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STOCHASTIC AND DELAY ANALYSIS OF TWO PREYS AND ONE PREDATOR ECOLOGICAL SYSTEM WITH COMPETITION AMONG PREYS AND SELF INTERACTION

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ABSTRACT: In this research article, a two prey-one predator system with intra specific competition and self-interaction is investigated and its dynamics are mathematically analyzed. The positivity of the solution and boundedness of the system is studied. The occurrence of possible equilibrium points and stability of the system at those points is examined. The necessary and sufficient condition for the existence of positive interior equilibrium point $E_6(x^*, y^*, z^*)$ is obtained. Also the point $E_6(x^*, y^*, z^*)$ is investigated for the local and global stability of the system. Stability of the delayed model is investigated and it is observed that stability of the system is dependent on time delay. Time delay drives the system from stable to unstable state. The system and its value is described by environmental stochasticity in the form of Gaussian white noise. Numerical simulation is performed to justify the analytical findings.

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1. INTRODUCTION

Mathematical modelling plays a vital role in many fields like ecological science, Applied Mathematics, Economics and Engineering Sciences. It plays a significant role in the novelty of various research works carried out by many scientists. It can simulate and simplify any sort of complex problems in to known tools and methodologies. In particular, by considering and studying dynamical system, modelling shows a suitable path for analysing and concluding some remarkable results effectively. In environmental ecology many authors simulated more structures with the help of modelling tools and studied in various dimensions among species. Mathematical modelling of interacting populations can provide valuable insights into variations of populations over time. Interaction of species may be in the form of competition, predation, parasitism, mutualism and prey-predator relationship. The interaction among species exists universally and so it is a very specific area of interest for mathematicians and biologists. These problems appear simple in the initial stage but are very challenging and complicated. Dynamical modelling of an ecological system is an evolving process. Ordinary differential equation has received significant attention from researchers after the innovative work of Lotka [4] and Volterra [42]. A systematic approach of the persuasive models and exposed discrepancies lead to necessary modification [5]. The basic Lotka-Volterra model was a milestone exploration in the study of predator-prey interaction. Functional response is known to be the key component and the heart of ecological modelling. Holling[7]-[8] classified the functional response into three kinds. In ecological models, the most commonly used functional responses are linear, hyperbolic and sigmoidal.

In author's knowledge, Parrish and Saila [18] were the first to propose a simple mathematical model for a two-prey and one-predator system, motivating the experimental results of Paine [32]. Fuji [22] showed the existence of globally stable limit cycle in the three species even when the equilibrium point is locally unstable.Klebanoff [1] showed the existence of chaotic behaviour in a three component food chain model consisting of one predator and two preys. Kumar [39] have investigated the harvesting of predator species predating over two preys. The constant harvesting rate is treated as control parameter and the system changes its stability to limit cycle, then harvesting exceeds a certain limit. Gakkar and Naji[35] have obtained the existence of chaotic dynamics in the food web comprising of two preys and predator without harvesting.Gakkar and Singh[37] studied the dynamics of food web consisting of two preys and a harvesting predator. They showed that density dependent harvesting on predator controls the chaos in the system Numerous systems with two-preys and one predator have been discussed by researchers (see [9], [15]-[16], [20], [34]-[36], and references cited therein)

Complex dynamical behaviour arises as a consequence of time delay in biological system may exhibit limit cycle oscillations and chaos [24]. The system becomes unstable due to the fluctuations caused in the individual population density by the larger value of gestational time delay. Feng [26] studied the dynamics of a delayed ratio-dependent model with Quadratic harvesting. In general, delay differential equation exhibit much more complicated dynamics than ordinary differential equation since a time delay can cause stable equilibrium to become unstable and then population fluctuate. The significance and application of time-delay in realistic models is elaborated in literature of Gopalsamy [23] and Kuang [43], [44].

The rest of the paper is structured as follows. In Section 2, we formulate a mathematical model with assumption. In Section 3, positivity and boundedness of deterministic model is discussed. Section 4 deals with the existence of equilibrium points with feasible condition. In Section 5, local stability analysis of equilibrium points is discussed. Section 6 deals with global stability analysis of interior equilibrium point $E_6\,(x^*, y^*, z^*)$. In Section 7, we introduce the gestational delay in predator response function. In Section 8, we computed the population intensity of fluctuation due to incorporation of noise which leads to chaos in reality. Numerical simulation of the proposed model is presented in Section 9. The conclusion and discussion is presented in the last section.

2. MATHEMATICAL MODEL AND STEADY STATE ANALYSIS

In this section, a mathematical model consists of two prey and one predator species. The predator exhibits a Holling type II response to one prey and a Holling type-I functional response to the other. It is also assumed that there is interaction between the two preys. One prey species grow logistically and second prey grows exponentially. Self interactionis considered in the second prey population. The model becomes:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - axy - \frac{\lambda_1 xz}{b+\alpha\eta y+x}, \tag{1}$$

$$\frac{dy}{dt} = \beta y - axy - \delta yz - \zeta y^2, \tag{2}$$

$$\frac{dz}{dt} = \frac{\lambda_2 xz}{b+\alpha\eta y+x} + \gamma yz - dz. \tag{3}$$

Assuming $r, K, \beta, c, b, \gamma, \delta, a, \alpha, \eta$ are positive constants. Here x, y denote population densities of prey species and z denote population density of the predator. We assume that the growth rate of first prey is logistic and second prey is exponential. Also assume that there is interaction between the prey species. In model (1-3) r and K denotes intrinsic growth rate, environmental carrying capacity of first prey respectively. a denotes interspecific competition on prey species, β denotes intrinsic growth rate of second prey respectively. d is the death of predator, ζ is self-limitation on second prey species, λ_1 and δ denotes searching efficiency of predator on first prey and second prey respectively, λ_2 and γ are the conversion factors denoting the number of newly born predators for each captured of first and second prey respectively.

$$x(0) \geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0. \tag{4}$$

3. POSITIVE INVARIANCE AND BOUNDEDNESS

Feasibility or biologically positivity studies aim to objectively and rationally uncover the strength of the proposed model in the given environment. Biologically positive insures the population never become negative and population always survive. The following theorems ensure that the positivity and boundedness of the system (1)-(3).

Theorem 1. *All solution* $(x(t), y(t), z(t))$ *of the system* (1)-(3) *with the initial condition* (4) *are positive for all* $t \geq 0$.

Proof. From (1) it is observed that

$$\frac{dx}{x} = \left[r\left(1 - \frac{x}{K}\right) - ay - \frac{\lambda_1 z}{b + \alpha\eta y + x} \right] dt = \phi_1(x, y, z)dt \text{ (say)},$$

where

$$\phi_1(x, y, z) = r\left(1 - \frac{x}{K}\right) - ay - \frac{\lambda_1 z}{b + \alpha\eta y + x}.$$

Integrating in the region $[0, t]$ we get $x(t) = x(0) \exp\left(\int \phi_1(x, y, z)dt\right) > 0$ for all t. From (2) it is observed that

$$\frac{dy}{y} = [\beta - ax - \delta z - \zeta y] \; dt = \phi_2(x, y, z)dt \text{ (say)},$$

where $\phi_2(z) = \beta - ax - \delta z - \zeta y$.

Integrating in the region $[0, t]$ we get $y(t) = y(0) \exp\left(\int \phi_2(x, y, z)dt\right) > 0$ for all t.

From (3) it is observed that

$$\frac{dz}{z} = \left[\frac{\lambda_2 x}{b + \alpha\eta y + x} + \gamma y - d \right] \; dt = \phi_3(x, y)dt \text{ (say)},$$

where $\phi_3(x, y) = \dfrac{\lambda_2 x}{b + \alpha\eta y + x} + \gamma y - d$.

Integrating in the region $[0, t]$ we get $z(t) = z(0) \exp\left(\int \phi_3(x, y)dt\right) > 0$ for all t.

Hence, all solutions starting from interior of the first octant (In R_+^3) remain positive in it for future time.

Theorem 2. *All the non-negative solutions of the model system ((1)-(3)) that initiate in $\Re^3_+$ are uniformly bounded.*

Proof. Let $x(t), y(t), z(t)$ be any solution of the system (1)-(3). Since, from (1) $\frac{dx}{dt} \leq rx(1-\frac{x}{K})$, we have $\lim_{t\to\infty} \sup x(t) \leq K$. Let $L = x+y+z$. Differentiate with respect to t we receive

$$\frac{dL}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}. \tag{5}$$

Substituting the equation ((1)-(3)) in equation (5), we obtain

$$\begin{aligned}\frac{dL}{dt} + \theta L =& x\left((r+\theta) - \frac{rx}{K}\right) + (\theta+\beta)y + z(\theta - d) \\ \leq& x\left((r+\theta) - \frac{rx}{K}\right), \\ \frac{dL}{dt} + \theta L \leq& \mu, \quad \text{since } (K(r+\theta)/r = \mu(say)).\end{aligned}$$

Applying Lemma on differential inequalities Birkoff [11], we obtain

$$0 \leq L(x,y,z) \leq (\mu/\theta)\left(1-e^{-\theta t}\right) + \left(L(x(0), y(0), z(0))/e^{\theta t}\right),$$

and for $t \to \infty$ we have $0 \leq L(x,y,z) \leq (\mu/\theta)$. Thus all solutions of system (1-3) enter into the region

$$\Gamma = \left\{(x,y,z) \in R^3_+ : 0 \leq x \leq K, 0 \leq L \leq (\mu/\theta) + \varepsilon,\ \forall \varepsilon > 0\right\}. \tag{6}$$

This completes the proof.

4. EXISTENCE AND STEADY STATE POINTS WITH FEASIBILITY CONDITIONS

The system 1-3 has seven feasible non negative steady states namely,

(i) $E_0(0,0,0)$ is a trivial steady state.

(ii) $E_1(K,0,0)$ is a axial steady state point on $x-$ axis.

(iii) $E_2\left(0, \frac{\beta}{\zeta}, 0\right)$ is the axial point on $y-$axis.

(iv) $E_3(\bar{x}, \bar{y}, 0)$ is the boundary steady state in $xy-$plane. The equilibrium level describes $\bar{x}$ and $\bar{y}$ are the positive solutions of the following equations

$$\begin{aligned} r\left(1-\frac{x}{K}\right) - ay =& 0, \\ \beta - ax - \zeta y =& 0. \end{aligned}$$

The positive solution is obtained as

$$\bar{x} = \left(\frac{\beta K a^2 - \zeta r a K}{a\left(K a^2 - r\zeta\right)}\right), \quad \bar{y} = \left(\frac{raK - r\beta}{Ka^2 - r\zeta}\right) \text{ with } raK > r\beta,$$

$$Ka^2 > r\zeta \text{ and } a\beta > r\zeta.$$

(v) $E_4\left(\tilde{x}, 0, \tilde{z}\right)$ is the boundary steady state in $xz-$ plane. Here $\hat{x}$ and $\hat{z}$ are the positive solutions of the following equations

$$\begin{aligned} r\left(1 - \frac{x}{K}\right) - \frac{\lambda_1 z}{b + x} =& 0, \\ \frac{\lambda_2 x}{b + x} - d =& 0. \end{aligned}$$

This gives

$$\begin{aligned} \tilde{x} =& \frac{bd}{\lambda_2 - d} \text{ with } \lambda_2 > d, \\ \tilde{z} =& r\left(1 - \frac{\tilde{x}}{K}\right)\left(\tilde{x} + b\right) \text{ with } K > \hat{x}. \end{aligned}$$

(vi) $E_5\left(0, \hat{y}, \hat{z}\right)$ is the boundary equilibrium point in the $yz-$ plane. Here $\hat{y}$ and $\hat{z}$ are the positive solutions of the following equations.

$$\begin{aligned} \beta - \delta z - \zeta y =& 0, \\ \gamma y - d =& 0. \end{aligned}$$

The solution yields that $\hat{y} = \dfrac{d}{\gamma}$, $\hat{z} = \dfrac{\beta\gamma - \zeta d}{\gamma\delta}$, with $\beta\gamma > \zeta d$.

(vii) $E_6\left(x^*, y^*, z^*\right)$ is the interior steady state of the system (1-3) and is obtained by solving the following equations

$$r\left(1 - \frac{x}{K}\right) - ay - \frac{\lambda_1 z}{b + \alpha\eta y + x} = 0, \tag{7}$$

$$\beta - ax - \delta z - \zeta y = 0, \tag{8}$$

$$\frac{\lambda_2 x}{b + \alpha\eta y + x} + \gamma y - d = 0. \tag{9}$$

Eliminating z from (7) and (8) we get

$$f(x, y) = 0, \tag{10}$$

where

$$f(x, y) = r\delta x^2 + Ka\alpha\eta\delta y^2 + x\left(Kay\delta - Ka\lambda_1 + r\delta b - Kr\delta\right)$$

$$+ y\left(Kab\delta - K\zeta\lambda_1 + r\delta\alpha\eta x - Kr\delta\alpha\eta\right) + \left(K\beta\lambda_1 - Kr\delta b\right). \quad (11)$$

From (9) we obtain

$$g(x, y) = 0. \quad (12)$$

Here

$$g(x, y) = \alpha\eta\gamma y^2 + y\left(x\gamma - d\alpha\eta + b\gamma\right) + x\left(\lambda_2 - d\right) - db \quad (13)$$

From (11) as $x \to 0, y \to y_a$:

$$Ka\alpha\eta\delta y^2 + y\left(Kab\delta - K\zeta\lambda_1 - Kr\delta\alpha\eta\right) + \left(K\beta\lambda_1 - Kr\delta b\right). \quad (14)$$

It may be noted that the above equation (14) has unique positive solution $y^* = y_a$ if following inequalities satisfied.

$$Kab\delta < K\zeta\lambda_1 + Kr\delta\alpha\eta; \quad (15)$$

$$K\beta\lambda_1 > Kr\delta b \quad (16)$$

Also from equation (11) we have

$$\frac{dy}{dx} = -\frac{P_1}{Q_1},$$

where $P_1 = 2r\delta x + (Ka\delta y - Ka\lambda_1 + r\delta b - Kr\delta) + r\delta\alpha\eta y$ and

$$Q_1 = 2Ka\alpha\eta\delta y + Ka\delta x + \left(Kab\delta - K\lambda_1\zeta - Kr\delta\alpha\eta + r\delta\alpha\eta x\right).$$

It is clear that $\dfrac{dy}{dx} > 0$ if $P_1 < 0$ and $Q_1 > 0$.

In (13), let $x \to 0, y \to y_b$, then

$$\alpha\eta\delta y^2 + \left(b\gamma - d\alpha\eta\right) y - db = 0.$$

Solving, we receive

$$y_b = \frac{-\left(b\delta - d\alpha\eta\right) \pm \sqrt{\left(b\delta - d\alpha\eta\right)^2 + 4\alpha\eta\delta db}}{2\alpha\eta\delta},$$

provided with $b\delta < d\alpha\eta$.

We also have $\frac{dy}{dx} = \left(\dfrac{\left(\frac{-\partial g}{\partial x}\right)}{\left(\dfrac{\partial g}{\partial y}\right)}\right)$. Let us also note that

$$\frac{dy}{dx} < 0 \text{ if } \frac{\partial g}{\partial x} < 0 \text{ and } \frac{\partial g}{\partial y} < 0 \text{ hold.} \quad (17)$$

From the above analysis, we note two isoclines (11) and (13) intersect at a unique (x^*, y^*) if in addition to condition (15), (16) along with

$$y_a < y_b. \tag{18}$$

Knowing values of x^*, y^*, the value of z^* have to be calculated by

$$z^* = \frac{\beta - ax^* - \zeta y^*}{\delta}.$$

It may be noted that for z^* be positive if

$$\beta > ax^* + \zeta y^*. \tag{19}$$

This completes the existence of $E_6\,(x^*, y^*, z^*)$.

5. LOCAL STABILITY ANALYSIS

In this section, we analysed the local stability of the system (1)-(3) is examined by constructing the jacobian matrix relating to every equilibrium point.

(i) The variational matrix for the equilibrium point at $E_0(0, 0, 0)$

$$J(E_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

The eigenvalues of $J(E_0)$ are $\lambda_1 = r, \lambda_2 = \beta, \lambda_3 = -d$.

Clearly two of the eigenvalue is positive and one will be negative. Hence the equilibrium point E_0 is unstable in $x - y$ direction and stable in z-direction.

(ii) The variational matrix for the equilibrium point at $E_1(K, 0, 0)$

$$J(E_1) = \begin{pmatrix} -r & 0 & \frac{-\lambda_1 K}{b+K} \\ 0 & \beta - aK & 0 \\ 0 & 0 & \frac{\lambda_2 K}{b+K} - d \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are $\lambda_1 = -r, \lambda_2 = \beta - aK, \lambda_3 = \frac{\lambda_2 K}{b+K} - d$.

Clearly, if $\beta < aK$ and $\lambda_2 K < d(b + K)$ all the eigenvalues are negative. Hence the equilibrium point E_1 is locally asymptotically stable in $x - y - z$-direction.

(iii) The variational matrix for the equilibrium point at $E_2(0, \frac{\beta}{\zeta}, 0)$

$$J(E_2) = \begin{pmatrix} r - \frac{a\beta}{\zeta} & 0 & 0 \\ -\frac{a\beta}{\zeta} & -\beta & -\frac{\delta\beta}{\zeta} \\ 0 & 0 & \frac{\gamma\beta}{\zeta} - d \end{pmatrix}.$$

Clearly If $r\zeta < a\beta$ and $\gamma\beta < d\zeta$ all the eigenvalues are negative. In this case the equilibrium point E_2 is locally asymptotically stable in $x-y-z$-direction.

(iv) The variational matrix for the equilibrium point at $E_3\left(\bar{x},\bar{y},0\right)$

$$J\left(E_3\right)=\begin{pmatrix} r-\frac{2r\overline{x}}{K}-a\overline{y} & -a\overline{x} & \frac{-\lambda_1\overline{y}}{b+\alpha\eta\overline{y}+\overline{x}} \\ -a\overline{y} & \beta-a\overline{x}-2\zeta\overline{y} & -\zeta\overline{y} \\ 0 & 0 & \frac{\lambda_2\overline{x}}{b+\alpha\eta\overline{y}+\overline{x}}+\gamma\overline{y}-d \end{pmatrix}.$$

All the eigenvalues of $J\left(E_3\right)$ are negative if it satisfies the condition

$$r\zeta <\beta a, \tag{20}$$

$$\beta <Ka. \tag{21}$$

In this case, the equilibrium point $E_3\left(\bar{x},\bar{y},0\right)$ is locally asymptotically stable.

(v) The variational matrix for the equilibrium point at $E_4\left(\widetilde{x},0,\widetilde{z}\right)$

$$J(E_4)=\begin{pmatrix} r-\frac{2r\tilde{x}}{K}-\frac{\lambda_1\tilde{z}b}{(b+\tilde{x})^2} & -a\tilde{x}+\frac{\lambda_1\tilde{x}\tilde{z}}{(b+\tilde{x})^2} & -\frac{\lambda_1\tilde{x}}{(b+\tilde{x})} \\ 0 & \beta-a\tilde{x}-\delta\tilde{z} & 0 \\ \frac{\lambda_2\tilde{z}b}{(b+\tilde{x})^2} & \frac{\gamma\tilde{z}-\lambda_1\tilde{x}\tilde{z}\alpha\eta}{(b+\tilde{x})^2} & 0 \end{pmatrix}$$

All the eigenvalues of $J(E_4)$ are negative if it satisfies the condition

$$K\left(\lambda_2-d\right)<2bd \tag{22}$$

$$\beta<a\tilde{x}+\delta\tilde{z} \tag{23}$$

In this case, the equilibrium point $E_4\left(\widetilde{x},0,\widetilde{z}\right)$ is locally asymptotically stable.

(vi) The variational matrix for the equilibrium point at $E_5\left(0,\hat{y},\hat{z}\right)$

$$J\left(E_5\right)=\begin{pmatrix} r-\frac{ad}{\gamma}+\frac{\lambda_1\gamma(\zeta d-\beta\gamma)}{\delta(b\gamma+\alpha\eta d)} & 0 & 0 \\ -\frac{ad}{\gamma} & \frac{-\zeta d}{\gamma} & \frac{-\delta d}{\gamma} \\ \frac{-\lambda_2\gamma(\zeta d-\beta\gamma)}{\delta(b\gamma+\alpha\eta d)} & \frac{\beta\gamma-\zeta d}{\delta} & 0 \end{pmatrix}$$

All the eigenvalues of $J(E_5)$ are negative if

$$\left(r\gamma\delta-ad\delta\right)\left(b\gamma+\alpha\eta d\right)<\lambda_1\gamma^2\left(\beta\gamma-\zeta d\right) \tag{24}$$

In this case, the equilibrium point E_5 is locally asymptotically stable.

Theorem 3. *The positive interior equilibrium point $E_6\left(x^*,y^*,z^*\right)$ is asymptotically locally stable if it satisfies the condition $\lambda_1Kx^*z^*<rx^*\left(b+\alpha\eta y^*+x^*\right)^2$.*

Proof. Let the Jacobian matrix of the system (1-3) evaluated at the equilibrium point E_6 be $J\left(E_6\left(x^*, y^*, z^*\right)\right) = \left(a_{ij}\right)_{3\times 3}$, where

$$a_{11} = \frac{-rx^*}{K} + \frac{\lambda_1 x^* z^*}{\left(b + \alpha\eta y^* + x^*\right)^2}, \qquad a_{12} = -ax^* + \frac{\alpha\eta x^* z^*}{\left(b + \alpha\eta y^* + x^*\right)^2}, \tag{25}$$

$$a_{13} = \frac{-\lambda_1 x^*}{\left(b + \alpha\eta y^* + x^*\right)^2}, \qquad a_{21} = -ay^*, a_{22} = -\zeta y^*, \quad a_{23} = -\delta y^* \tag{26}$$

$$a_{31} = \frac{\lambda_2\left(b + \alpha\eta y^*\right) z^*}{\left(b + \alpha\eta y^* + x^*\right)^2}, \qquad a_{32} = \gamma z^* + \frac{\lambda_2 \alpha\eta x^* z^*}{\left(b + \alpha\eta y^* + x^*\right)^2}, \quad a_{33} = 0. \tag{27}$$

Thus the characteristic equation of the Jacobian matrix at E_6 is obtained as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where

$$\begin{aligned} A_1 &= -\left(a_{11} + a_{22}\right), \\ A_2 &= a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}, \\ A_3 &= a_{11}a_{23}a_{32} + a_{13}a_{22}a_{31} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}. \end{aligned}$$

Using Routh-Hurwitz criteria, the condition for local stability of equilibrium point $E_6\left(x^*, y^*, z^*\right)$ is

$$A_1 > 0, \quad A_2 > 0, \quad A_1A_2 - A_3 > 0. \tag{28}$$

Note that if $A_1 > 0$ requires

$$\lambda_1 K x^* z^* < rx^*\left(b + \alpha\eta y^* + x^*\right)^2. \tag{29}$$

Also $A_2 > 0$ and $A_1A_2 - A_3 > 0$ for the condition (29). Thus the interior equilibrium point $E_6\left(x^*, y^*, z^*\right)$ is locally asymptotically stable.

6. GLOBAL STABILITY ANALYSIS

In this section, we investigated the Global stability behaviour of the system (1)-(3) at the interior equilibrium $E_6\left(x^*, y^*, z^*\right)$ by using Lyapunov stability theorem.

Theorem 4. *If* $\frac{x^* z}{z^*} < x < x^*$ *and* $y^* > y$*, then* $E_6\left(x^*, y^*, z^*\right)$ *is globally asymptotically stable.*

Proof. Let us define

$$\begin{aligned} V = L\left(x - x^* - x^*\ln\left(\frac{x}{x^*}\right)\right) + M\left(y - y^* - y^*\ln\left(\frac{y}{y^*}\right)\right) \\ + N\left(z - z^* - z^*\ln\left(\frac{z}{z^*}\right)\right), \end{aligned} \tag{30}$$

where L, M, N are positive constants to be determined. Since the derivative of V along the trajectories of the system (1-3) can be written as

$$\begin{aligned}
\frac{dV}{dt} =& L\left(\frac{x-x^*}{x}\frac{dx}{dt}\right)+M\left(\frac{y-y^*}{y}\frac{dy}{dt}\right)+N\left(\frac{z-z^*}{z}\frac{dz}{dt}\right)\\
=& L\left(r\left(1-\frac{x}{K}\right)-\frac{\lambda_1 z}{b+\alpha\eta y+x}-ay\right)(x-x^*)\\
&+M(\beta-ax-\delta z-\zeta y)(y-y^*)\\
&+N\left(\frac{\lambda_2 x}{b+\alpha\eta y+x}+\gamma y-d\right)(z-z^*)\\
=& L\left(-\frac{r}{K}(x-x^*)-\lambda_1\left(\frac{z}{b+\alpha\eta y+x}-\frac{z^*}{b+\alpha\eta y+x}\right)-a(y-y^*)\right)\\
&(x-x^*)+M(-a(y-y^*)-\zeta(y-y^*)-\delta(z-z^*))(y-y^*)\\
&+N\left(\lambda_2\left(\frac{x}{b+\alpha\eta y+x}-\frac{x^*}{b+\alpha\eta y+x^*}\right)+\gamma(y-y^*)\right)(z-z^*).
\end{aligned}$$

After simple computation we choose $L=\lambda_2$, $M=\frac{\lambda_1\gamma}{\delta}$ and $N=\lambda_1$, then simplifying we get

$$\begin{aligned}
\frac{dV}{dt}=-\lambda_2\frac{r}{K}(x-x^*)^2-\frac{\lambda_2\lambda_1(x^*z-xz^*)(\alpha\eta(y-y^*)+(x-x^*))}{(a+\alpha\eta y+x)(a+\alpha\eta y+x^*)}\\
-(\lambda_2\delta a+a\lambda_1\gamma)(y-y^*)(x-x^*)-\zeta(y-y^*)^2. \quad (31)
\end{aligned}$$

Now $\frac{dV}{dt}<0$ if $\frac{x^*z}{z^*}<x<x^*$ and $y^*>y$.

Then $\frac{dV}{dt}$ is negative definite and consequently,V is a Lyapunov function with respect to all solutions in the interior of the positive octant.

7. DELAY ANALYSIS

Time-delay occurs in any manmade or natural phenomenon. More realistic and importance models of population ecology should be taken into account with the time delay and the stability of an ecological systems with time delays has been studied by many authors [3, 6, 10, 33, 38].In this section we analyze the model system (1-3) with delay τ(discrete time delay in predator response function). Then the model system (1-3) takes the following form

$$\frac{dx}{dt}=rx\left(1-\frac{x}{K}\right)-axy-\frac{\lambda_1 xz}{b+\alpha\eta y+x} \quad (32)$$

$$\frac{dy}{dt}=\beta y-axy-\delta yz-\zeta y^2 \quad (33)$$

$$\frac{dz}{dt} = \frac{\lambda_2 x(t-\tau)z}{b+\alpha\eta y(t-\tau)+x(t-\tau)} + \gamma y(t-\tau)z - dz \tag{34}$$

With the initial densities

$$x(\theta) \geq 0, y(0) \geq 0, z(0) \geq 0, \theta \in (-\tau, 0), \tau \neq 0 \tag{35}$$

The main purpose of this section to study the stability behavior of $E_6(x^*, y^*, z^*)$ in the presence of discrete delay ($\tau \neq 0$). Now to prove the stability behavior of $E_6(x^*, y^*, z^*)$ for the system (32-34), first we linearise the system (32-34) by using following transformation,

$$x(t) = x^* + x_1(t)$$
$$y(t) = y^* + y_1(t)$$
$$z(t) = z^* + z_1(t)$$

The linear system is given by

$$\dot{x_1}(t) = a_{11}x_1(t) + a_{12}y_1(t) + a_{13}z_1(t)$$
$$\dot{y_1}(t) = a_{21}x_1(t) + a_{22}y_1(t) + a_{23}z_1(t)$$
$$\dot{z_1}(t) = c_{31}x_1(t-\tau) + c_{32}y_1(t-\tau) + a_{33}z_1(t)$$

$$\begin{aligned}
a_{11} &= \tfrac{-rx^*}{K} + \tfrac{\lambda_1 x^* z^*}{(b+\alpha\eta y^*+x^*)^2}, & a_{12} &= \tfrac{\lambda_1 \alpha\eta x^* z^*}{(b+\alpha\eta y^*+x^*)^2} - ax^*,\\
a_{13} &= \tfrac{-\lambda_1 x^*}{(b+\alpha\eta y^*+x^*)}, & a_{21} &= -ay^* a_{22} = -\zeta y^*,\\
a_{23} &= -\delta y^*, & c_{31} &= \tfrac{\lambda_2(b+\alpha\eta y^*)z^*}{(b+\alpha\eta y^*+x^*)^2},\\
c_{32} &= \tfrac{\lambda_2 \alpha\eta x^* z^*}{(b+\alpha\eta y^*+x^*)^2} + \gamma z^*, & a_{33} &= -d
\end{aligned}$$

We look for solution of the model (32-34) of the form $A(\tau) = \rho e^{-\lambda\tau}$,$\rho \neq 0$ this leads to the characteristic equation

$$\begin{aligned}\Delta(\lambda, \tau) &= \left(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3\right)\\ &\quad + (p_4\lambda + p_5)e^{-\lambda\tau} = 0\end{aligned} \tag{36}$$

where

$$\begin{aligned}
&p_1 = -a_{11} - a_{22} - a_{33}, \quad p_2 = a_{11}a_{22} - a_{21}a_{12} + a_{11}a_{33} + a_{33}a_{22},\\
&p_3 = a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33}, \ p_4 = -a_{13}c_{31} - a_{23}c_{32},\\
&p_5 = a_{13}a_{22}c_{31} + a_{23}a_{11}c_{32} - a_{12}a_{23}c_{31} - a_{11}a_{21}c_{32}
\end{aligned}$$

The eigenvalues are the roots of the characteristic equation (36) of the system (32-34) that has infinitely many solutions. We wish to find periodic solution of the system (32-34), for the periodic solution eigenvalues will be purely imaginary. Substituting $\lambda = i\omega$ in equation (36) we get

$$\left[-i\omega^3 - p_1\omega^2 + ip_2\omega + p_3\right] + \left[ip_4\omega + p_5\right]e^{-i\omega\tau} = 0$$

Comparing real and imaginary parts, we get

$$p_1\omega^2 - p_3 = (p_5 cos\omega\tau + \omega p_4 sin\omega\tau)$$

$$p_2\omega - \omega^3 = -\omega p_4 cos\omega\tau + (p_5 sin\omega\tau)$$

Squaring and adding we get

$$\omega^6 + S_1\omega^4 + S_2\omega^2 + S_3 = 0 \tag{37}$$

where

$$S_1 = p_1^2 - 2p_2, S_2 = p_2^2 - 2p_3p_5 - p_4^2, S_3 = p_3^2 - p_5^2$$

putting $\omega^2 = \delta$ in equation (37) we get

$$f(\delta) = \delta^3 + S_1\delta^2 + S_2\delta + S_3 = 0 \tag{38}$$

Now equation (38) will be positive if

$$S_1 > 0, S_3 < 0 \tag{39}$$

By Descartes rule of sign, the cubic equation (39), has at least one positive root. Consequently the stability criteria of the system for τ=0, will not necessarily ensure the stability of system for $\tau \neq 0$.

The critical value of delay that is given as

$$\cos\omega\tau = \frac{(\omega^4(p_4) - p_5p_3) + \omega^2(p_1p_5 - p_2p_4)}{(p_5^2 + p_4^2\omega^2)}$$

So corresponding to $\lambda = i\omega_0$ there exists τ^* such that

$$\tau_{0n}^* = \frac{1}{\omega_0}\left[\cos^{-1}\left[\frac{(\omega^4(p_4) - p_5p_3) + \omega^2(p_1p_5 - p_2p_4)}{(p_5^2 + p_4^2\omega^2)}\right]\right] + \frac{2n\pi}{\omega_0}, n = 0, 1, 2, 3.......$$

8. HOPF BIFURCATION

We observe that the condition's for Hopf bifurcation (Hale [19]) are satisfied yielding the required periodic solution, that is

$$\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_0} \neq 0$$

This signifies that there exists at least one eigenvalue with positive real part for $\tau > \tau^*$.Now, we show the existence of Hopf bifurcation near $E_6(x^*, y^*, z^*)$ by taking τ as bifurcating parameter.

Differentiating equation (36) with respect to τ

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2+2p_1\lambda+p_2}{\lambda(p_4\lambda+p_5)e^{-\lambda\tau}} + \frac{p_4}{\lambda\,(p_4\lambda+p_5)} - \frac{\tau}{\lambda}$$

$$= \frac{2\lambda^3+p_1\lambda^2-p_3-(p_4\lambda+p_5)e^{-\lambda\tau}}{\lambda^2(p_4\lambda+p_5)e^{-\lambda\tau}} + \frac{p_4\lambda}{\lambda^2\,(p_4\lambda+p_5)} - \frac{\tau}{\lambda}$$

$$= \frac{(2\lambda^3+p_1\lambda^2-p_3)}{\lambda^2(p_4\lambda+p_5)e^{-\lambda\tau}} - \frac{1}{\lambda^2} + \frac{p_4\lambda}{\lambda^2\,(p_4\lambda+p_5)} - \frac{\tau}{\lambda}$$

$$= \frac{(2\lambda^3+p_1\lambda^2-p_3)}{-\lambda^2(\lambda^3+p_1\lambda^2+p_2\lambda+p_3)} + \frac{-p_5}{\lambda^2\,(p_4\lambda+p_5)} - \frac{\tau}{\lambda}$$

Taking $\lambda = i\omega_0$ in above equation, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} = \frac{2(i\omega_0)^3+p_1\,(i\omega_0)^2-p_3}{-(i\omega_0)^2\left((i\omega_0)^3+p_1(i\omega_0)^2+p_2\,(i\omega_0)+p_3\right)}$$

$$+\frac{-p_5}{(i\omega_0)^2\,(p_4\,(i\omega_0)+p_5)} + \frac{i\tau}{\omega_0}$$

$$= \left[\frac{-[(p_1\omega_0^2)+2i\omega_0^3+p_3]}{\omega_0^2[(p_3-p_1\omega_0^2)+i\,(p_2w_0-w_0^3)}.\frac{(p_3-p_1\omega_0^2)-i\,(p_2w_0-w_0^3)}{(p_3-p_1\omega_0^2)-i\,(p_2w_0-w_0^3)}\right]$$

$$+\left[\frac{p_5}{\omega_0^2\,((p_5+ip_4\omega_0)}.\frac{(p_5-ip_4\omega_0}{(p_5-ip_4\omega_0}\right] + \frac{\tau i}{w_0}$$

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=iw_0} = \left[\frac{2\omega_0^3\,(\omega_0^3-p_2\omega_0)-((p_1\omega_0^2)^2-p_3^2)}{\left[\omega_0^2\,(p_3-p_1\omega_0^2)^2+(p_2\omega_0-\omega_0^3)^2\right]}\right]$$

$$+\frac{(p_5)^2}{\omega_0^2[(p_5)^2+p_4^2\omega_0^2]}$$

Thus, we obtain $Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} > 0$. Therefore transversity condition holds and hence Hopf bifurcation occurs at $\tau=\tau^*$. This signifies that there exits at least or equal value with positive real part for $\tau > \tau^*$

Theorem 5. *If E_6 exists with the condition* (29) *and $\delta = \omega_0^2$ be positive root of* (37)*,then there exists a $\tau = \tau^*$ such that*

(i) E_6 is locally asymptotically stable for $0 \le \tau < \tau^*$

(ii) E_6 is unstable for $\tau > \tau^*$

(iii) The system (32-34) undergoes a Hopf –bifurcation around E_6 at $\tau=\tau^*$,$\tau^* =$ $\min h(\omega_0)$

where

$$h(\omega_0) = \tau^* = \frac{1}{\omega_0}\left[\cos^{-1}\left[\frac{(\omega^4(p_4) - p_5p_3) + \omega^2(p_1p_5 - p_2p_4)}{(p_5^2 + p_4^2\omega^2)}\right]\right]$$
$$+\frac{2n\pi}{\omega_0}, n = 0, 1, 2, 3...,$$

and the minimum taken over all positive ω_0 *such that* $\delta = \omega_0^2$ *is a solution of* (37).

9. STOCHASTIC ANALYSIS

External noise may arise from random fluctuations of finite number of parameters around some known mean values of the population densities. Since the aquatic ecosystem which always has unsystematic fluctuations of the environment, it is difficult to define the usual phenomenon as a deterministic ideal. The stochastic investigation benefits us to get an extra intuition about the continuous changing aspects of any ecological unit. Numerous examples of analysis of stochastic model by the researches [2, 5, 12, 13,17,21,27-31].

This section is meant for the extension of the deterministic model of [14], which is formed by adding noisy term. There are several ways in which environmental noise may be incorporated in an ecological system. The deterministic model given by [14] is extended with the effect of random noise of the environmental results in a stochastic system given below.

$$x'(t) = rx\left(1 - \frac{x}{K}\right) - axy - \frac{\lambda_1 xz}{b + \alpha\eta y + x} + \alpha_1\xi_1(t) \tag{40}$$

$$y'(t) = \beta y - axy - \delta yz - \zeta y^2 + \alpha_2\xi_2(t) \tag{41}$$

$$z'(t) = \frac{\lambda_2 xz}{b + \alpha\eta y + x} + \gamma yz - dz + \alpha_3\xi_3(t) \tag{42}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the real constants and $\xi_i(t) = [\xi_1(t), \xi_2(t), \xi_3(t)]$ is a three dimensional Gaussian white noise process satisfying $E(\xi_i(t)) = 0$; $i = 1, 2, 3$; $E[\xi_i(t)\xi_j(t)] = \delta_{ij}\delta(t - t')$; $i = j = 1, 2, 3$ where δ_{ij} is the Kronecker delta function; δ is the Dirac delta function. Where δ_{ij} is the Kronecker symbol; δ is the δ-dirac function. All other parameters have their usual meanings (see Section 1)

$$Let\ \ x(t) = u_1(t) + S^*\,; y(t) = u_2(t) + P^*\,; z(t) = u_3(t) + T^*. \tag{43}$$

Then

$$x'(t) = u_1^{'}(t);\ y'(t) = u_2^{'}(t); z'(t) = u_3^{'}(t) \tag{44}$$

Using (43) and (44), the linear parts of (40-42) are

$$u_1^{'}(t) = -\frac{r}{K}u_1(t)S^* - au_2(t)S^* - \lambda_1 u_3(t)S^* + \alpha_1\xi_1(t) \quad (45)$$

$$u_2^{'}(t) = -au_1(t)P^* - \zeta u_2(t)P^* - \delta u_3(t)P^* + \alpha_2\xi_2(t) \quad (46)$$

$$u_3^{'}(t) = \lambda_2 u_1(t)T^* + \gamma u_2(t)T^* + \alpha_3\xi_3(t) \quad (47)$$

Taking the Fourier transform on both sides of (45-47) we get,

$$\left(i\omega + \frac{rS^*}{k}\right)\tilde{u}_1(\omega) + aS^*\tilde{u}_2(\omega) + \lambda_1 S^*\tilde{u}_3(\omega) = \alpha_1\tilde{\xi}_1(\omega) \quad (48)$$

$$aP^*\tilde{u}_1(\omega) + (i\omega + \zeta P^*)\tilde{u}_2(\omega) + \delta P^*\tilde{u}_3(\omega) = \alpha_2\tilde{\xi}_2(\omega) \quad (49)$$

$$-\lambda_2 T^*\tilde{u}_1(\omega) - \gamma T^*\tilde{u}_2(\omega) + i\omega\tilde{u}_3(\omega) = \alpha_3\tilde{\xi}_3(\omega) \quad (50)$$

The matrix form of (48)-(50) is

$$M(\omega)\tilde{u}(\omega) = \tilde{\xi}(\omega) \quad (51)$$

where $M(\omega) = \begin{pmatrix} A_1(\omega) & B_1(\omega) & C_1(\omega) \\ A_2(\omega) & B_2(\omega) & C_2(\omega) \\ A_3(\omega) & B_3(\omega) & C_3(\omega) \end{pmatrix}$; $\tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \\ \tilde{u}_3(\omega) \end{bmatrix}$;
$\tilde{\xi}(\omega) = \begin{bmatrix} \alpha_1\tilde{\xi}_1(\omega) \\ \alpha_2\tilde{\xi}_2(\omega) \\ \alpha_3\tilde{\xi}_3(\omega) \end{bmatrix}$; where

$$A_1(\omega) = i\omega + \frac{rS^*}{K}, B_1(\omega) = aS^*, C_1(\omega) = \lambda_1 S^*, A_2(\omega) = aP^*,$$
$$B_2(\omega) = i\omega + \zeta P^*, C_2(\omega) = \delta P^*, A_3(\omega) = -\lambda_2 T^*, B_3(\omega) = -\gamma T^*,$$
$$C_3(\omega) = i\omega$$

Equation (51) can also be written as

$$\tilde{u}(\omega) = [M(\omega)]^{-1}\tilde{\xi}(\omega) \quad (52)$$

where

$$[M(\omega)]^{-1} = \frac{1}{R(\omega) + iI(\omega)}\begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix} \quad (53)$$

and where $D_1 = -\omega^2 + i\zeta\omega P^* + \gamma\delta P^* T^*$, $D_2 = -ia\omega S^* - \lambda_1\gamma S^* T^*$,
$D_3 = a\delta S^* P^* - i\lambda_1\omega S^* - \zeta\lambda_1 S^* P^*$,
$E_1 = -ia\omega P^* - \lambda_2\delta T^* P^*$,$E_2 = -\omega^2 + \frac{i\omega r S^*}{K} + \lambda_1\lambda_2 S^* T^*$,

$$E_3 = -i\omega\delta P^* - \frac{r\delta S^* P^*}{K} + a\lambda_1 S^* P^*, F_1 = -a\gamma P^* T^* + i\omega\lambda_2 T^* + \zeta\lambda_2 P^* T^*,$$

$$F_2 = i\omega\gamma T^* + \frac{\gamma r S^* T^*}{K} - a\lambda_2 S^* T^*,$$
$$F_3 = \omega^2 - i\omega\zeta P^* - \frac{i\omega r S^*}{K} - \frac{r\zeta S^* P^*}{K} + a^2 S^* P^*$$

Here $|D_1|^2 = X_1^2 + Y_1^2$; $|D_2|^2 = X_2^2 + Y_2^2$; $|D_3|^2 = X_3^2 + Y_3^2$;
$|E_1|^2 = X_4^2 + Y_4^2$; $|E_2|^2 = X_5^2 + Y_5^2$; $|E_3|^2 = X_6^2 + Y_6^2$;
$|F_1|^2 = X_7^2 + Y_7^2$; $|F_2|^2 = X_8^2 + Y_8^2$; $|F_3|^2 = X_9^2 + Y_9^2$;
where $X_1 = -\omega^2 + \gamma\delta P^* T^*$; $Y_1 = \zeta\omega P^*$;$X_2 = -\lambda_1\gamma S^* T^*$; $Y_2 = -a\omega S^*$;$X_3 = a\delta S^* P^* - \zeta\lambda_1 S^* P^*$;$Y_3 = -\lambda_1\omega S^*$;$X_4 = -\lambda_2\delta T^* P^*$; $Y_4 = -a\omega P^*$;$X_5 = -\omega^2 + \lambda_1\lambda_2 S^* T^*$;$Y_5 = \dfrac{\omega r S^*}{K}$;$X_6 = -\frac{r\delta S^* P^*}{K} + a\lambda_1 S^* P^*$; $Y_6 = -\omega\delta P^*$;$X_7 = -a\gamma P^* T^* + \zeta\lambda_2 P^* T^*$; $Y_7 = \omega\lambda_2 T^*$; $X_8 = \frac{\gamma r S^* T^*}{K} - a\lambda_2 S^* T^*$; $Y_8 = \omega\gamma T^*$;

$$X_9 = \omega^2 - \frac{r\zeta S^* P^*}{K} + a^2 S^* P^*; Y_9 = \omega\zeta P^* - \frac{\omega r S^*}{K} \tag{54}$$

$|M(\omega)|^2 = [R(\omega)]^2 + [I(\omega)]^2$ where $R(\omega) = -\zeta\omega^2 P^* - \dfrac{\omega^2 r S^*}{K} + \dfrac{\delta\gamma r S^* P^* T^*}{K} + a\lambda_2\delta S^* P^* T^* - a\lambda_1\gamma S^* P^* T^* + \zeta\lambda_1\lambda_2 S^* P^* T^*$ and $I(\omega) = \omega^3 + \gamma\delta\omega P^* T^* + \frac{r\zeta\omega S^* P^*}{K} + a^2\omega S^* P^* + \lambda_1\lambda_2\omega S^* T^*$.

If the function $Y(t)$ has a zero mean value, then the fluctuation intensity (variance) of its components in the frequency interval $[\omega, \omega + d\omega]$ is $S_Y(\omega)d\omega$. where $S_Y(\omega)$ is spectral density of Y and is defined as

$$S_Y(\omega) = \lim_{\tilde{T}\to\infty} \frac{\left|\tilde{Y}(\omega)\right|^2}{\tilde{T}}. \tag{55}$$

If Y has a zero mean value, the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)\, e^{i\omega\tau} d\omega \tag{56}$$

The corresponding variance of fluctuations in $Y(t)$ is given by

$$\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega) d\omega \tag{57}$$

and the auto correlation function is the normalized auto covariance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)} \tag{58}$$

For a Gaussian white noise process, it is
$S_{\xi_i\xi_j}(\omega)$

$$\begin{aligned} &= \lim_{\tilde{T}\to+\infty} \frac{E\left[\tilde{\xi}_i(\omega)\,\tilde{\xi}_j(\omega)\right]}{\tilde{T}} \\ &= \lim_{\hat{T}\to+\infty} \frac{1}{\hat{T}} \int_{-\frac{\tilde{T}}{2}}^{\frac{\tilde{T}}{2}} \int_{-\frac{\tilde{T}}{2}}^{\frac{\tilde{T}}{2}} E\left[\tilde{\xi}_i(t)\,\tilde{\xi}_j\left(t'\right)\right] e^{-i\omega(t-t')} dt\, dt' = \delta_{ij} \end{aligned} \tag{59}$$

From (53), we have

$$\tilde{u}_i(\omega) = \sum_{j=1}^{3} K_{ij}(\omega)\ \tilde{\xi}_j(\omega)\ ;\ i = 1, 2, 3. \tag{60}$$

From (56) we have

$$S_{u_i}(\omega) = \sum_{j=1}^{3} \eta_j\ |K_{ij}(\omega)|^2\ ;\ i = 1, 2, 3 \tag{61}$$

where $K_{ij}(\omega) = [M(\omega)]^{-1}$.

Hence by (60) and (61), the intensities of fluctuations in the variable u_i ; $i = 1, 2, 3$ are given by

$$\sigma_{u_i}^2 = \frac{1}{2\pi} \sum_{j=1}^{3} \int_{-\infty}^{\infty} \eta_j\, |K_{ij}(\omega)|^2\, d\omega, \quad i = 1, 2, 3, \tag{62}$$

and from (53), (54), (62) we obtain:

$$\sigma_{u_1}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)}\left[\alpha_1\left(X_1^2 + Y_1^2\right) + \alpha_2\left(X_2^2 + Y_2^2\right) + \alpha_3\left(X_3^2 + Y_3^2\right)\right] d\omega\right\}, \tag{63}$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)}\left[\alpha_1\left(X_4^2 + Y_4^2\right) + \alpha_2\left(X_5^2 + Y_5^2\right) + \alpha_3\left(X_6^2 + Y_6^2\right)\right] d\omega\right\}, \tag{64}$$

$$\sigma_{u_3}^2 = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)}\left[\alpha_1\left(X_7^2 + Y_7^2\right) + \alpha_2\left(X_8^2 + Y_8^2\right) + \alpha_3\left(X_9^2 + Y_9^2\right)\right] d\omega\right\}, \tag{65}$$

where $|M(\omega)| = R(\omega) + iI(\omega)$.

If we are interested in the dynamics of system (40)-(42) with either $\alpha_1 = 0$ or $\alpha_2 = 0$ or $\alpha_3 = 0$, then the population variances are:

If $\alpha_1 = 0$, $\alpha_2 = 0$, then $\sigma_{u_1}^2 = \frac{\alpha_3}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_3^2+Y_3^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$; $\sigma_{u_2}^2 = \frac{\alpha_3}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_6^2+Y_6^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$, $\sigma_{u_3}^2 = \frac{\alpha_3}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_9^2+Y_9^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$.

If $\alpha_2 = 0$, $\alpha_3 = 0$, then $\sigma_{u_1}^2 = \frac{\alpha_1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_1^2+Y_1^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$, $\sigma_{u_2}^2 = \frac{\alpha_1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_4^2+Y_4^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$, $\sigma_{u_3}^2 = \frac{\alpha_1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_7^2+Y_7^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$.

If $\alpha_3 = 0$, $\alpha_1 = 0$, then $\sigma_{u_1}^2 = \frac{\alpha_2}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_2^2+Y_2^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$, $\sigma_{u_2}^2 = \frac{\alpha_2}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_5^2+Y_5^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$;, $\sigma_{u_3}^2 = \frac{\alpha_2}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\left(X_8^2+Y_8^2\right)}{R^2(\omega)+I^2(\omega)}\, d\omega$.

The equations in (63)-(65) give three variations of inhabitants. The integrations over the real line can be estimated which gives the variations of inhabitants.

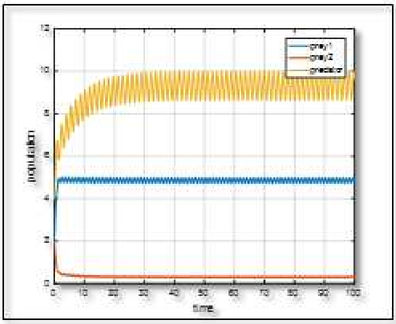

Figure 1: Represents the variations of populations against time

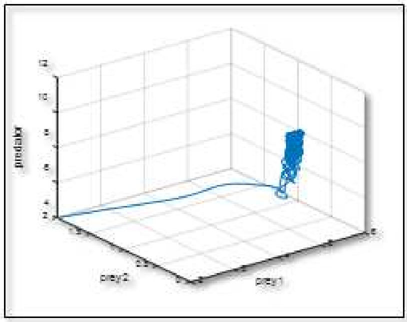

Figure 2: Represents phase portrait diagram among species

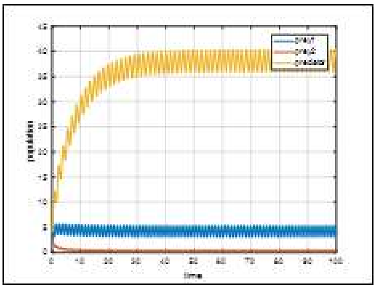

Figure 3: Represents the variations of populations against time

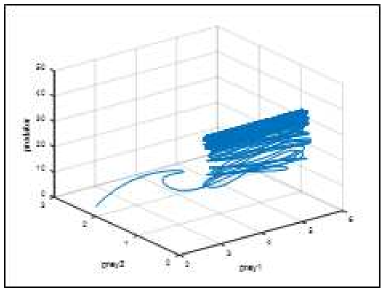

Figure 4: Represents phase portrait diagram among species

10. NUMERICAL SIMULATIONS

In this segment we validate our analytical findings through numerical simulations by using MATLAB software considering the following parameters:

Example 1. For the parameters $\mathrm{r} = 3.5; \mathrm{K} = 1.5; \mathrm{a} = 0.41; \mathrm{b} = 0.7; \lambda_2 = 0.9; \lambda_1 = 1.2; \alpha = 0.1; \beta = 0.7; \gamma = 0.36; \delta = 0.6; \eta = 0.2; \mathrm{d} = 0.2; \zeta = 0.25$; with $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3$

Example 2. For the parameters $\mathrm{r} = 3.5; \mathrm{K} = 1.5; \mathrm{a} = 0.41; \mathrm{b} = 0.7; \lambda_2 = 0.9; \lambda_1 = 1.2; \alpha = 0.1; \beta = 0.7; \gamma = 0.36; \delta = 0.6; \eta = 0.2; \mathrm{d} = 0.2; \zeta = 0.25$;with $\alpha_1 = 8,\ \alpha_2 = 9,\ \alpha_3 = 10$,

Example 3. For the parameters $\mathrm{r} = 3.5; \mathrm{K} = 1.5; \mathrm{a} = 0.41; \mathrm{b} = 0.7; \lambda_2 = 0.9; \lambda_1 = 1.2; \alpha = 0.1; \beta = 0.7; \gamma = 0.36; \delta = 0.6; \eta = 0.2; \mathrm{d} = 0.2; \zeta = 0.25$;with $\alpha_1 = 10,\ \alpha_2 = 20,\ \alpha_3 = 30$.

Example 4. For the parameters $\mathrm{r} = 1.5; \mathrm{K} = 1.23; \mathrm{a} = 0.41; \mathrm{b} = 0.7; \lambda_2 = 0.9; \lambda_1 = 1.2; \alpha = 0.1; \beta = 0.7; \gamma = 0.36; \delta = 0.6; \eta = 0.2; \mathrm{d} = 0.2; \zeta = 0.3$;and initial densi-

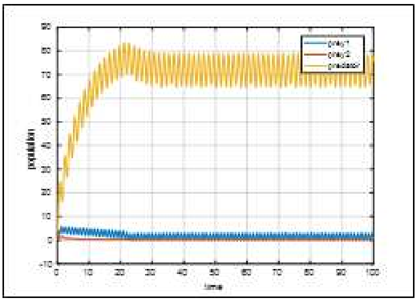

Figure 5: Represents the variations of populations against time

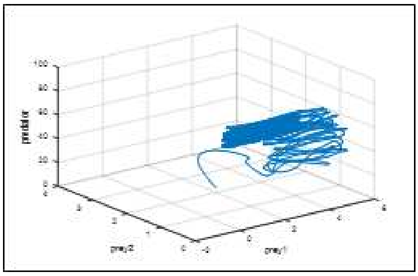

Figure 6: Represents phase portrait diagram among species

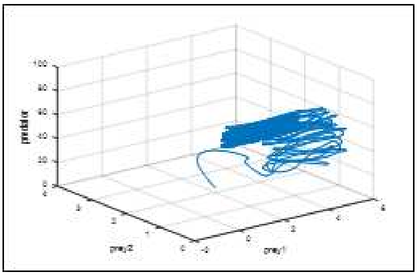

Figure 7: Represents the variations of populations against time

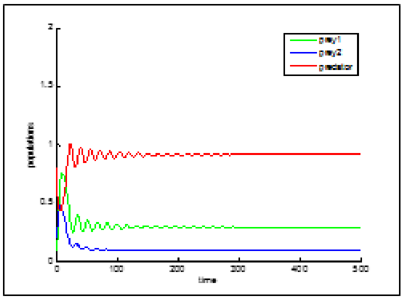

Figure 8: Represents phase portrait diagram among species

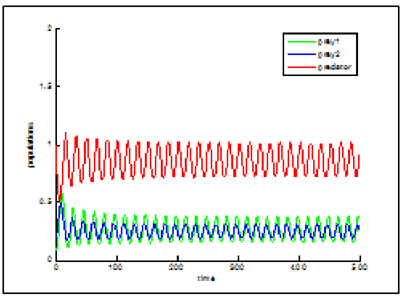

Figure 9: Represents the variations of populations against time

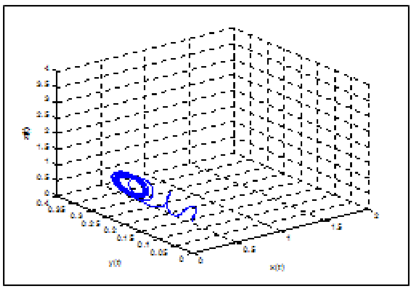

Figure 10: Represents phase portrait diagram among species

ties[0.1 0.20 0.80]

Example 5. For the parametersr $= 1.5; \mathrm{K} = 1.23; \mathrm{a} = 0.41; \mathrm{b} = 0.7; \lambda_2 = 0.9; \lambda_1 = 1.2; \alpha = 0.1; \beta = 0.7; \gamma = 0.36; \delta = 0.6; \eta = 0.2; \mathrm{d} = 0.2; \zeta = 0.3;$

Example 6. For the parameters $\mathrm{r} = 1.5; \mathrm{K} = 1.23; \mathrm{a} = 0.41; \mathrm{b} = 0.7; \lambda_2 = 0.9; \lambda_1 = 1.2; \alpha = 0.1; \beta = 0.7; \gamma = 0.36; \delta = 0.6; \eta = 0.2; \mathrm{d} = 0.2; \zeta = 0.3;$

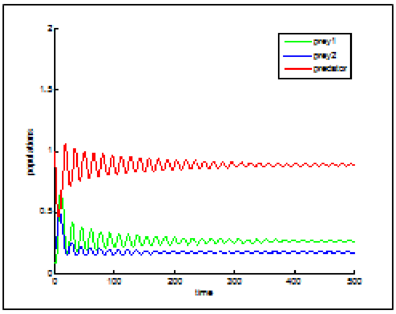

Figure 11: Represents the variations of populations against time

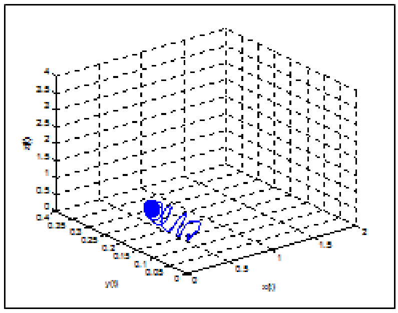

Figure 12: Represents phase portrait diagram among species

11. CONCLUDING REMARKS

In this article, we checked the positivity, boundedness, existence of equilibrium points with feasible condition of deterministic model is discussed. We also analysed local & global stabilities about steady states from figures (8)-(12). It is verified that the impact of the gestational delay in predator response function. The stability criteria in the absence of delay ($\tau = 0$) will not necessarily guarantee the stability of the system in presence of delay ($\tau \neq 0$). For the above choice of Example-5 there is a unique positive root of the equation for which Hopf-bifurcation occurs $\tau = \tau^* = 4.65$ (see Fig.9 and 10).Therefore By theorem 5 $E_6(x^*, y^*, z^*)$ loses its stability as τ passes through critical value of τ^*. We verify that $\tau = 4.5 < \tau^*$,E_6 is locally asymptotically stable (see Fig.7 and 8), Keeping other parameter fixed, if we take $\tau = 4.83 > \tau^*$, it is seen that E_6 is unstable and there is bifurcating periodic solution near E_6(See Fig 12) Periodic oscillations of x, y and z in finite time are shown in Fig (11).Thus using the time delay as control, it is possible to break stable behaviour of system and drive it to an unstable state. Also it is possible to keep population at a desired level. Also we computed the population intensity of fluctuation due to incorporation of noise which leads to chaos in reality. In this paper we have studied stochastic stability of two prey and one predator (by inclusion of self-interaction prey species and competition between prey species) of around interior steady state.We also conclude that the inclusion of stochastic perturbation creates a significant change in the intensity of populations due to change of responsive parameters causes chaotic dynamics with low, medium and high variances of oscillations from figures (1), (2), (3), (4), (5), (7).

REFERENCES

[1] A. Klebanoff, and A.Hastings, *Chaos in one-predator, two-prey models; general*

results from bifurcation theory, Math, Bioscience, **122 (2)**, 221, 1994.

[2] A. Maiti, M.M. Janaand, G.P. Samanta, *Deterministic and stochastic analysis of a ratio dependent predator-prey system with delay*, Nonlinear Anal. Model. Control, **12 (3)**, 383–398, 2007.

[3] A.F.Nindjin, M.A. Aziz –Alaoui, *Analysis of a predator-prey model with modified Leslie-Gower and Holling type II schemes with time delay*, Nonlinear Anal: Real World Appl. **7**, 1104-1118, 2006.

[4] A.J. Lotka, Elements of Physical Biology. Baltimore: Williams and Willkins, 1925.

[5] A.Maiti, G.P. Samanta, *Deterministic and stochastic analysis of ratio dependent prey-predator system*, Int. J. syst. sci, **37**, 317-326, 2006.

[6] B.Patra, A.Maiti, Samantha, *Effect of time delay on a Ration dependent food chain model nonlinear analysis modeling and control*, **14**, 199-216, 2009.

[7] C.S.Holling, *Some characteristic of simple types of predation and parasitism*, Canadian Entomologist, **91**, 385-398, 1959.

[8] C.S.Holling, *The functional response of predators to prey density and its role in mimicry and population regulations*, Mem.Entomol.Soc Can **97**, 5-60, 1965.

[9] D. Xiao,S. Ruan, *Global dynamics of a ratio-dependent predator-prey system*, Journal of Mathematical Biology, **43** 268-290, 2001.

[10] F.Karagoglu, H.Merdan, *Hopf Bifurcation of a ratio-dependent predator-prey model involving two discrete maturation time delays haos*, Solit., Fract., **68**, 159-168, 2014.

[11] G.Birkoff, G.C.Rota, *Ordinary differential equation*, Ginn, Bosta, 1982.

[12] G.P. Samanta, *Stochastic analysis of a prey-predator system*, Int. J. Math. Edu. Sci. Technol., **25 (6)**, pp.797–803, 1994.

[13] G.P. Samanta, S.Bera, A.Maiti,. *Stochastic analysis of prey-predator model with herd behaviour prey*, Nonlinear analysis, modelling and control, **21 (3)**, 345-361, 2015

[14] G.P.Samantha, Sharma, *Dynamical behavior of a two prey and one predator system*, Differential equations and dynamical systems App; **22**, 125-145, 2014.

[15] H.Beak, Y.Do and D. Lim,(2011), *A three species food chain system with two types of functional response*, Abstract and Applied Analysis, **1085/3375**, 155-159, 2011.

[16] H.I. Freedman and PWaltman, *Mathematical analysis of some three species food chain models*, Mathematical Bioscience, **33 (3/4)**, 257-276, 1977.

[17] I.B.Schwartz, L.Billings, T.W.Carr, M.I.Dykman, *Noise induced switching and extinction of the systems in delay*, Phys., Rev., E 91, 012139, 2015.

[18] J.D.Parrish. and S.B. Saila, *Inter specific competition, predation and species diversity*, Journal of Theoretical Biology, **27 (2)**, 207-220, 1970.

[19] J.K. Hale, *Ordinary differential equation*, John Willey and sons, New York, 1969.

[20] J.P. Tripathi, S. Abbas, M. Thauker, *Stability analysis of two prey one predator model*, AIP control, 1479-95, 2012.

[21] J. Ripa, P. Lundbeg, V. Kaitala, *A general theory of environmental noise in ecological food webs*, Am., Nat 256-263, 1998.

[22] K. Fujii, *Complexity-stability relationship of two-prey-one predator species system model: local and global stability*, Journal of Theoretical Biology, **69 (4)**, 613-623, 1977.

[23] K.Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers, Netherlands, 1992.

[24] M. Mackey, Ld chaos in ph.Glass, Oscillations and chaos in physiological control systems, science 197, 287-289, 1977.

[25] M.W. Hirsch., Smale, S. and Devaney, R., *Differential Equations, Dynamical Systems and an Introduction to Chaos*, Academic Press, USA, 2004

[26] P.Feng, *Analysis of a delayed predator-prey model with ratio-dependent functional response and quadratic harvesting*, J. Appl. Math.Comp, 44, (1-2), 2014.

[27] R.K. Upadhyay, A. Mukhopadhyay, S.R.K. Iyengar, *Influence of environmental noise on the dynamics of a realistic ecological model*, Fluct.Noise. Lett, **7(1)**, 61-77, 2001.

[28] R.Lande, *Risks of Population Extinction From Demographic And Environmental Stochasticity And Random Catastrophes*, Am., Nat. 1422, 911-927, 1993.

[29] R.M.May, *Stability in randomly fluctuating deterministic environments*, Am., Nat, **107**, 621-650, 1973.

[30] R.M. Nisbet and W.S.C. Gurney., *Modelling Fluctuating Populations*, John Wiley, New York, 1982

[31] R.M.Nisbet, and W.S.C. Gurney., *The systematic formulation of population models for insects with dynamically varying in star duration*, Theoret. Popul. Biol., **23 (1)**, 114–135, 1983.

[32] R.T.Paine, *The Pisaster-Tegula interaction: preypatches, predator food preference and inter tidal community structure*, Ecology, **50 (6)**, 950-961, 1969.

[33] R.Xu, O.Gan, Z.Ma, *Stability and bifurcation analysis on a ratio-dependent predator-prey model with time delay*, J. Comp. App. Math. **230(1)**, 187-203, 2009.

[34] S. Baek,, W. Ko,. and I.Ahn,. (2012), *Coexistence of a one-prey two-predator model with ratio-dependent functional responses*, Applied Mathematics Computation, **219 (4)**, 1897-1908.

[35] S. Gakkar, R.K. Naji, *Existence of chaos in two prey and one predator system*, Chaos, Solitons & Fractals; 17 (**??**):639-649, 2003.

[36] S. Gakkhar, A. Singh, *Control of choas due to additional predator in the hasting-powell food chain model*, J.Math, Anal, Appl, **385**, 423-438, 2012.

[37] S.Gakkhar, and B.Singh, *The Dynamics of food web consisting of two preys and a harvesting predator*, ChaosSolitons and Fractals, **34 (4)**, 1346-1356, 2007.

[38] S.Jana, M.Chakraborty .T.K.Kar, *Global stability and bifurcation of time delayed prey-predator system incorporating prey refuge*, Math, Comp., Simul, **85**, 57-77, 2012.

[39] S.Kumar.,S.K. Srivastava, and P. Chingakham, *Hopf bifurcation and stability analysis in a harvested one-predator-two-prey model*, Applied Mathematics Computation, **129 (1)**, 107-118, 2002.

[40] T.K. Kar, and A.Batabyal, *Persistence and Stability of a Two Prey One Predator System*, International Journal of Engineering Science and Technology, **2 (2)**, 174-190, 2010.

[41] T.K. Kar, S.K. Chattopadhyay, and C.K.Pati, *A bio-economic model of two-prey one-predator system*, Journal Applied Mathematics Information, **27 (5/6)**, 1141-1427, 2009.

[42] V.Volterra, *Variazioni E fluttazionidelnumero di individui in species animaliconviventi*, Memorie Accademia Nazionaledei Lincei, **2**, 31-313, 1926.

[43] Y.Kuang, *Delay Differential Equations: with Applications in population Dynamics*, Academic Press, New York, 1993.

[44] Y.Kuang, E.Berratta, *Global analysis of Gauss-type ratio-dependent predator-prey system*, J. Math. Biol, **36**, 389-406, 1998.