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A FRACTIONAL VERSION OF THE HESTON MODEL WITH HURST PARAMETER $H \in (1/2, 1)$

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ABSTRACT. We consider a fractional version of the Heston model where the two standard Brownian motions are replaced by two fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$. We show that the stochastic differential equation admits a unique positive solution by adapting and generalizing some results of Y. Hu, D. Nualart and X. Song on singular equations driven by rough paths. Moreover, we show that the fractional version of the variance, which is a version of the fractional Cox-Ingersoll-Ross model, is still a mean-reverting process.

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1. INTRODUCTION

In classical quantitative finance, it is usual to suppose that risky asset price's dynamics are driven by Brownian motions, as proposed for the first time by Bachelier in 1900 in his Ph.D thesis. Developed by Fisher Black, Robert Merton and Myron Scholes in the seventies, the famous Black and Scholes model remains popular nowadays. Indeed, as closed pricing formulae are provided for European call and put options, the model is easy to implement. Although, the constant volatility assumption of the model [7] contradicts the empirical observations, i.e. the implied volatility generally depends on time [14, 8]. This led to consider more sophisticated models, e.g. dynamics with local volatilities [13], but also stochastic volatility models [10, 1].

Despite these improvements, we may observe in practice a long-term correlation between the underlying asset prices, see [20]. To address this issue for the Heston model, a natural idea is to replace the two Brownian motions by fractional Brownian motions (FBM), see [5, 21]. Indeed, heavier tail distributions and long-range dependence are some of the interesting features of the FBM models that confirms their relevance, see [2, 4]. The fractional Black-Scholes (FBS) model, one of the first FBM models, appears to be more efficient and flexible than the classical Black and Scholes

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model to reproduce the behaviour of the stock dynamics despite its limited capacity to fit the market data, see [3].

In this paper, we introduce and study the stochastic differential equation defined as the fractional version of the Heston model (FHM) when the Hurst parameter $H \in (1/2, 1)$. Precisely, we first recall the definition of a fractional Brownian motion and we formulate the existence and uniqueness theorem for the stochastic differential equation in the fractional Heston model. We show that there exists a unique solution which is positive. The proof is based on singular equations driven by rough paths which are studied in Section 3. This section generalises results of [11] to a larger class of drivers. We then deduce the existence of solutions to singular equations driven by a FBM in Section 4, which is also a generalisation of [11]. Moreover, we show that these solutions are not necessary stationary. For specific drivers, we give an explicit expression of the expectation. This allows to deduce an explicit expression of the expectation of the fractional Cox-Ingersoll-Ross (CIR) process, which may be seen a generalisation of the case $H = 1/2$ to the case $H > 1/2$. In particular, the fractional CIR process is mean-reverting.

2. FRACTIONAL HESTON MODEL

2.1. **Reminder on the fractional Brownian motion.** Fractional Brownian motions were first introduced by Kolmogorov in 1940 [12].

Definition 2.1. A Gaussian stochastic process $(B^H_t)_{t\geq 0}$ of Hurst parameter $H \in (0,1)$ is called (standard) fractional Brownian motion, if

1. The paths of B^H are continuous and satisfy $B^H_0 = 0$.
2. $\mathbb{E}[B^H_t] = 0$ and $Var[B^H_t] = t^{2H}$, for any $t \geq 0$.
3. The increments of B^H are stationary.
4. The process B^H admits the covariance function

$$\rho_{t,s} := \mathbb{E}[B^H_t B^H_s] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{2.1}$$

Existence of such Gaussian processes satisfying (2.1) is discussed in [17]. The class of *FBM* processes may be splitted into two categories apart from the standard Brownian motion, i.e. when $H = \frac{1}{2}$. If $H \in (\frac{1}{2}, 1)$, the increments of *FBM* are positively correlated so that the process B^H satisfies a long dependence behaviour useful to describe phenomenon with memory and persistence. When $H \in (0, \frac{1}{2})$, as the increments of B^H are negatively correlated, it may be used to model intermittency and anti-persistency, see [18]. In this paper, we only consider *FBM* processes with

Hurst parameter $H \in (\frac{1}{2}, 1)$. In 1968, Mandelbrot and Van Ness [15] gave the following stochastic integral representation of a FBM process:

(2.2)
$$B_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_0^t (t-s)^{H-\frac{1}{2}} dB_s + \int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB_s\right), \quad t \geq 0$$

where B is a standard Brownian motion. The first term models the current innovation or shock and the second part contains a moving average of historical shocks, see [15]. We recall the following properties, see [15]:

Theorem 2.2. *A fractional Brownian motion B^H satisfies the following properties*

1. *The process B^H is self-similar.*
2. *The trajectories of B^H are almost surely nowhere differentiable.*

The following definition may be found in [21].

Definition 2.3. A stationary process $(Y_t)_{t\geq 0}$ with finite variance is said to have long range dependance if its autocorrelation function $C_t(\tau) := \operatorname{cor}(Y_t, Y_{t+\tau})$ decays as a power of the lag τ: $C_t(\tau) \sim \frac{L(\tau)}{\tau^\alpha}$, as $\tau \to \infty$, where $\alpha \in (0,1)$ and L is slowly varying at infinity, i.e. for all $a > 0$, $L(at)/L(t) \to 1$ as $t \to \infty$.

As shown in [15, 19], a fractional Brownian motion B^H such that $H \in (\frac{1}{2}, 1)$ admits a long-range dependence.

2.2. **Main result.** Let us consider the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_t^{1,H}, \quad t \in [0, \infty), \tag{2.3}$$

$$dV_t = \kappa(\theta - V_t) dt + \sigma\sqrt{V_t} dB_t^{2,H}, \quad t \in [0, \infty), \tag{2.4}$$

where $(B^{1,H}, B^{2,H})$ is a two dimensional FBM process with the Hurst parameter $H \in (1/2, 1)$ and $\kappa > 0$, $\theta > 0$, $\mu \in \mathbf{R}$ and $\sigma > 0$ are constants. We also suppose that $V_0 > 0$ and $S_0 > 0$ are given. Recall that, as in the classical Heston model, we could suppose that there exists a constant $\rho \geq 0$ satisfying

$$\begin{aligned} \operatorname{cov}(B_{t+dt}^{1,H} - B_t^{1,H}, B_{t+dt}^{2,H} - B_t^{2,H}) &:= E(B_{t+dt}^{1,H} - B_t^{1,H})(B_{t+dt}^{2,H} - B_t^{2,H}) \\ &= \rho(dt)^{2H}, \quad t,\ dt \geq 0. \end{aligned} \tag{2.5}$$

This implies that the correlation between two increments of $B^{1,H}$ and $B^{2,H}$ per unit of time is the constant ρ. Nevertheless, we do not need this assumption is this paper.

The following theorem, which is the main goal of this paper, states that the system of SDEs above admits a unique solution. By definition, we call it the price dynamics of the risky asset S in the fractional Heston model. Moreover, Equation (2.4) defines a fractional version of the CIR model. We shall prove below that it is a mean-reverting stochastic process, i.e. $\lim_{t\to\infty} EV_t = \theta$.

The proof of existence and uniqueness is based on the study of singular equations driven by rough paths given in Section 3. This generalizes results of [11] that allow to consider singular equations driven by a FBM as done in Section 4.

Theorem 2.4. *The S.D.E's* (2.3) *and* (2.4) *admit unique positive solutions.*

Proof. Let us introduce $W = \sigma^{-2}V$. Then, (2.4) reads as

$$dW_t = \kappa(\sigma^{-2}\theta - W_t)dt + \sqrt{W_t}dB_t^{2,H}, \quad t \in [0,T].$$

This is the singular equation (4.2) of Section 4 with $g(t,x) = \kappa\sigma^{-2}(\theta - \sigma^2 x)$. This function satisfies the required Condition C_g hence the S.D.E. (2.4) admits a unique positive solution by Theorem 4.6. Notice that the integral with respect to $B^{2,H}$ is a pathwise Young integral, see [9]. Moreover, V is almost surely continuous and admits finite moments of all orders.

In order to show that (2.3) admits a unique positive solution, we shall apply the results of [16, Section 5.3.3] to the S.D.E.

$$dY_t = \mu dt + \sqrt{V_t}dB_t^{1,H}, \quad Y_0 = 0. \tag{2.6}$$

It suffices to verify that the two functions $\sigma(t,x) = \sqrt{V_t}$ and $b(t,x) = \mu$ satisfy [16, Conditions H_1, H_2, Section 5.3.2] for some constants that may depend on $\omega \in \Omega$. The only difficulty is to show that the process $\sqrt{V}$ is Hölder continuous of order H. As V is positive and a.s. bounded on every interval $[0,T]$, $T > 0$, this is equivalent to show that V or W are Hölder continuous of order H. By the proof of Theorem 4.6, $W = X^2/4$ where X is the positive process

$$X_t = X_0 + \int_0^t 2X_s^{-1}g(s, X_s^2/4)ds + B_t^{2,H}, \quad t \in [0,T]. \tag{2.7}$$

As V is positive and a.s. bounded, the process $s \mapsto X_s^{-1}g(s,X_s^2/4)$ is a.s. bounded on $[0,T]$ hence the integral process in the expression of X is Lipschitz. We deduce that X is Hölder continuous of same order H than $B^{2,H}$. Since X is a.s. bounded on $[0,T]$, we get that W is Hölder continuous of same order H. We conclude that the S.D.E. (2.6) admits a unique solution Y on each interval $[0,T]$ hence it is possible to conclude on $[0,\infty)$.

Therefore, by the change of variable formula [16, Section 5.2.2], we deduce that the process $S := S_0 e^Y$ satisfies (2.3) and is positive. Reciprocally, suppose that S is a.s. positive and satisfies (2.3). Consider $\omega \in \Omega$ such that $\alpha(\omega) := \min_{t\in[0,T]} S_t(\omega)/S_0 > 0$ where $T > 0$. We construct a function γ on $\mathbb{R}$ which is twice differentiable and satisfies $\gamma(x) = \log(x)$ for all $x \geq \alpha(\omega)$. Applying the change of variable formula to the deterministic function $\gamma(S_.(\omega)/S_0)$, i.e. for ω fixed, we deduce that the function $t \mapsto \gamma(S_t(\omega)/S_0)$ satisfies the same S.D.E. (2.6) than Y in the pathwise sense hence $\gamma(S(\omega)/S_0) = Y(\omega)$ by [16, Theorem 5.3.1] and [16, Section 5.3.3]. As

$\gamma(S(\omega)/S_0) = \log(S(\omega)/S_0)$, we deduce that $S = S_0 e^Y$ a.s. We then conclude as $T \to \infty$. □

In the classical Heston model, the volatility process satisfies the stochastic differential equation (2.4) with $H = 1/2$. We then deduce that $t \mapsto EV_t$ satisfies an o.d.e. of first order and finally

$$EV_t = (V_0 - \theta)e^{-\kappa t} + \theta, \quad t \geq 0,$$

so that $\theta = \lim_{t\to\infty} EV_t$. We say that V is mean-reverting. Note that the increments of V are not weakly stationary by Lemma 4.2. By Corollary 4.8, we deduce the following result:

Proposition 2.5. *The fractional volatility process V, solution to Equation* (2.4)*, satisfies*

$$EV_t = (V_0 - \theta)e^{-\kappa t} + \theta + e^{-\kappa t}\delta_H(t), \tag{2.8}$$

where δ_H is a differentiable function which satisfies $\delta_H(t) \in (0, \frac{\sigma^2 t^{2H}}{2})$ for all $t \geq 0$.

This implies that the variance V of the fractional Cox-Ingersoll-Ross (CIR) process (2.4) is larger when $H > 1/2$ than it is when $H = 1/2$. Nevertheless, we still have $\lim_{t\to\infty} EV_t = \theta$.

We leave for further research a deeper study of the FHM regarding long range dependence, as well as discretization of the process and pricing in finance with this model.

3. SINGULAR EQUATIONS DRIVEN BY ROUGH PATHS

For any $s \leq t$, we consider the Banach space of continuous functions $\mathcal{C}([s,t])$ equipped with the topology of the supremum norm we denote by $\|f\|_{[s,t]}$, $f \in \mathcal{C}([s,t])$. When a continuous function is defined on a subset I of $\mathbf{R}$, we naturally extend the notation by denoting its supremum by $\|f\|_I$. The space of Holder continuous functions of order $\beta > 0$ is denoted by $\mathcal{C}^\beta([s,t])$ and its norm is

$$\|f\|_{[s,t],\beta} := \sup\left\{\frac{|f(x) - f(y)|}{|x-y|^\beta},\ x, y \in [s,t]\right\}, \quad f \in \mathcal{C}^\beta([s,t]).$$

We consider the deterministic differential equation driven by a rough path φ of [11]:

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \varphi(t), \quad t \in [0, \infty), \tag{3.1}$$

where $x_0 > 0$ is a constant, φ is continuous and $\varphi(0) = 0$. We impose conditions on f.

Definition 3.1. We say that a function $f = f(t,x)$, $(t,x) \in \mathbf{R}_+ \times (0,\infty)$ is locally Lipschitz with respect to the space variable x on $x \in (0,\infty)$ if for all $c, C > 0$ such that $c < C$, and $T > 0$, we have $|f(t,x) - f(t,y)| \le L_{T,c,C}|x-y|$ whatever $x, y \in [c, C]$ and $t \le T$ for some Lipschitz constant $L_{T,c,C}$ depending on T, c, C.

Condition C_f^1**:**

1. f is a locally Lipschitz function with respect to the space variable x on $x \in (0,\infty)$.
2. There exists constants $m, \gamma \ge 0$ such that $f(t,x) \ge -m - \gamma x$ for all $x > 0$ and $t \ge 0$.
3. The mapping $t \mapsto f(t,x)$ is continuous for all $x > 0$.

Note that the conditions we impose on f are weaker than the conditions of [11]. In particular, f is not necessarily differentiable with respect to $x \in (0,\infty)$ and $x \mapsto f(t,x)$ is not necessarily non increasing. The following condition is also considered when $\varphi \in \mathcal{C}^\beta([0,T])$ for all $T \ge 0$, where $\beta \in (0,1)$, to obtain a solution to (3.1) on the whole interval $[0,\infty)$.

Condition C_f^β**:** For every $T > 0$, there exists $\epsilon > 0$, $\gamma_0 > 0$, $\alpha \in (\beta^{-1} - 1, \beta^{-1})$ and positive constants c, d, such that $f(t,x) \ge cx^{-\alpha} - d$ for all $t \in [T-\epsilon, T]$ and $x \in (0, \gamma_0]$.

Theorem 3.2. *Let $f = f(t,x)$, $(t,x) \in \mathbf{R}_+ \times (0,\infty)$, be a function satisfying condition C_f^1. Then, there exists a unique positive solution to (3.1) on some maximal interval $[0, T^*)$ such that $x_{T^*} := \lim_{t\to T^*} x_t = 0$ exists if $T^* < \infty$. Moreover, if $\varphi \in \mathcal{C}^\beta([0,T])$ for all $T > 0$, then $T^* = \infty$ under Condition C_f^β.*

Proof. By the assumptions on f, note that for all c, C such that $0 < c \le C$, and for all $\Delta \ge 0$, we have $\|f\|_{[0,\Delta]\times[c,C]} < \infty$. To see it, we use the local Lipschitz property of f with respect to the space variable as well as the continuity of f with respect to the time variable. Note that we may reformulate the problem if we replace f by $f + m \vee d$ and φ by $\tilde{\varphi}(t) = \varphi - (m \vee d)t$ so that we may assume without loss of generality that $m = d = 0$. Let us consider a fixed constant C such that $C \ge 3x_0$. It is also possible to find $T_0 \in (0,1)$ small enough so that, by uniform continuity, $\|\varphi\|_{[0,T_0]} \le 4^{-1}x_0$ and, as $T_0 \to 0$, we have $C^{-1}\|f\|_{[0,1]\times[2^{-1}x_0,C]} \le (2T_0)^{-1}$ and $T_0 \le x_0/(4\gamma C)$ in the case where $\gamma > 0$ in Condition C_f^1. Consider the following iterative scheme:

$$x_t^{n+1} = x_0 + \int_0^t f(s, x_s^n)ds + \varphi(t), \quad t \in [0, T_0], \quad n \ge 0 \tag{3.2}$$

where $x_t^0 = x_0$ for all $t \in [0, T_0]$. Let us show by induction that $x_t^n \in [2^{-1}x_0, C]$ for all $t \in [0, T_0]$. This is the case with $n = 0$. Suppose that this holds with x^n. As $x^n > 0$, we deduce by assumption with $m = 0$ that $f(s, x^n_s) \geq -\gamma x^n_s$ for all $s \in [0, T_0]$ hence

$$\int_0^t f(s, x^n_s)ds \geq -\gamma \int_0^t x^n_s ds \geq -\gamma \int_0^{T_0} x^n_s ds \geq -\gamma C T_0 \geq -4^{-1}x_0, \quad t \in [0, T_0].$$

Moreover, $\varphi(t) \geq -4^{-1}x_0$ by construction of T_0 hence

$$x^{n+1}_t \geq x_0 - 4^{-1}x_0 + \varphi(t) \geq 2^{-1}x_0.$$

Let us show that $\|x^n\|_{[0,T_0]} \leq C$. We have

$$\begin{aligned} |x^{n+1}_t| &\leq |x_0| + \|f\|_{[0,1]\times[2^{-1}x_0,C]}T_0 + \|\varphi\|_{0,T_0}, \\ &\leq C\left(\frac{|x_0| + 2^{-1}x_0}{C} + T_0\frac{\|f\|_{[0,1]\times[2^{-1}x_0,C]}}{C}\right) \leq C. \end{aligned}$$

The last inequality is deduced from the conditions imposed on C and T_0. Let us define

$$g_n(t) = |x^{n+1}_t - x^n_t|, \quad t \in [0, T_0], \quad n \geq 0.$$

By continuity, note that $\|g_1\|_{[0,T_0]} < \infty$. Since f is locally Lipschitz with respect to the space variable on $(0, \infty)$, let us consider a constant $k = k(T_0, x_0, C)$ such that $|f(s, x) - f(s, y)| \leq k|x - y|$ for all $x, y \in [2^{-1}x_0, C]$ and $s \in [0, T_0]$. By (3.2), we get that

$$g_{n+1}(t) \leq k \int_0^t g_n(s)ds, \quad t \in [0, T_0], \quad n \geq 1.$$

By induction, we then deduce that $g_{n+1}(t) \leq \|g_1\|_{[0,T_0]}k^n t^n/n!$ for all $t \in [0, T_0]$. Therefore, the sequence $g_{n+1} = g_1 + \sum_{i=0}^n (g_{i+1} - g_i)$ is absolutely convergent with respect to the supremum norm hence uniformly converges to $x^{(T_0)}$ on $[0, T_0]$. Moreover, it is trivial that $x^{(T)}$ satisfies (3.1) on $[0, T_0]$.

Let us now prove that the equation (3.1) admits a unique positive solution on every compact $[0, T]$, $T > 0$. To see it, consider two solutions x and y and let us consider $c = c_T = \min\{|x_t|, |y_t| : t \in [0, T]\} > 0$ and $C = C_T = \max(\|x\|_{[0,T]}; \|y\|_{[0,T]})$. We then deduce that

$$\|x - y\|_{[0,t]} \leq k \int_0^t \|x - y\|_{[0,s]}ds,$$

where $k = k(T, c, C)$ is a constant such that $|f(s, x) - f(s, y)| \leq k|x - y|$ for all $x, y \in [c, C]$ and $s \in [0, T]$. By induction, we deduce that $\|x - y\|_{[0,t]} \leq 2Ck^n t^n/n!$ for all $t \leq T$. As $n \to \infty$, we get that $\|x - y\|_{[0,T]} = 0$ hence $x = y$.

Consider the set $\Lambda \ni T_0$ of all $T > 0$ such that (3.1) admits a unique positive solution $x^{(T)}$ on $[0, T]$. Note that if $T_1, T_2 \in \Lambda$ satisfy $T_1 \leq T_2$, then $x^{(T_1)}$ and $x^{(T_2)}$ coincides on $[0, T_1]$ by uniqueness. Therefore, we may define the function $x_t := x^{(t)}_t$ on $t \in [0, T^*)$ where $T^* = \sup \Lambda$. This function satisfies the equation (3.1) on $[0, T^*)$ and is uniquely defined and positive.

If $T^* = \infty$, we may conclude about the lemma. Otherwise, we show that $\lim_{t\to T^*} x_t$ exists and $\lim_{t\to T^*} x_t = 0$. Indeed, in the contrary case, there exists $l > 0$

and a sequence $(t_n)_{n\geq 1}$ such that $t_n \to T^*$ and $x_{t_n} \to 2l$ as $n \to \infty$. We may assume without loss of generality that $x_{t_n} \in (l, 3l)$. Fix a constant $C \geq 6l$ and $T_0^* > 0$ small enough such that $T_0^* \leq l(4\gamma C)^{-1}$ if $\gamma > 0$, such that $|\varphi(u)-\varphi(t)| \leq 4^{-1}l$ if $|t-u| \leq T_0^*$ and $C^{-1}\|f\|_{[0,T^*]\times[2^{-1}l,C]} \leq (2T_0^*)^{-1}$. Note that T_0^* does not depend on n. Consider an arbitrary $n \geq 1$. Let us introduce the following scheme: $x^0 = x_{t_n}$ and for $m \geq 1$:

$$x_t^{m+1} = x_{t_n} + \int_{t_n}^{t} f(s, x_s^m)ds + \varphi(t) - \varphi(t_n), \quad t \in [t_n, t_n + T_0^*], \quad n \geq 0. \tag{3.3}$$

As above, we may show by induction that $x_t^m \in [2^{-1}l, C]$ for all $t \in [t_n, t_n + T_0^*]$ and all $m \geq 1$. Then, we also deduce that the sequence $(x^m)_{m\geq 1}$ uniformly converges to some function z^n on $[t_n, t_n + T_0^*]$ such that

$$z_t^n = x_{t_n} + \int_{t_n}^{t} f(s, z_s^n)ds + \varphi(t) - \varphi(t_n), \quad t \in [t_n, t_n + T_0^*], \quad n \geq 0. \tag{3.4}$$

We then define $\tilde{x}_t^n = x_t 1_{[0,t_n)}(t) + z_t^n 1_{[t_n, t_n+T_0^*]}(t)$ on $[0, t_n + T_0^*]$ where we recall that x is the solution to Equation (3.1) on $[0, T^*)$. Note that $\tilde{x}^n$ satisfies Equation (3.1) on $[0, t_n + T_0^*]$. Therefore, by definition of T^*, we deduce that $t_n + T_0^* \leq T^*$ for all n. As T_0^* does not depend on n, we deduce as $n \to \infty$ that $T_0^* \leq 0$ hence a contradiction.

Let us now suppose that Condition C_β holds and $T^* < \infty$. Then, $x_{T^*} = 0$ and, as $x_t \in (0, \gamma_0)$ if $t \in [T^* - \epsilon, T^*)$, for some $\epsilon > 0$, we get that $f(s, x_s) \geq 0$ for all $s \in [T^* - \epsilon, T^*)$ since we assume without loss of generality that $d = 0$. We deduce a contradiction by repeating the arguments in the proof of [11, Theorem 2.1]. □

We may reproduce the proof of [11, Theorem 2.1] under our assumptions and under a weaker assumption than [11, (iii)]:

Condition C_f^2: For all $T > 0$, there exists positive constants h_T and $\epsilon_T^1, \epsilon_T^2 > 0$ such that $|f(t,x)| \leq h_T(x^{-1} + 1)$ for all $x \in (0, \epsilon_T^1)$ and $|f(t,x)| \leq h_T(x+1)$ for all $x \in (\epsilon_T^2, \infty)$ and $t \leq T$.

Theorem 3.3. *Let f be a function defined on $\mathbf{R}_+ \times (0,\infty)$ which satisfies Conditions C_β and C_f^i, $i = 1, 2$. Then, for any $\gamma > 2$ and for all $T > 0$, the unique solution x to* (3.1) *satisfies*

$$\|x\|_{[0,T]} \leq C_{1,\gamma,\beta,T}(1 + x_0)\exp\left\{C_{1,\gamma,\beta,T}(1 + \|\varphi\|_{[0,T],\beta}^{\frac{\gamma}{\beta(\gamma-1)}})\right\}, \tag{3.5}$$

where $C_{1,\gamma,\beta,T}$ and $C_{1,\gamma,\beta,T}$ are constants depending on γ, β, T and f.

Proof. We exactly follow the proof of [11, Theorem 2.1] but we do not use the assumption (iii) of [11]. With $y = x^\gamma$, recall that (iii) is used in [11] to obtain the following inequality: $|f(u, y_u^{\frac{1}{\gamma}})y_u^{1-\frac{1}{\gamma}}| \leq c(y_u^{1-\frac{2}{\gamma}} + y_u^{1-\frac{1}{\gamma}})$ where c is a constant depending on the function h of (iii). Instead, since f is locally Lipschitz, there exists under Conditions C_f^i, $i = 1, 2$, a constant $L_{T,f}$ depending on f and T such that $|f(t,x)| \leq L_{T,f}(x+1)$ for all $x \in [\epsilon_T^1, \infty)$ and $t \leq T$. Therefore, we deduce that

$|f(u, y_u^{\frac{1}{\gamma}})y_u^{1-\frac{1}{\gamma}}| \leq L_{T,f}(y_u + y_u^{1-\frac{1}{\gamma}})$ if $y_u^{\frac{1}{\gamma}} \geq \epsilon_T^1$. Otherwise, if $y_u^{\frac{1}{\gamma}} \leq \epsilon_T^2$, we have the inequality $|f(u, y_u^{\frac{1}{\gamma}})y_u^{1-\frac{1}{\gamma}}| \leq h_T(y_u^{1-\frac{2}{\gamma}} + y_u^{1-\frac{1}{\gamma}})$. Finally, for all $u \leq T$, we deduce the inequality $|f(u, y_u^{\frac{1}{\gamma}})y_u^{1-\frac{1}{\gamma}}| \leq L(f,T)(y_u + y_u^{1-\frac{2}{\gamma}} + y_u^{1-\frac{1}{\gamma}})$ for some constant $L(f,T)$ depending on f and T. This implies that, in the proof of [11, Theorem 2.1], we replace $\|y\|_{[s,t]}^{1-\frac{2}{\gamma}}$ by the sum $\|y\|_{[s,t]}^{1-\frac{2}{\gamma}} + \|y\|_{[s,t]}$. This substitution does not change the desired inequality because, as in the proof of [11, Theorem 2.1], we use the inequality $x^\alpha \leq 1 + x$ for all $\alpha \in (0,1)$ to bound from above the three terms $\|y\|_{[s,t]}$, $\|y\|_{[s,t]}^{1-\frac{2}{\gamma}}$ and $\|y\|_{[s,t]}^{1-\frac{1}{\gamma}}$ by $1 + \|y\|_{[s,t]}$. □

4. SINGULAR EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION WITH HURST PARAMETER $H \in (1/2, 1)$

Let us consider the singular stochastic differential equation

$$X_t = x_0 + \int_0^t f(s, X_s)ds + B_t^H, \quad t \geq 0, \tag{4.1}$$

where $x_0 > 0$, B^H is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and f is function which satisfies Conditions C_β and C_f^i, $i = 1, 2$. Repeating the proof of [11, Theorem 3.1] with the results of Section 3, we obtain the following:

Theorem 4.1. *Suppose that $x_0 > 0$ and f is a function which satisfies Conditions C_β and C_f^i, $i = 1, 2$. Then, there exists a unique positive pathwise solution X to Equation (4.1) such that $E(\|X\|_{[0,T]}^p) < \infty$ for all $p > 0$ and $T \geq 0$.*

The following result is classical; we use it to show Theorem 4.3, which implies that the increments of X are not stationary at least when $f(0, X_0) < 0$.

Lemma 4.2. *Let X be an integrable process on $\mathbf{R}_+$ and consider the function $\varphi(t) = EX_t - X_0$, $t \geq 0$. If the increments are weakly stationary, then φ is additive. Therefore, if φ is continuous on $\mathbf{R}_+$, $\varphi(t) = \alpha t$ for all $t \geq 0$ where α is a constant.*

Proof. As the increments are weakly stationary, we easily deduce that φ is additive, i.e. $\varphi(t+h) = \varphi(t) + \varphi(h)$. By induction , we deduce that $\varphi(nt) = n\varphi(t)$ for all $n \in \mathbb{N}$. If $a, b \in \mathbb{N}$, with $b > 0$, we get that $b\varphi((a/b)t) = \varphi(at) = a\varphi(t)$. This implies that $\varphi(qt) = q\varphi(t)$ for all non negative rational numbers q. We then conclude by density and continuity with $\varphi(1) = \alpha$. □

Theorem 4.3. *Suppose that $x_0 > 0$ and f is a function which satisfies Conditions C_β and C_f^i, $i = 1, 2$. Consider the positive solution X to Equation (4.1). If the increments of X are weakly stationary and, if the mapping $t \mapsto Ef(t, X_t)$ is continuous at 0, then $EX_t = X_0 + f(0, X_0)t$ for all $t \geq 0$. In particular, $f(0, X_0) \geq 0$.*

Proof. The function φ defined by

$$\varphi(t) := E\int_0^t f(s,X_s)ds = \int_0^t Ef(s,X_s)ds$$

is differentiable at zero and we have $\varphi'(0) = f(0,X_0)$. Moreover, $EX_t = X_0+\varphi(t)$. As $E(\|X\|_{[0,T]}) < \infty$ for all $T \geq 0$, we deduce that the mapping $t \mapsto EX_t$ is continuous and so is φ. Suppose that the increments of X are weakly stationary. By Lemma 4.2, φ is linear hence we have $\varphi(t) = f(0,x_0)t$ for all $t \geq 0$. □

Note that the mapping $t \mapsto Ef(t,X_t)$ is continuous at 0 in the following case.

Corollary 4.4. *Suppose that $f(t,x) = f(x) = ax^{-1} - bx$ where $a > 0$ and $b \in \mathbf{R}$. Then, the mapping $t \mapsto Ef(t,X_t)$ is continuous at 0. Therefore, when $b > 0$ and $x_0^2 > ab^{-1}$, the increments are not weakly stationary.*

Proof. As $E(\|X\|_{[0,T]}) < \infty$ for all $T \geq 0$, we deduce that the mapping $t \mapsto EX_t$ is continuous at zero. Moreover, following the arguments used in the proof of [11, Proposition 3.4], we get that $EX_t^{-1} \leq X_0^{-1}$ if $t \in [0,t_0]$ where $t_0 > 0$ is small enough. Indeed, it suffices to notice that $f'(x)$ is bounded from above, which implies that the Malliavin derivative $(D_sX_t)_{0\leq s\leq t}$ is a bounded process so that Proposition 3.4 is still valid in our more general case. Finally, by the Jensen inequality, we get that $(EX_t)^{-1} \leq EX_t^{-1} \leq X_0^{-1}$ if $t \in [0,t_0]$. Since the mapping $t \mapsto EX_t$ is continuous at zero, we finally deduce that $\lim_{t\to 0} EX_t^{-1} = X_0^{-1}$. This implies that the mapping $t \mapsto Ef(t,X_t)$ is continuous at 0. When $b > 0$ and $x_0^2 > ab^{-1}$, we get that $f(x_0) < 0$ so that the increments are not weakly stationary by Theorem 4.3. □

Remark 4.5. In the case where $t \mapsto EX_t^{-1}$ is continuous and $f(t,x) = f(x) = ax^{-1} - bx$, we may show that

$$EX_t = x_0e^{-bt} + ae^{-bt}\int_0^t EX_u^{-1}e^{bu}du, \quad t \geq 0.$$

Let us now consider the singular stochastic differential equation

$$Y_t = y_0 + \int_0^t g(s,Y_s)ds + \int_0^t \sqrt{Y_s}dB_s^H, \quad t \geq 0, \tag{4.2}$$

where $y_0 > 0$ and g is a function which satisfies the following conditions denoted by Condition C_g. Notice that the conditions imposed on g in [11] are not correctly formulated as the authors made a small error when defining f in terms of g. When corrected, our conditions remain weaker.

Condition C_g:

1. The mapping $t \mapsto g(t,x)$ is continuous on $[0,\infty)$ for all $x > 0$.

2. The function g is locally Lipschitz with respect to the space variable x on $x \in (0,\infty)$ and there exists constant $m, \alpha \geq 0$ such that $g(t,x) \geq -mx^{1/2} - \alpha x$ for all $x > 0$ and $t \geq 0$.
3. For every $T > 0$, there exists $\epsilon > 0$, $\gamma_0 > 0$, $\alpha \in (\beta^{-1} - 1, \beta^{-1})$ and positive constants c, d, such that $g(t,x) \geq cx^{\frac{1-\alpha}{2}} - dx^{1/2}$ for all $t \in [T-\epsilon, T]$ and $x \in (0, \gamma_0]$.
4. For all $T > 0$, there exists positive constants h_T and $\epsilon_T^1, \epsilon_T^2$ such that we have $|g(t,x)| \leq h_T(1 + x^{1/2})$ for all $x \in (0, \epsilon_T^1)$, and $|g(t,x)| \leq h_T(x + x^{1/2})$ for all $x \in (\epsilon_T^2, \infty)$ and $t \leq T$.

Theorem 4.6. *Let g be a function satisfying Condition C_g. Then, Equation (4.2) admits a unique positive pathwise solution Y such that $E(\|Y\|_{[0,T]}^p) < \infty$ for all $p > 0$.*

Proof. Let us consider the function $f^g(t,x) = 2x^{-1}g(t, x^2/4)$ for all $t \geq 0$ and $x > 0$, i.e. $g(t,y) = f^g(t, 2y^{1/2})y^{1/2}$. Using the change of variable $x = 2y^{1/2}$, the chain rule for young integrals yields that Y is a positive solution to (4.2) if and only if $X = 2Y^{1/2}$ is a positive solution to (4.1) with the driver function f^g. As f^g satisfies the conditions of Theorem 3.2 as well as Conditions C_β and C_f if and only if g satisfies Condition C_g, we deduce that Equation (4.2) admits a unique positive pathwise solution Y given by $Y = X^2/4$ where X is the unique positive solution to (4.1) with the driver function f^g. ☐

Note by [16, Proposition 5.2.3], under Condition C_g, we may estimate the expectation of Y_t as follows:

$$EY_t = y_0 + \int_0^t Eg(r, Y_r)dr + \delta(t), \quad \delta(t) := \frac{H(2H-1)}{2}\int_0^t p(r)dr, \tag{4.3}$$

$$p(r) := \int_0^r ED_sX_r(r-s)^{2H-2}ds, \quad t, r \geq 0, \tag{4.4}$$

where D is the Malliavin derivative operator, $X = 2\sqrt{Y}$ is the solution of Equation (4.1) with $f = f^g$ and

$$f^g(t,x) = 2x^{-1}g(t, x^2/4), \quad t \geq 0,\ x > 0.$$

Moreover, if $f^g(t,x) = f(x)$ is differentiable, we have

$$D_sX_t = \exp\left\{\int_s^t f'(X_r)dr\right\} 1_{s\leq t}. \tag{4.5}$$

We deduce the following:

Theorem 4.7. *Let $g = g(x)$ be a function satisfying Condition C_g and let Y be the unique positive solution to Equation (4.2). Suppose that the derivative of f^g is bounded from above and the mapping $t \mapsto Eg(Y_t)$ is continuous at zero. Then, if the increments of Y are weakly stationary, we necessarily have $EY_t = y_0 + \nu t$ for all $t \geq 0$, where $0 \leq \nu \leq g(Y_0)$.*

Proof. As $E(\|Y\|^p_{[0,T]}) < \infty$ for every $T \geq 0$, the function $\varphi(t) = EY_t - X_0$, $t \geq 0$ is linear hence $EY_t = y_0 + \nu t$ for some $\nu \geq 0$ by Lemma 4.2. Moreover, by (4.3) and (4.5), we deduce a constant $C > 0$ such that $EY_t \leq y_0 + \int_0^t Eg(Y_r)dr + Ct^{2H}$ for all $t \leq 1$. We divide this inequality by $t > 0$ so that

$$\nu \leq \frac{1}{t}\int_0^t Eg(Y_r)dr + Ct^{2H-1}, \quad t > 0.$$

As $t \to 0$, we deduce that $\nu \leq g(y_0)$. □

The following result implies that the increments of Y are not weakly stationary if g is an affine function such that $g(0) > 0$.

Lemma 4.8. *Suppose that $g(x) = \alpha - \beta x$, $x > 0$, where $\alpha > 0$ and $\beta \in \mathbf{R}$. Then,*

$$EY_t = y_0 e^{-\beta t} + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + e^{-\beta t}\delta(t), \tag{4.6}$$

with the convention $(1 - e^{-\beta t})/\beta = t$ when $\beta = 0$. Moreover, when $\beta \geq 0$, $0 \leq \delta(t) \leq 2^{-1}t^{2H}$ for all $t \geq 0$.

Proof. Note that the derivative of $f^g(x) = ax^{-1} - bx$, $a = 2\alpha$ and $b = 2^{-1}\beta$ is bounded from above on $(0, \infty)$. We deduce that $(D_s X_t)_{s \leq t}$ is bounded on any interval $[0, T]$, $T > 0$. Moreover, note that $t \mapsto D_s X_t$ is continuous except at the point $t = s$. By the dominated convergence theorem, we deduce that the mapping p is continuous on any interval $[0, T]$. Therefore, δ is differentiable and $\delta' = p$. Moreover, by (4.3), $\varphi(t) = EY_t$ satisfies the o.d.e. $\varphi'(t) + \beta\varphi(t) = p(t) + \alpha$. We then conclude. When $\beta \geq 0$, $0 \leq D_s X_t \leq 1$ hence we deduce that $0 \leq \delta(t) \leq 2^{-1}t^{2H}$ for all $t \geq 0$. □

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