have

$$\begin{aligned} & \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta \in [t-p,t]} E\|\Psi_n(\theta)\|^2 \right) \rho(t) dt \\ \leq \quad & m_0^{-1} e^{-2\delta(n-p)} \frac{M^2(1+e^{2\delta})}{2\delta} \\ & \times \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta \in [t-p,t]} \left(\int_{\theta-n}^{\theta-n+1} \gamma(r-\theta+n) E\|\beta(r)\|^2 dr \right) \right) \rho(t) dt. \end{aligned}$$

Since $\beta^b \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$, according to the inequality above, it follows that

$$E\|\Psi_n(t)\|^2 \leq m_0^{-1} e^{-2\delta(n-p)} \frac{M^2(1+e^{2\delta})}{2\delta} \|\beta\|^2_{S^2_\gamma}.$$

Note that $m_0^{-1} e^{2\delta p} \frac{M^2(1+e^{2\delta})}{2\delta} \Sigma_{n=1}^{\infty} e^{-2\delta n} < \infty$, therefore, we deduce from the well-known Weierstrass theorem that the series $\Sigma_{n=1}^{\infty} \Psi_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$\Psi(t) := \int_{-\infty}^{t} U(t,r)\beta(r) dr = \sum_{n=1}^{\infty} \Psi_n(t),$$

$\Psi \in C(\mathbb{R}, L^2(P,\mathbb{H}))$ and $E\|\Psi(t)\|^2 \leq \Sigma_{n=1}^{\infty} E\|\Psi_n(t)\|^2 \leq L(M, m_0, \delta, p)\|\beta\|^2_{S^2_\gamma}$, where $L(M, m_0, \delta, p) > 0$ is a constant. Since $\Psi_n \in PAA_0(\mathbb{R}, \rho)$ and the following inequality

$$\begin{aligned} & \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta \in [t-p,t]} E\|\Psi(\theta)\|^2 \right) \rho(t) dt \\ \leq \quad & \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta \in [t-p,t]} E\|\Psi(\theta) - \sum_{k=1}^{n} \Psi_k(\theta)\|^2 \right) \rho(t) dt \\ & + \sum_{k=1}^{n} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta \in [t-p,t]} E\|\Psi_k(\theta)\|^2 \right) \rho(t) dt. \end{aligned}$$

Consequently, the uniform limit $\Psi(t) = \Sigma_{n=1}^{\infty} \Psi_n(t) \in PAA_0(\mathbb{R}, \rho)$. Therefore, $v'(t) = \Upsilon(t) + \Psi(t) \in WPAA(\mathbb{R}, L^2(P,\mathbb{H}))$. $\square$

Lemma 4.5. Let $u'' \in WPAAS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$. Under assumptions (H1), (H2), (H3), (H5) and provided that the series $\Sigma_{n=1}^{\infty} \left[\int_{n-1}^{n} e^{-2\omega s} H^2(s) ds \right]$ converges and $H_s = \sup_{s \in [0,\infty)} H(s)$, the function $v''(\cdot)$ is defined by

$$v''(t) := \int_{-\infty}^{t} A(s) U(t,s) u''(s) ds$$

then $v'' \in WPAA(\mathbb{R}, L^2(P,\mathbb{H}))$.

Proof. Since $u'' \in WPAAS^2_\gamma(\mathbb{R}, L^2(P,\mathbb{H}))$, there exist $\mu \in AS^2_\gamma$ and

$$\nu^b \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$$

such that $u' = \mu + \nu$. Therefore,

$$v''(t) := \int_{-\infty}^{t} A(s)U(t,s)\mu(s)ds + \int_{-\infty}^{t} A(s)U(t,s)\nu(s)ds = \Phi(t) + Z(t).$$

Let us show that $\Phi(t) \in AA(\mathbb{R}, L^2(P,\mathbb{H}))$. For $n = 1, 2, \cdots$, consider the integral

$$\Phi_n(t) = \int_{t-n}^{t-n+1} A(r)U(t,r)\mu(r)dr.$$

Now, by using the Cauchy-Schwartz inequality and the exponential dichotomy, it follows that

$$\begin{aligned}
E\|\Phi_n(t)\|^2 &\leq E\left(\int_{t-n}^{t-n+1} \|A(r)U(t,r)\mu(r)\|dr\right)^2 \\
&\leq E\Bigg(\int_{t-n}^{t-n+1} \gamma^{-\frac{1}{2}}(r-t+n)e^{-\omega(t-r)}H(t-r) \\
&\quad \times \|\mu(r)\|\gamma^{\frac{1}{2}}(r-t+n)dr\Bigg)^2 \\
&\leq \left[\int_{t-n}^{t-n+1} \gamma^{-1}(r-t+n)e^{-2\omega(t-r)}H^2(t-r)dr\right] \\
&\quad \times \left[\int_{t-n}^{t-n+1} \gamma(r-t+n)E\|\mu(r)\|^2 dr\right] \\
&\leq m_0^{-1}\left[\int_{n-1}^{n} e^{-2\omega s}H^2(s)ds\right]\|\mu\|^2_{S^2_\gamma}.
\end{aligned}$$

Using the fact that the series given by

$$m_0^{-1}\left[\int_{n-1}^{n} e^{-2\omega s}H^2(s)ds\right]$$

converges, we deduce from the well-known Weierstrass theorem that the series $\Sigma_{n=1}^{\infty}\Phi_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$\Phi(t) := \int_{-\infty}^{t} A(r)U(t,r)\mu(r)dr = \sum_{n=1}^{\infty}\Phi_n(t),$$

$\Phi \in C(\mathbb{R}, L^2(P,\mathbb{H}))$ and $E\|\Phi(t)\|^2 \leq \Sigma_{n=1}^{\infty} E\|\Phi_n(t)\|^2 \leq L(m_0,\omega)\|\mu\|^2_{S^2_\gamma}$. Since $\mu \in AS^2_\gamma$, then for every sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_m\}_{m\in\mathbb{N}}$ and functions U_3 and $\widetilde{\mu} \in L^2_{loc}(\mathbb{R}, \gamma ds)$ such that

$$\lim_{m\to\infty} A(s+s_m)U(t+s_m, s+s_m)x = U_3(t,s)x, \quad t,s \in \mathbb{R}, \quad x \in L^2_{loc}(\mathbb{R}, \gamma ds) \qquad (4.9)$$

$$\lim_{m\to\infty} U_3(t-s_m, s-s_m)x = A(s)U(t,s)x, \quad t,s\in\mathbb{R}, \quad x\in L^2_{loc}(\mathbb{R},\gamma ds) \tag{4.10}$$

$$\lim_{m\to\infty}\int_t^{t+1}\gamma(s-t)E\|\mu(s+s_m)-\widetilde{\mu}(s)\|^2 ds = 0, \tag{4.11}$$

$$\lim_{m\to\infty}\int_t^{t+1}\gamma(s-t)E\|\widetilde{\mu}(s-s_m)-\mu(s)\|^2 ds = 0. \tag{4.12}$$

Let $\widetilde{\Phi}_n(t) = \int_{t-n}^{t-n+1} U_3(t,r)\widetilde{\mu}(r)dr$. By using the Cauchy-Schwartz inequality and the exponential dichotomy, one has

$$\begin{aligned}
& E\|\Phi_n(t+s_m)-\widetilde{\Phi}_n(t)\|^2 \\
= \;& E\left\|\int_{t+s_m-n}^{t+s_m-n+1} A(r)U(t+s_m,r)\mu(r)dr - \int_{t-n}^{t-n+1} U_3(t,r)\widetilde{\mu}(r)dr\right\|^2 \\
= \;& E\left\|\int_{t-n}^{t-n+1} [A(r+s_m)U(t+s_m,r+s_m)\mu(r+s_m) - U_3(t,r)\widetilde{\mu}(r)]dr\right\|^2 \\
\le \;& 2E\left\|\int_{t-n}^{t-n+1} A(r+s_m)U(t+s_m,r+s_m)[\mu(r+s_m)-\widetilde{\mu}(r)]dr\right\|^2 \\
& +2E\left\|\int_{t-n}^{t-n+1} [A(r+s_m)U(t+s_m,r+s_m) - U_3(t,r)]\widetilde{\mu}(r)dr\right\|^2
\end{aligned}$$

For the first term of the inequality above

$$\begin{aligned}
& 2E\left\|\int_{t-n}^{t-n+1} A(r+s_m)U(t+s_m,r+s_m)[\mu(r+s_m)-\widetilde{\mu}(r)]dr\right\|^2 \\
\le \;& 2E\bigg(\int_{t-n}^{t-n+1}\gamma^{-\frac{1}{2}}(r-t+n)e^{-\omega(t-r)}H(t-r) \\
& \times\|\mu(r+s_m)-\widetilde{\mu}(r)\|\gamma^{\frac{1}{2}}(r-t+n)dr\bigg)^2 \\
\le \;& 2\left[\int_{t-n}^{t-n+1}\gamma^{-1}(r-t+n)e^{-2\omega(t-r)}H^2(t-r)dr\right] \\
& \times\left[\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|\mu(r+s_m)-\widetilde{\mu}(r)\|^2dr\right] \\
\le \;& 2m_0^{-1}\left[\int_{n-1}^{n} e^{-2\omega s}H^2(s)ds\right] \\
& \times\int_{t-n}^{t-n+1}\gamma(r-t+n)E\|h(r+s_m)-\widetilde{h}(r)\|^2dr
\end{aligned}$$

Consequently,

$$\begin{aligned}
& E\|\Phi_n(t+s_m)-\widetilde{\Phi}_n(t)\|^2 \\
\le \;& 2m_0^{-1}\left[\int_{n-1}^{n} e^{-2\omega s}H^2(s)ds\right]
\end{aligned}$$

$$\times \int_{t-n}^{t-n+1} \gamma(r-t+n)E\|\mu(r+s_m)-\widetilde{\mu}(r)\|^2 dr$$
$$+2E\left\|\int_{t-n}^{t-n+1} [A(r+s_m)U(t+s_m,r+s_m)-U_3(t,r)]\widetilde{\mu}(r)dr\right\|^2$$

Now by using (4.9), (4.11) and also the Lebesgue Dominated Convergence theorem, for each $t \in \mathbb{R}$, it follows that

$$\lim_{m\to\infty} E\|\Phi_n(t+s_m)-\widetilde{\Phi}_n(t)\|^2 = 0.$$

Similarly, by using (4.10) and (4.12) it can be shown that

$$\lim_{m\to\infty} E\|\widetilde{\Phi}_n(t-s_m)-\Phi_n(t)\|^2 = 0.$$

Therefore, for each $\Phi_n \in AA(\mathbb{R}, L^2(P,\mathbb{H}))$ for each n and their uniform limit $\Phi(t) \in AA(\mathbb{R}, L^2(P,\mathbb{H}))$.

Next we prove $Z(t) \in PAA_0(\mathbb{R},\rho)$. Similarly, for $n = 1, 2, \cdots$, consider the integral

$$Z_n(t) = \int_{t-n}^{t-n+1} A(r)U(t,r)\nu(r)dr.$$

For $p > 0$, we can obtain

$$\begin{aligned}
&\sup_{\theta\in[t-p,t]} E\|Z_n(\theta)\|^2 \\
\leq\ & \sup_{\theta\in[t-p,t]} E\left(\int_{\theta-n}^{\theta-n+1} \|A(r)U(\theta,r)\nu(r)\|dr\right)^2 \\
\leq\ & \sup_{\theta\in[t-p,t]} E\Bigg(\int_{\theta-n}^{\theta-n+1} \gamma^{-\frac{1}{2}}(r-\theta+n) \\
& \times e^{-\omega(\theta-r)}H(\theta-r)\|\nu(r)\|\gamma^{\frac{1}{2}}(r-\theta+n)dr\Bigg)^2 \\
\leq\ & m_0^{-1}e^{2\delta p}H_s^2\left(\int_{n-1}^{n} e^{-2\omega s}ds\right) \\
& \times\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta-n}^{\theta-n+1}\gamma(r-\theta+n)E\|\nu(r)\|^2dr\right)\right) \\
\leq\ & \left[m_0^{-1}e^{-2\omega(n-p)}\frac{H_s^2(1+e^{2\omega})}{2\omega}\right] \\
& \times\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta-n}^{\theta-n+1}\gamma(r-\theta+n)E\|\nu(r)\|^2dr\right)\right).
\end{aligned}$$

for $q > 0$, we have

$$\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]} E\|Z_n(\theta)\|^2\right)\rho(t)dt$$

$$
\begin{aligned}
\leq\ & \left[m_0^{-1}e^{-2\omega(n-p)}\frac{H_s^2(1+e^{2\omega})}{2\omega}\right] \\
& \times \frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}\left(\int_{\theta-n}^{\theta-n+1}\gamma(r-\theta+n)E\|\nu(r)\|^2 dr\right)\right)\rho(t)dt.
\end{aligned}
$$

Since $\nu^b \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$, according to the inequality above, it follows that

$$
E\|Z_n(t)\|^2 \leq \left[m_0^{-1}e^{-2\omega(n-p)}\frac{H_s^2(1+e^{2\omega})}{2\omega}\right]\|\nu\|^2_{S^2_\gamma}.
$$

Note that $m_0^{-1}e^{2\omega p}\frac{H_s^2(1+e^{2\omega})}{2\omega}\Sigma_{n=1}^{\infty}e^{-2\omega n} < \infty$, therefore, we deduce from the well-known Weierstrass theorem that the series $\Sigma_{n=1}^{\infty}Z_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$
Z(t) := \int_{-\infty}^{t} A(r)U(t,r)\nu(r)dr = \sum_{n=1}^{\infty} Z_n(t),
$$

$Z \in C(\mathbb{R}, L^2(P,\mathbb{H}))$ and $E\|Z(t)\|^2 \leq \Sigma_{n=1}^{\infty}E\|Z_n(t)\|^2 \leq L(m_0,\omega,p,H_s)\|\nu\|^2_{S^2_\gamma}$, where $L(m_0,\omega,p,H_s) > 0$ is a constant. Since $Z_n \in PAA_0(\mathbb{R},\rho)$ and the following inequality

$$
\begin{aligned}
& \frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}E\|Z(\theta)\|^2\right)\rho(t)dt \\
\leq\ & \frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}E\|Z(\theta)-\sum_{k=1}^{n}Z_k(\theta)\|^2\right)\rho(t)dt \\
& +\sum_{k=1}^{n}\frac{1}{m(q,\rho)}\int_{-q}^{q}\left(\sup_{\theta\in[t-p,t]}E\|Z_k(\theta)\|^2\right)\rho(t)dt.
\end{aligned}
$$

Consequently, the uniform limit $Z(t) = \Sigma_{n=1}^{\infty}Z_n(t) \in PAA_0(\mathbb{R},\rho)$. Which completes the proof. □

Next, we consider the existence and uniqueness results of weighted pseudo almost automorphic solutions in distribution to (1.1) with S^2_γ-weighted pseudo almost automorphic coefficients. We make the following assumptions:

(H6) The functions f, g, σ satisfy Lipschitz conditions in the second variable and uniformly in the first variable, that is there exist a positive constant L_f and continuous functions $L_g, L_\sigma : \mathbb{R} \to [0,\infty)$ such that

$$
E\|f(t,\phi_1)-f(t,\phi_2)\|^2 \leq L_f E\|\phi_1-\phi_2\|^2_{\mathcal{B}}
$$

$$
E\|g(t,\phi_1)-g(t,\phi_2)\|^2 \leq L_g(t) E\|\phi_1-\phi_2\|^2_{\mathcal{B}}
$$

$$
E\|\sigma(t,\phi_1)-\sigma(t,\phi_2)\|^2 \leq L_\sigma(t) E\|\phi_1-\phi_2\|^2_{\mathcal{B}}
$$

for all $t \in \mathbb{R}$, $\phi_i \in \mathcal{B}$, $i = 1, 2$.
(H7) For $p > 0$ and every $\chi^b \in PAA_0(\mathbb{R}, L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$ such that

$$\limsup_{r\to\infty} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} L_g(\theta) \right) \rho(t)dt < \infty,$$

$$\lim_{r\to\infty} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} L_g(\theta) \right) \chi(t)\rho(t)dt = 0.$$

$$\limsup_{r\to\infty} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} L_\sigma(\theta) \right) \rho(t)dt < \infty,$$

$$\lim_{r\to\infty} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} L_\sigma(\theta) \right) \chi(t)\rho(t)dt = 0.$$

(H8) The function $f = h_1 + \varphi_1 \in WPAA(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$, $\varphi_1 \in PAA_0(\mathbb{R} \times L^2(P,\mathbb{H}), \rho)$ and $h_1 \in AA(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ is uniformly continuous in any bounded subset $K' \subset L^2(P,\mathbb{H})$ uniformly for $t \in \mathbb{R}$.
(H9) The function $g = h_2 + \varphi_2 \in WPAAS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$, $\varphi_2^b \in PAA_0(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$, and also assume that

$$h_2 \in AS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$$

is uniformly continuous in any bounded subset $K' \subset L^2(P,\mathbb{H})$ uniformly for $t \in \mathbb{R}$.
(H10) The function $\sigma = h_3 + \varphi_3 \in WPAAS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$, $\varphi_3^b \in PAA_0(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(0,1; L^2(P,\mathbb{H}), \gamma ds), \rho)$, and also assume that

$$h_3 \in AS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$$

is uniformly continuous in any bounded subset $K' \subset L^2(P,\mathbb{H})$ uniformly for $t \in \mathbb{R}$.

Theorem 4.1. Fix $\gamma \in \mathbb{U}$ and $\rho \in \mathbb{V}_\infty$. Under the assumptions (H1)-(H10), if

$$\begin{aligned}\Theta := \Re\Big(& 4L_f + 4L_f H_s^2 + \frac{4M^2}{\delta} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s)ds \\ & + 4M^2 \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\delta(t-s)} L_\sigma(s)ds \Big) < 1,\end{aligned}$$

where $H_s = \sup_{s\in[0,\infty)} H(s)$ and $\Re$ is defined as before. Then there exists a unique weighted pseudo almost automorphic solution in distribution to the problem (1.1).

Proof. For any $u(t)$, define the operator Π by

$$(\Pi u)(t) \quad = \quad -f(t,u_t) - \int_{-\infty}^{t} A(s)U(t,s)f(s,u_s)ds + \int_{-\infty}^{t} U(t,s)g(s,u_s)ds$$

$$+\int_{-\infty}^{t} U(t,s)\sigma(s,u_s)dW(s),$$

for each $t \in \mathbb{R}$. It is easy to see that Πu is well defined and continuous. Moreover, from Lemma 2.4 and Lemma 4.1, we have $f(t,u_t) \in WPAA(\mathbb{R}, L^2(P,\mathbb{H}))$, i.e. we can let $f = f_1 + f_2$, where $f_1 \in AA(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H}))$ and $f_2 \in PAA_0(\mathbb{R} \times L^2(P,\mathbb{H}), \rho)$. Then, we suppose $\Pi u(t) = \Pi_1(t) + \Pi_2(t)$, where

$$\Pi_1(t) = f_1(t) + X(t) + \Upsilon(t) + \Phi(t), \quad \Pi_2(t) = f_2(t) + Y(t) + \Psi(t) + Z(t),$$

from the Theorem 3.2, Lemma 4.2 and the proofs of Lemmas 4.3-4.5, we can easily find a stochastic process $\widetilde{\Pi}_1(t) = \widetilde{f}_1(t) + \widetilde{X}(t) + \widetilde{\Upsilon}(t) + \widetilde{\Phi}(t)$ such that

$$\begin{aligned}
& \lim_{n\to\infty} E\|\Pi_1(t+s_n) - \widetilde{\Pi}_1(t)\|^2 \\
\leq\ & \lim_{n\to\infty} 4E\|f_1(t+s_n) - \widetilde{f}_1(t)\|^2 + \lim_{n\to\infty} 4E\|X(t+s_n) - \widetilde{X}(t)\|^2 \\
& + \lim_{n\to\infty} 4E\|\Upsilon(t+s_n) - \widetilde{\Upsilon}_1(t)\|^2 + \lim_{n\to\infty} 4E\|\Phi(t+s_n) - \widetilde{\Phi}_1(t)\|^2 \\
=\ & 0,
\end{aligned}$$

by [19, Remark 2.12], we get that $\Pi_1(t+s_n) \to \widetilde{\Pi}_1(t)$ in distribution as $n \to \infty$. Similarly, we have that $\widetilde{\Pi}_1(t-s_n) \to \Pi_1(t)$ in distribution as $n \to \infty$ for each $t \in \mathbb{R}$.

On the other hand, we obtain that

$$\begin{aligned}
& \lim_{q\to\infty} \frac{1}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} E\|\Pi_2(\theta)\|^2 \right) \rho(t)dt \\
\leq\ & \lim_{q\to\infty} \frac{4}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} E\|f_2(\theta)\|^2 \right) \rho(t)dt \\
& + \lim_{q\to\infty} \frac{4}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} E\|Y(\theta)\|^2 \right) \rho(t)dt \\
& + \lim_{q\to\infty} \frac{4}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} E\|\Psi(\theta)\|^2 \right) \rho(t)dt \\
& + \lim_{q\to\infty} \frac{4}{m(q,\rho)} \int_{-q}^{q} \left(\sup_{\theta\in[t-p,t]} E\|Z(\theta)\|^2 \right) \rho(t)dt \\
=\ & 0,
\end{aligned}$$

that is to say $\Pi_2(t) \in PAA_0(\mathbb{R}, \rho)$. Then by the definition, Πu is weighted pseudo almost automorphic solution in distribution. Next, we prove that Π is a contraction mapping. Indeed, for each u, v, by [29, Theorem 3.9] we can get Π is a contraction mapping. Hence, by Banach fixed point theorem, we can conclude that Π has a unique fixed point such that $\Pi u* = u*$. Then there exists a unique weighted pseudo almost automorphic solution in distribution to the problem (1.1). $\square$

Example 4.1. We consider the following stochastic neutral partial differential equation which is inspired by [30]

$$\begin{cases} d\left[u(t,x)+\displaystyle\int_{-\infty}^{t}\int_{0}^{\pi}b(t-s,\eta,x)u(s,\eta)d\eta ds\right] \\ =\displaystyle\int_{-\infty}^{t}a_2(t-s)u(s,x)dsdW(t) \\ +\left[\dfrac{\partial^2 u(t,x)}{\partial x^2}+a_0(t,x)u(t,x)+\displaystyle\int_{-\infty}^{t}a_1(t-s)u(s,x)ds\right]dt, \\ u(t,0)=u(t,\pi)=0, \quad t\in\mathbb{R}, \quad x\in I=[0,\pi], \end{cases} \tag{4.13}$$

where $a_1, a_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy appropriate conditions, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Take $\mathbb{H} = L^2[0,\pi]$ and $\mathcal{B} = C_0 \times L^p(\rho, \mathbb{H})$. Define the linear operator A by

$$D(A) := \{\varphi \in L^2[0,\pi] : \varphi'' \in L^2[0,\pi], \varphi(0) = \varphi(\pi) = 0\}, \ A\varphi = \varphi'', \forall \varphi \in D(A).$$

It is well-known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $L^2[0,\pi]$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(x) := \sqrt{\frac{1}{2}}\sin(nx)$. Also, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{H}$;

(b) For $\varphi \in \mathbb{H}$, $T(t)\varphi = \Sigma_{n=1}^{\infty}\exp(-n^2 t)\langle\varphi, z_n\rangle z_n$ and $A\varphi = -\Sigma_{n=1}^{\infty} n^2\langle\varphi, z_n\rangle z_n$, for all $\varphi \in D(A)$.

Define the class of operators $A(t)$ by:

$$A(t)\varphi(x) = A\varphi(x) + a_0(t,x)\varphi, \quad \text{for each } \varphi \in D(A(t)) = D(A).$$

By assuming that $x \to a_0(t,x)$ is continuous for each $t \in \mathbb{R}$ with $a_0(t,x) \leq -\delta_0 (\delta_0 > 0)$ for all $t \in \mathbb{R}$, $x \in [0,\pi]$. Clearly, the system

$$\begin{cases} u'(t) = A(t)u(t), \quad t \geq s, \\ u(s) = x \in \mathbb{H}, \end{cases} \tag{4.14}$$

has an associated evolution family $(U(t,s))_{t\geq s}$ on $\mathbb{H}$, which is given by

$$U(t,s)\varphi = T(t-s)\exp\left(\int_s^t a_0(\tau,x)d\tau\right)\varphi.$$

Moreover,

$$\|U(t,s)\| \leq \exp\{-(1+\delta_0)(t-s)\} \ \text{ for every } t \geq s.$$

Further, we define $f : \mathbb{R} \times \mathcal{B} \to \mathbb{H}$, $g : \mathbb{R} \times \mathcal{B} \to \mathbb{H}$ and $\sigma : \mathbb{R} \times \mathcal{B} \to \mathbb{H}$ by

$$f(t,\psi)(x) = \int_{-\infty}^{0}\int_{0}^{\pi} b(\theta,\eta,x)u(\theta,\eta)d\eta d\theta,$$

$$g(t,\psi)(x) = \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds,$$

$$\sigma(t,\psi)(x) = \int_{-\infty}^{0} a_2(s)\psi(s,\xi)ds.$$

In view of the above and from Lemma 3.3, we have

$$f, g, \sigma \in WPAAS^2_\gamma(\mathbb{R} \times L^2(P,\mathbb{H}), L^2(P,\mathbb{H})),$$

then the system (4.13) can be rewritten in the abstract form of (1.1). Hence, the conditions of Theorem 4.1 are satisfied. Therefore, the system (4.13) has a unique weighted pseudo almost automorphic solution in distribution under S^2_γ-weighted pseudo almost automorphic coefficients.

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