FAIR PRICING OF REVERSE MORTGAGE WITHOUT REDEMPTION RIGHT

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ABSTRACT. We derive in this article the pricing of a reverse mortgage without redemption right. In this, the underlying model employs a jump-diffusion process to represent the dynamics of the housing price, the Vasicek model to drive the instantaneous interest rate, and the force mortality model to describe the longevity risk. The said pricing is based on the Principle of Balance between the expected gain and expected payment. We compute the expected gain and the expected payment respectively under the continuous and discrete framework. We also present, with the above model, explicit formulas for the increasing (or decreasing) perpetual annuity and the level perpetual annuity. Furthermore, we discuss the monotonicity property of the annuities, lump sum, and annuity payment factors with respect to the parameters associated with the house price, the interest rate, and the force of mortality model. Finally, some numerical results for the lump sum, the annuity, and the annuity payment factors are presented, and also the sensitivity with respect to the above parameters is discussed. Based on the average change rate, we evaluate all parameters’ degree of impact on the annuity, the lump sum, and the annuity payment factors.

Keywords: Reverse mortgage; Fair pricing; Perpetual annuity; Jump-diffusion; Vasicek model; Force of mortality

1. INTRODUCTION

Reverse Mortgage is an inviting financial lending product offered to any senior citizen who owns a house. It is normally categorized by law into two categories, viz., (i) collateral reverse mortgage and (ii) ownership conversion reserve mortgage

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The collateral reverse mortgage can be redeemed and ownership conversion reverse mortgage not. In the case of the *collateral reverse mortgage*, the borrower is able to redeem the reverse mortgage by repaying the loan amount with the accumulated interests through property sale at any time from the mortgage’s effective date to due date. Of course, when the reserve mortgage contract is due, the borrower can choose a financial institution to auction off the pledged property to repay loan and due interests. In a collateral reverse mortgage, the elderly householder borrows annuity like periodical installment mortgage on his/her residential house. Home Equity Conversion Mortgage System is a typical collateral reverse mortgage in USA. In the case of the *ownership conversion reverse mortgage*, the borrower enters into a contract with a lending institution to obtain an annuity until his/her death, and at death the pledged property is transferred to the lender. *Rente Viager* is a typical ownership conversion reverse mortgage offered in France (Ohgaki, 2003).

Since the introduction of reverse mortgage, earlier research mainly included the basic principle, operation modes, feasibility, effectiveness, policies, laws, risks, and pricing. The literature on pricing reverse mortgage is not as rich as those on other aspects. The pricing of reverse mortgage mainly refers to how to determine a lump sum and annuity payments that the lender can pay. The main pricing techniques include two areas: (a) *the actuarial pricing technique* and (b) *the option pricing technique*. Generally, the former technique is employed to price the reverse mortgage when the redemption right has not been taken into account, and in the opposite case the latter technique is applied. The main idea of the former is to employ the principle of balance between the expected gain and expected payment under the assumption of perfect competition market. This makes the discounted present value of payment of the lender to be equal to a certain proportion of discounted present value of the mortgaged property, (see DiVenti and Herzog (1990), Tse (1995), Mitchell and Piggott (2004)). The main idea behind the latter is to apply the option pricing technique, which regards the mortgaged property (the pledged property is usually assumed to follow a stochastic process or stochastic series) as the underlying asset, and the loan principal and accumulated interests as the strike price of underlying asset. When the contract expires, the lender or its successor determines whether or not to execute the option (i.e., redeem the pledged property) according to the difference between the price of pledged property and the loan principle and accumulated interests, (see Li et al. (2010), Chen et al. (2010b), Lee et al. (2012), and Tsay et al. (2014)).

The main risks involved with reverse mortgage, as pointed out by Szymanoski (1994), include property value risk, interest rate risk, and longevity risk. In order to rationally price the reverse mortgage, one must build an appropriate model that takes into account the above risks. In general, the risk of housing price is modeled in two ways. The first one is to assume directly that the dynamics of housing price is driven
by a forward stochastic differential equation, as in Bardhan et al. (2006), Wang et al. (2008), Mizrahi (2012), Huang et al. (2011), Chen et al. (2010a), Lee et al. (2012), and Tsay et al. (2014). The second one is to fit the time series model based on the historical data of the housing price, as discussed by Nothaft et al. (1995), Chinloy et al. (1997), Chen et al. (2010b), and Li et al. (2010).


There are usually several ways to describe the longevity risk, such as a life table, force of mortality model. The classical force of mortality model can refer to de Moivre (1724), Gompertz (1825), Makeham (1860, 1867), Weibull (1951), Heligman and Pollard (1980), and Lee-Carter (1992).

In the model we study, we use a jump diffusion process to represent the dynamics of the housing price, the Ornstein-Uhlenbeck process is utilized to derive the instantaneous interest rate, and appeal to the force of mortality to describe the longevity risk. With this model we price the reverse mortgage without redemption right.

This article is organized as follows. Section 2 presents the models of risk factors. In Section 3, we first design the reverse mortgage without redemption right with fixed yearly payment until death, and then derive the pricing model for the lump sum and annuity payments by the principle of balance between expected gain and expected payment. In Section 4, we analyze the monotonicity of the lump sum, annuity payments, and annuity payment factors with respect to the parameters involved in housing price, interest rate and force of mortality models. Section 5 provides numerical results to examine how the housing price risk, interest rate risk, and longevity risk impact the lump sum, the annuity payment, and the annuity payment factors. Finally, in Section 6 we draw some conclusions from our findings.

2. RISK FACTORS

In order to obtain a suitable model to value the annuity of reverse mortgage without redemption right, we must first explore how to describe the risk factors that the reverse mortgage enforces. In this section we employ the jump-diffusion model to simulate the dynamics of house price, the Vasicek model to drive the instantaneous interest rate, and a force of mortality model to describe the longevity risk.

2.1. House Price. Our stochastic quantities are defined on a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\). We assume that the house price \(H(t), t \geq 0\), follows the special exponential Lévy process (Lee et al., 2012), namely the generalized Merton
jump diffusion model (Merton, 1976). First we set up the notations needed to define the said Merton equation. Let \( \{W_h(t), t \geq 0\} \) denote a \( P \)-standard Brownian motion capturing the unanticipated instantaneous change of house price, (but, this may not work so well for abnormal shocks); \( \{N(t), t \geq 0\} \) be the Poisson process with intensity \( \lambda_h \), describing the total number of jumps (including the house price sudden rise and drop event) during the time interval of \((0, t]\); \( \{J_i, i \geq 0\} \) be a sequence of independent normal random variables modeling the size of the jumps, with mean \( \mu_j \) and variance \( \sigma_j^2 \); and let \( k_h = \exp (\mu_j + \left(\frac{1}{2}\sigma_j^2\right)) - 1 \) with \( \sigma_j \) being some positive constant. With these we model the house price by the generalized Merton jump diffusion process given by

\[
h(t) = h(0) \exp \left[ \int_0^t \mu_h(s)ds - \left( \frac{1}{2}\sigma_h^2 + \lambda_h k_h \right) t + \sigma_h W_h(t) + \sum_{i=1}^{N(t)} J_i \right], \quad h(0) = h_0.
\]

Here the standard assumption is that \( \{W_h(t), t \geq 0\}, \{N(t), t \geq 0\} \) and \( \{J_i, i \geq 0\} \) are independent. Note that \( \mu_h(t) \) is the annual average return rate function w.r.t time \( t \), and \( \sigma_h \) is the annual volatility of the house price, assuming \( \sigma_h > 0 \).

2.2. Interest Rate. We take the instantaneous short-rate dynamics as the Vasicek model (Vasicek, 1977). Specifically, the interest rate process \( \{r(t), t \geq 0\} \) is governed by the following stochastic differential equation

\[
dr(t) = \alpha_r (\mu_r - r(t))dt + \sigma_r dW_r(t), \quad r(0) = r_0,
\]

where \( \{W_r(t), t \geq 0\} \) is a \( P \)-standard Brownian motion, and \( r_0, \alpha_r, \mu_r, \sigma_r \) are positive constants. Denote the correlation coefficient between \( W_r(t) \) and \( W_h(t) \) by \( \rho_{hr} \).

Applying Itô's formula to \( e^{\alpha_r t} r(u) \) we obtain

\[
r(t) = e^{-\alpha_r t} r(0) + \mu_r (1 - e^{-\alpha_r t}) + \sigma_r \int_0^t e^{-\alpha_r (t-u)} dW_r(u), \quad t \geq 0.
\]

The discount factor at time \( t \) is denoted by \( d(t) \) and is defined as

\[
d(t) := \exp \left( - \int_0^t r(s)ds \right).
\]

With some trivial computations, we have

\[
E[d(t)] = \exp \left\{ \left( \frac{\sigma_r^2}{2\alpha_r^2} - \mu_r \right) t + \frac{1}{\alpha_r} (\mu_r - r_0)(1 - e^{-\alpha_r t}) + \frac{\sigma_r^2}{4\alpha_r^3} [1 - (2 - e^{-\alpha_r t})^2] \right\}.
\]

We refer to Norberg (2004) for the derivation of Equation (2.5).

2.3. Longevity. We designate time \( t = 0 \) to be the time at which the reverse mortgage without redemption right is signed. Assume that the homeowner’s age is \( x_0 \) years old at time \( t = 0 \). Let \( X \) represent the life span of the new born infant. Let

\[
T(x_0) := X - x_0
\]
be the residual life of a home owner at his/her age $x_0$. The force of mortality at age $x$ ($x \geq 0$) is $\lambda(x)$. Then, the density function of $T(x_0)$ is

$$f_T(t) = \lambda(x_0 + t) \cdot \exp \left\{- \int_0^t \lambda(x_0 + u) du \right\}.$$  

In our numerical experiment, the force of mortality $\lambda(x)$ will be characterized by the Gompertz-Makeham force of mortality (see Carrière, 1994, Frees et al., 1996, or Huang et al., 2013)

$$\lambda(x) = a + \frac{1}{b} \exp \left( \frac{x - c}{b} \right),$$

where $a \geq 0$ denotes the constant hazard rate (independent of age); $b \geq 0$ is the dispersion; and $c \geq 0$ denotes a modal value. Note that the Gompertz-Makeham force of mortality model reduces to a constant force of mortality $a$ as $c \to \infty$.

3. FAIR PRICING OF REVERSE MORTGAGE

In this section, we will first design a reverse mortgage without redemption right with fixed annual payment until the death of the house owner. Then, the pricing model of the reverse mortgage without redemption right are built based on the principle of balance between expected gain and expected payment. Under the two-dimensional Gaussian distribution and independence assumptions, we obtain the explicit pricing formulas for lump sum and annuity payments for the reverse mortgage, particularly, the increasing (decreasing) perpetuity annuity and the level annuity.

3.1. Reverse Mortgage without Redemption Right. In this section, we will design a reverse mortgage without redemption right with fixed annual payment paid to the house owner until his/her death. The product that we design has the following basic features:

(I) The lender starts the payment of annuity to the house owner at the end of the year of signing the contract. The annuity payment is terminated upon the death of the house owner. More precisely, had the house owner survived through the $k$-th year, ($k \geq 1$), the lender would have paid the annuity payment $A_1, A_2, \ldots, A_k$ to the house owner at the end of the first, second, $\ldots$, $k$-th year, respectively.

(II) When the house owner dies, the lender will take over the house-owner’s pledged property, sell it in the market, and keep all of the proceeds from the sale of the property.

The essence of the reverse mortgage without redemption right is to exchange the profit from selling the mortgaged house with the house-owner’s annuity paid until his/her death. When the house owner dies, the lender will take over the house-owner’s
mortgaged property and sell it. The cash that is acquired from the sale of the house-
owner’s house is used to repay loan (including annuities and accumulated interests)
that the house owner owes to the lender. Since the reverse mortgage possesses the
non-recourse clauses (that is, the lender may not reclaim the loan against the house-
owner’s other assets or cash income except for his/her pledged property), the lender
will suffer a loss when the cash out of selling the mortgaged property is less than the
total annuity paid plus the accumulated interest; otherwise the lender will make a
profit.

We shall illustrate how the reverse mortgage product functions. Assume that
the homeowner dies at the age of $X = 68.7$ and the contract was signed at her
age of $x_0 = 65$. This means that the house owner lives in her house and claims
three cash payments $A_1$, $A_2$, $A_3$ at the end of the first, second and third year of the
contract, respectively. The annuity payment going to the house owner need not be a
fixed amount implied by the Feature (I) above; that is, $A_1$, $A_2$, $A_3$ can be unequal
amounts. When the house owner dies at the age of 68.7, the lender will take over
the pledged house and sell it in the market. Most of the time, it may not be possible
to sell the pledged house as soon as the lender take over it. Thus the time at which
the pledged house is sold is usually much later than that of taking over the pledged
house.

3.2. Fair Pricing Model. We assume that we are in the perfectly competitive mar-
ket. We price the reverse mortgage by the principle of balance between the expected
gain and expected payment. That is, the pricing is determined under the principle
where the expected discounted present value of future sale of the pledged property
balances out the expected discounted present value of annuities paid by the lender.

At time $T(x_0)$, the lender takes over the home-owner’s mortgaged property, and
sells it at time $T(x_0) + t_0$, where $t_0 \geq 0$ is the delay time between the lender taking
over the mortgaged property and the sale of the mortgaged property. We assume that
$t_0$ is fixed and not a random variable. Then the expectation of discounted present
value of the sale price of the property (i.e., the lender’s expected gain) is
\begin{equation}
E[h(T(x_0) + t_0)d(T(x_0) + t_0)],
\end{equation}
where recall that $h(t)$ is the value of the mortgaged property at time $t$ given by the
Stochastic Differential Equation (2.1), and $d(t)$ is the discount factor at time $t$ given
by Equation (2.4).

The expectation of discounted present value of the home-owner’s annuities (i.e.,
the lender’s expected payment) is
\begin{equation}
E\left[\mathbf{1}_{\{1 \leq T(x_0) < +\infty\}} \sum_{i=1}^{[T(x_0)]} A_i d(i)\right],
\end{equation}
where the function \([x]\) gives the largest integer not greater than \(x\). Then, the principle of balance between the expected gain and expected payment yields

\[
E[h(T(x_0) + t_0)d(T(x_0) + t_0)] = E \left[ 1_{\{1 \leq T(x_0) < +\infty\}} \sum_{i=1}^{[T(x_0)]} A_i d(i) \right].
\]

Though the analytic formula of annuity payment is difficult to obtain from the Equation (3.3), we can obtain the analytic formula under the two-dimension Gauss distribution and independence assumptions. The following Proposition 3.1 presents the analytic formula for the expected discounted present value of the mortgaged property at any time \(t\).

**Proposition 3.1.** Define

\[
Y(t) := \int_0^t \left[ \sigma_r e^{-\alpha_r s} \int_0^s e^{\alpha_r u} dW_r(u) \right] ds.
\]

Assume that the dynamics of home price follows the exponential Lévy process given by the Equation (2.1), the instantaneous short interest rate is governed by the Equation (2.2), the joint distribution of \((W_h(t), Y(t))\) follows the two dimensional normal distribution, and that \(\sigma_h W_h(t) - Y(t)\) is independent of \(\sum_{i=1}^{N(t)} J_i\). Then the expectation of discounted present value of the mortgaged property at time \(t\) is given by

\[
E[h(t)d(t)] = G(t)D(t),
\]

where

\[
G(t) = h_0 \exp \left\{ \int_0^t \mu(s) ds - \sigma_h \sigma_y \rho_{hr} \frac{1}{\alpha_r} \left( t + \frac{1}{\alpha_r} e^{-\alpha_r t} - \frac{1}{\alpha_r} \right) \right\},
\]

and

\[
D(t) = \exp \left\{ \left( \frac{\sigma^2_r}{2\alpha_r^2} - \mu_r \right) t + \frac{1}{\alpha_r} (\mu_r - r_0) (1 - e^{-\alpha_r t}) + \frac{\sigma^2_r}{4\alpha_r^2} \left[ 1 - (2 - e^{-\alpha_r t})^2 \right] \right\}.
\]

**Proof.** We begin by noting that \(Y(t)\) follows the normal distribution with the mean 0 and variance

\[
\sigma^2_y(t) = \frac{\sigma^2_r}{2\alpha_r^2} t + \frac{\sigma^2_r}{2\alpha_r^2} \left[ 1 - (2 - e^{-\alpha_r t})^2 \right].
\]

Recalling that the correlation coefficient between \(W_h(t)(t \geq 0)\) and \(W_r(t)(t \geq 0)\) is \(\rho_{nr}\), we also note that the covariance between \(W_h(t)\) and \(Y(t)\) is

\[
Cov(W_h(t), Y(t)) = \sigma_r \sigma_y \rho_{hr} \frac{1}{\alpha_r} \left( t + \frac{1}{\alpha_r} e^{-\alpha_r t} - \frac{1}{\alpha_r} \right).
\]

Thus the correlation coefficient between \(W_h(t)\) and \(Y(t)\), denoted by \(\rho(t)\), is

\[
\rho(t) = \frac{\sigma_r \rho_{hr} \sigma_y(t)}{\alpha_r \sigma_y(t) \sqrt{t} \left( t + \frac{1}{\alpha_r} e^{-\alpha_r t} - \frac{1}{\alpha_r} \right)}.
\]
Since the joint distribution of \((W_h(t), Y(t))\) follows the two-dimensional normal distribution with the correlation coefficient \(\rho(t)\) obtained above, we have from Equation (3.8) that

\[
E \{\exp[\sigma_h W_h(t) - Y(t)]\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(\sigma_h x - y)f(x, y)dxdy, \tag{3.10}
\]

where

\[
f(x, y) = \frac{1}{2\pi\sigma_y(t)\sqrt{t(1 - \rho^2(t))}} \exp\left\{-\frac{1}{2(1 - \rho^2(t))}S_{xy}\right\}, \tag{3.11}
\]

and

\[
S_{xy} = \left(\frac{x}{\sqrt{t}}\right)^2 - 2\rho(t)\frac{x}{\sqrt{t}\sigma_y(t)} + \left(\frac{y}{\sigma_y(t)}\right)^2.
\]

Under the substitutions \(u = \frac{x}{\sqrt{t}}\) and \(v = y - \rho(t)\sigma_y(t)u\) we obtain

\[
E \{\exp[\sigma_h W_h(t) - Y(t)]\}
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left\{\left[\sigma_h\sqrt{t} - \rho(t)\sigma_y(t)\right] u - v\right\} g(u, v)dudv
= \exp\left[\frac{1}{2}\sigma_h^2 t - \rho(t)\sigma_y(t)\sigma_h\sqrt{t} + \frac{1}{2}\sigma_y^2(t)\right], \tag{3.12}
\]

where

\[
g(u, v) = \frac{1}{2\pi\sigma_y(t)\sqrt{1 - \rho^2(t)}} \exp\left\{-\frac{1}{2} \left[ u^2 + \frac{v^2}{\sigma_y^2(t)(1 - \rho^2(t))}\right]\right\}.
\]

Noting that \(\{N(t), t \geq 0\}\) and the jumps \(\{J_i, i \geq 1\}\) are independent, and that \(J_i\) are Gaussian with mean \(\mu_J\) and variance \(\sigma_J^2\), we obtain

\[
E \left[\exp\left(\sum_{i=1}^{N(t)} J_i\right)\right] = e^{k_h \lambda_h t}, \tag{3.13}
\]

where \(k_h := \exp(\mu_J + \frac{1}{2}\sigma_J^2) - 1\), as defined in the interest rate model. Defining

\[
m_1 := \int_0^t \mu_h(s)ds - \left(\frac{1}{2}\sigma_h^2 + \lambda_h k_h\right)t,
\]

\[
m_2 := \mu_r t + \frac{1}{\alpha_r}(\mu_r - r_0)\left(e^{-\alpha_r t} - 1\right),
\]

it is easy to obtain

\[
h(t) = h_0 \exp \left[ m_1 + \sigma_h W_h(t) + \sum_{i=1}^{N(t)} J_i \right], \tag{3.14}
\]

\[
\int_0^t r(u)du = m_2 + Y(t). \tag{3.15}
\]
Noting that $\sigma h W_h(t) - Y(t)$ is independent of $\sum_{i=1}^{N(t)} J_i$, it follows from the Equations (3.12)-(3.15) that

\begin{equation}
E[h(t)d(t)] = h_0 e^{m_1 - m_2} E[\exp\{\sigma h W_h(t) - Y(t)\}] \left[ \exp\left( \sum_{i=1}^{N(t)} J_i \right) \right] = G(t) D(t),
\end{equation}

where $G(t)$ and $D(t)$ are respectively defined by Equations (3.6) and (3.7). This concludes the proof of Proposition 3.1. \qed

The following Proposition 3.2 presents an explicit expressions for the expected lump sum that the house owner can borrow in average at time 0 and the pricing equation that the annuity payments satisfy.

**Proposition 3.2.** Assume that $h(t)d(t)$ and $r(t)$, $(t \geq 0)$, are independent of $T(x_0)$, where recall that $h(t)$, $d(t)$, $r(t)$ and $T(x_0)$ are defined by Equations (2.1), (2.4), (2.3) and (2.6), respectively. If the pledged property is sold at time $T(x_0) + t$, then:

1. The expectation of the lump sum $\tilde{G}$ that the householder can borrow, in average, at the time of signing the reverse mortgage contract is given by

\begin{equation}
\tilde{G} = \int_0^{+\infty} G(x + t_0) D(x + t_0) f_T(x) dx.
\end{equation}

and

2. The annuity payments $A_k \ (k = 1, 2, \ldots)$ satisfy the following pricing equation

\begin{equation}
\int_0^{+\infty} G(x + t_0) D(x + t_0) f_T(x) dx = \sum_{k=1}^{+\infty} A_k D(k) \exp \left\{ - \int_0^k \lambda(x_0 + u) du \right\},
\end{equation}

where $f_T(x)$ is given by the Equation (2.7), and $G(\cdot)$ and $D(\cdot)$ are as in Equations (3.6) and (3.7).

**Proof.** Since the lender’s only gain can result from the proceeds of selling the pledged house, (subject to the principle of balance between expected gain and expected payment), the expectation of lump sum that the house owner can borrow at time 0 of signing the contract is equal to $E[h(T(x_0) + t_0)d(T(x_0) + t_0)]$. Noting that $h(t)d(t)$ $(t \geq 0)$ is independent of $T(x_0)$, we get

\begin{equation}
\tilde{G} = E[h(T(x_0) + t_0)d(T(x_0) + t_0)]
= \int_0^{+\infty} E[h(x + t_0)d(x + t_0)] f_T(x) dx
= \int_0^{+\infty} G(x + t_0) D(x + t_0) f_T(x) dx.
\end{equation}
From the independence of \( r(t) \) and \( T(x_0) \), we have

\[
E \left[ \sum_{k=1}^{[T(x_0)]} A_k d(k) \right] 
= E \left[ \sum_{i=1}^{+\infty} \sum_{k=1}^{i} A_k d(k) \mathbf{1}_{\{T(x_0) = i\}} \right]
= \sum_{i=1}^{+\infty} \sum_{k=1}^{i} A_k E[d(k)] P(T(x_0) = i) 
= \sum_{k=1}^{+\infty} A_k D(k) P(T(x_0) \geq k),
\]

where \( D(k) \) is as in the Equation (3.7). Recalling that the probability density function for \( T(x_0) \) is given by the Relation (2.7), we get the Equation (3.18). This proves Proposition 3.2.

The claims in the following Proposition 3.3 are special cases of the Proposition 3.2, and they present the valuation formulas for the increasing (or decreasing) perpetuity annuity and the level annuity.

**Proposition 3.3.** The payments for the increasing (or decreasing) perpetuity annuity are characterized as follows. At the end of \( k \)-th period, the annuity payment is \( A_k := A_0 + d \cdot k \), \( k = 1, 2, \ldots, n \), with \( A_0 \) and \( d \) positive constants (as the house owner is alive). Here, \( A_0 \) and \( d \) are determined by the simultaneous equations

\[
A_0 = \frac{\tilde{G} - d \cdot \tilde{F}_2}{\tilde{F}_1},
\]
\[
d = \frac{\tilde{G} - A_0 \cdot \tilde{F}_1}{\tilde{F}_2},
\]

where

\[
\tilde{F}_1 = \sum_{k=1}^{+\infty} D(k) \exp \left\{ - \int_0^k \lambda(x_0 + u) du \right\},
\]
\[
\tilde{F}_2 = \sum_{k=1}^{+\infty} k D(k) \exp \left\{ - \int_0^k \lambda(x_0 + u) du \right\},
\]

and \( G(x + t_0) \), \( D(x + t_0) \), and \( D(k) \) are as above.

For the level annuity, a fixed amount \( A \) of annuity is paid during the entire loan period and is given by

\[
A = \frac{\tilde{G}}{\tilde{F}_1},
\]

where \( \tilde{G} \) and \( \tilde{F}_1 \) are defined by the Equations (3.17) and (3.23), respectively.
It is easy to see that \( \tilde{F}_1 \) and \( \tilde{F}_2 \) can affect the amount of each annuity payment, and therefore we shall hereafter call them the annuity payment factors. The \( \tilde{G} \) is equal to the expectation of the discounted present value of the cumulative amount that the lender can lend to the house owner at time 0; we shall henceforth call it the lump sum for short.

4. MONOTONICITY PROPERTIES

In this section, we assume that the function \( \mu_h(s) \) representing the average rate of return of house price is constant, that is \( \mu_h(s) \equiv \mu_h \). The Propositions 4.1A–4.1D proved below analyze the monotonicity of the annuity payment, lump sum, and annuity payment factors with respect to the parameters involved in the house price model, the interest rate model, the force of mortality model, and the delay time of selling the pledged house.

4.1. Monotonicity w.r.t Parameters of House Price. The following Proposition 4.1 analyzes the monotonicity of the annuity payment, lump sum and annuity payment factors with respect to the house price model related parameters such as the constant rate of return \( \mu_h \), the volatility \( \sigma_h \), the initial house price \( h_0 \), the correlation coefficient between the Brownian motions driving the house price and those driving the interest rate \( \rho_{hr} \), and the delay time of selling the pledged house \( t_0 \). Proposition 4.1 is established in four parts via Propositions 4.1A–4.1D.

Proposition 4.1. With respect to the parameters of the house price, the basic annuity \( A_0 \) and the fixed increment \( d \), the annuity payment \( A \), the lump sum \( \tilde{G} \), and the annuity payment factors \( \tilde{F}_i \) (\( i = 1, 2 \)) have the monotonicity properties presented in the following Propositions 4.1A–4.1D.

Proposition 4.1A (Parameter \( \mu_h \)): (i) The annuity payment factors \( \tilde{F}_1 \) and \( \tilde{F}_2 \) are independent of \( \mu_h \).
(ii) The lump sum \( \tilde{G} \) is an increasing function of the average rate of return \( \mu_h \) of house price.
(iii) The quantities \( A_0 \), \( d \) and \( A \) appearing in Proposition 3.3 are increasing functions of \( \mu_h \).

Proof of Proposition 4.1A: From the definitions of the annuity payment factors \( \tilde{F}_1 \) and \( \tilde{F}_2 \) (see Relations (3.23) and (3.24)), we note that these two annuity payment factors are independent of \( \mu_h \).

Note that \( D(x + t_0) \) and \( f_T(x) \) do not depend on \( \mu_h \), and from Relation (3.17) that

\[
\frac{\partial}{\partial \mu_h} [G(x + t_0)D(x + t_0)f_T(x)] = (x + t_0)G(x + t_0)D(x + t_0)f_T(x),
\]
Since $G(x + t_0) > 0, D(x + t_0) > 0, f_T(x) \geq 0$, and $x + t_0 \geq 0$, we see from the above that the lump sum $\tilde{G}$ is increasing function of $\mu_r$.

Furthermore, from the Equations (3.21), (3.22) and (3.25), we have the quantities $A_0, d$ and $A$ appearing in Proposition 3.3 as increasing functions of $\mu_r$. This proves Proposition 4.1A.

**Proposition 4.1B (PARAMETER $\sigma_h$):** (i) The annuity payment factors $\tilde{F}_1$ and $\tilde{F}_2$ are independent of the volatility $\sigma_h$ of the house price.

(ii) In case of $\rho_{hr} > 0, \sigma_r > 0$ and $\alpha_r \neq 0$, the lump sum $\tilde{G}$ is a decreasing function of $\sigma_h$.

(iii) In case of $\rho_{hr} < 0, \sigma_r > 0$ and $\alpha_r \neq 0$, the lump sum $\tilde{G}$ is an increasing function of $\sigma_h$.

(iv) In case of $\rho_{hr} > 0, \sigma_r > 0$ and $\alpha_r \neq 0$, the quantities $A_0, d$ and $A$ are decreasing functions of $\sigma_h$.

(v) In case of $\rho_{hr} < 0, \sigma_r > 0$ and $\alpha_r \neq 0$, the quantities $A_0, d$ and $A$ are increasing functions of $\sigma_h$.

**Proof of Proposition 4.1B:** We note the definitions of the annuity payment factors $\tilde{F}_1$ and $\tilde{F}_2$ that these annuity payment factors are independent of $\sigma_h$.

Define

$$g_1(z) := \frac{1}{\alpha_r} \left[ -z + \frac{1}{\alpha_r} (1 - e^{-\alpha_r z}) \right]. \quad (4.1)$$

Note that $D(x + t_0)$ and $f_T(x)$ do not depend on $\sigma_h$, and that

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \sigma_h} = \sigma_r \rho_{hr} G(x + t_0)D(x + t_0)f_T(x) g_1(x + t_0).$$

In the case of $\alpha_r \neq 0$ and $z \geq 0$, we have $g_1(z) \leq 0$ and hence the Proposition 4.1B.

**Proposition 4.1C (PARAMETER $\rho_{hr}$):** (i) The annuity payment factors $\tilde{F}_1$ and $\tilde{F}_2$ are independent of $\rho_{hr}$.

(ii) In case of $\sigma_h > 0, \sigma_r > 0$ and $\alpha_r \neq 0$, the lump sum $\tilde{G}$ is a decreasing function of $\rho_{hr}$.

(iii) In case of $\sigma_h > 0, \sigma_r > 0$ and $\alpha_r \neq 0$, the quantities $A_0, d$ and $A$ are decreasing functions of $\rho_{hr}$.

**Proof of Proposition 4.1C:** Note that $D(x + t_0)$ and $f_T(x)$ are free of $\rho_{hr}$, and that

$$\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \mu_r} = \sigma_h \sigma_r G(x + t_0)D(x + t_0)f_T(x) g_1(x + t_0).$$

It follows from this and the definition of $g_1(z) \leq 0$ (see Equation (4.1)) when $z \geq 0, \alpha_r \neq 0$ that we have the Proposition 4.1C.

From the trend of proof of the Propositions 4.1A–4.1C, the proof of the following Proposition is clear, and hence we omit the proof.
**Proposition 4.1D (Parameter \( h_0 \))**: (i) The annuity payment factors \( \tilde{F}_1 \) and \( \tilde{F}_2 \) are independent of the initial house price \( h_0 \).
(ii) The lump sum \( \tilde{G} \) is an increasing function of \( h_0 \).
(iii) The quantities \( A_0, d \) and \( A \) are increasing functions of \( h_0 \).

**Proposition 4.2.** With respect to the delay time \( t_0 \) between taking over of the pledged property and the sale of that property, the basic annuity \( A_0 \), the fixed increment \( d \), the annuity payment \( A \), the lump sum \( \tilde{G} \), and the annuity payment factors \( \tilde{F}_i \) \((i = 1, 2)\) have the following monotonicity properties:

(a) The annuity payment factors \( \tilde{F}_1 \) and \( \tilde{F}_2 \) do not depend on \( t_0 \).
(b) Define

\[
\Delta := \left( \mu_r - r_0 + \frac{\sigma_h \sigma_r \rho h r}{\alpha_r} \right)^2 + \frac{2 \sigma_r^2 (r_0 - \mu_h)}{\alpha_r^2},
\]

\[
z_1 := -\frac{\alpha_r^2}{\sigma_r^2} \left[ \mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} + \frac{\sigma_h \sigma_r \rho h r}{\alpha_r} + \sqrt{\Delta} \right],
\]

and

\[
z_2 := -\frac{\alpha_r^2}{\sigma_r^2} \left[ \mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} + \frac{\sigma_h \sigma_r \rho h r}{\alpha_r} - \sqrt{\Delta} \right].
\]

(b-1) In the case of any one of the following conditions

\[\Delta \leq 0,\]
\[\Delta \geq 0, \alpha_r > 0, z_1 \geq 1,\]
\[\Delta \geq 0, \alpha_r > 0, z_2 \leq 0,\]

the lump sum \( \tilde{G} \) is an increasing function of \( t_0 \). Also, the quantities \( A_0, d \) and \( A \) (appearing in Proposition 3.3) are increasing functions of \( t_0 \).

(b-2) If

\[
\Delta \geq 0, \quad \alpha_r > 0, \quad z_1 \leq 0, \quad z_2 \geq 1,
\]

holds, then the lump sum \( \tilde{G} \) is a decreasing function of \( t_0 \). The quantities \( A_0, d \) and \( A \) are decreasing functions of \( t_0 \).

**Proof.** Define

\[
g_2(z) := \frac{\sigma_r^2}{2 \alpha_r^2} z^2 + \beta_1 z + \beta_0, \quad (-\infty < z < +\infty),
\]

where

\[
\beta_0 = \mu_h - \mu_r - \frac{\sigma_h \sigma_r \rho h r}{\alpha_r} + \frac{\sigma_r^2}{2 \alpha_r^2},
\]

\[
\beta_1 = \mu_r - r_0 + \frac{\sigma_h \sigma_r \rho h r}{\alpha_r} - \frac{\sigma_r^2}{\alpha_r^2}.
\]
It is easy to get that
\[
\frac{\partial}{\partial t_0} \left[ G(x + t_0) D(x + t_0) f_T(x) \right] = G(x + t_0) D(x + t_0) f_T(x) g_2(e^{-\alpha_r(x + t_0)}).
\]

Since the minimum of \( g_2(z) \) is \(-\frac{\alpha^2_r}{2\beta^2} \Delta \) (\( \Delta \) given by Equation (4.2)), when the condition \( \Delta \leq 0 \) holds, we then have \( g_2(z) \geq 0 \). Thus \( \tilde{G} \) is an increasing function of \( t_0 \).

Recall the definitions of \( z_1 \) and \( z_2 \) given above by the Relations (4.3) and (4.4), respectively. Now, if the condition \( \Delta \geq 0 \) holds, then \( g_2(z_i) = 0 \), \( i = 1, 2 \). Moreover, it is obvious that \( 0 < \exp(-\alpha_r(x + t_0)) \leq 1 \) in case of \( \alpha_r > 0 \) and \( x + t_0 \geq 0 \). Thus the lump sum \( \tilde{G} \) is a decreasing function of \( t_0 \) whenever the Condition (4.5) holds. One similarly obtains the rest of the properties, thereby concluding the proof of Proposition 4.2. \( \square \)

4.2. Monotonicity w.r.t Parameters of Interest Rate. The following Proposition 4.3 analyzes how the lump sum and annuity payment factors vary with the parameters involved in the interest rate model, such as the initial interest rate \( r_0 \), the mean reversion level \( \mu_r \) and the volatility \( \sigma_r \).

**Proposition 4.3.** With respect to the parameters of the interest model, the basic annuity \( A_0 \), the fixed increment \( d \), the annuity payment \( A \), and the annuity payment factors \( \tilde{F}_i \) \( (i = 1, 2) \) have the following properties:

(1). (Parameter \( r_0 \)): If \( \alpha_r \neq 0 \), then \( \tilde{F}_1, \tilde{F}_2 \) and \( \tilde{G} \) are a decreasing functions of \( r_0 \).

(2). (Parameter \( \mu_r \)): If \( \alpha_r > 0 \), then \( \tilde{F}_1, \tilde{F}_2 \) and \( \tilde{G} \) are a decreasing functions of \( \mu_r \). If the opposite case \( \alpha_r < 0 \) holds, then \( \tilde{F}_1, \tilde{F}_2 \) and \( \tilde{G} \) are an increasing functions of \( \mu_r \).

(3). (Parameter \( \sigma_r \)): (a) In case of \( \alpha_r \neq 0, \sigma_r > 0 \), \( \tilde{F}_1, \tilde{F}_2 \) are increasing functions of \( \sigma_r \).

(b) In the case of \( \alpha_r > 0, \sigma_h > 0 \) and \( \rho_{hr} \geq 0 \), \( \tilde{G} \) is a decreasing function of \( \sigma_r \) in the interval \( \sigma_r \in (0, \sigma_h \rho_{hr} \alpha_r] \), and that the \( A_0, d \) and \( A \) are decreasing functions of \( \sigma_r \).

(c) In the case of \( \alpha_r > 0, \sigma_h > 0 \) and \( \rho_{hr} \leq 0 \), \( \tilde{G} \) is an increasing function of \( \sigma_r \).

**Proof.** Note that both \( G(x + t_0) \) and \( f_T(x) \) are independent of \( r_0 \), that the partial derivative of the integrand in the definition of \( \tilde{G} \) is
\[
\frac{\partial}{\partial r_0} \left[ G(x + t_0) D(x + t_0) f_T(x) \right] = -\frac{1}{\alpha_r} \left[ 1 - e^{-\alpha_r(x + t_0)} \right] G(x + t_0) D(x + t_0) f_T(x),
\]
and that
\[
\frac{\partial F_1}{\partial r_0} = -\frac{1}{\alpha_r} \sum_{k=1}^{+\infty} (1 - e^{-\alpha_r k}) D(k) \exp \left\{ - \int_0^k \lambda(x_0 + u)du \right\},
\]
\[
\frac{\partial F_2}{\partial r_0} = -\frac{1}{\alpha_r} \sum_{k=1}^{+\infty} k(1 - e^{-\alpha_r k}) D(k) \exp \left\{ - \int_0^k \lambda(x_0 + u)du \right\}.
\]

Now, since \(\frac{1}{\alpha_r} (1 - e^{-\alpha_r z}) \geq 0\) whenever \(\alpha_r \neq 0\) and \(z \geq 0\), we obtain Part 1 of the Proposition.

Define now
\[(4.6) \quad g_3(z) := -z + \frac{1}{\alpha_r} (1 - e^{-\alpha_r z}).\]

Since \(G(x + t_0)\) and \(f_T(x)\) are free of \(\mu_r\), we have
\[
\frac{\partial \left[ G(x + t_0) D(x + t_0) f_T(x) \right]}{\partial \mu_r} = G(x + t_0) D(x + t_0) f_T(x) g_3(x + t_0).
\]

Also,
\[
\frac{\partial F_1}{\partial \mu_r} = \sum_{k=1}^{+\infty} D(k) g_3(k) \exp \left\{ - \int_0^k \lambda(x_0 + u)du \right\},
\]
\[
\frac{\partial F_2}{\partial \mu_r} = \sum_{k=1}^{+\infty} kD(k) g_3(k) \exp \left\{ - \int_0^k \lambda(x_0 + u)du \right\}.
\]

Since \(g_3(z) \leq 0\) in case of \(\alpha_r > 0, z \geq 0\), and \(g_3(z) \geq 0\) in case of \(\alpha_r < 0, z \geq 0\), we obtain Part 2.

For \(-\infty < z < +\infty\), define
\[
g_4(z) := -\frac{\sigma_r}{2\alpha_r^2} e^{-2\alpha_r z} + \left( \frac{2\sigma_r}{\alpha_r} - \frac{\sigma_h \rho_h}{\alpha_r} \right) e^{-\alpha_r z}
+ \left( \frac{\sigma_r}{\alpha_r^2} - \frac{\sigma_h \rho_h}{\alpha_r} \right) z + \left( \frac{\sigma_h \rho_h}{\alpha_r} - \frac{3\sigma_r}{2\alpha_r^2} \right),
\]
\[
g_5(y) := \frac{1}{\alpha_r} \left[ \frac{\sigma_r}{\alpha_r} y^2 + \left( \sigma_h \rho_h - \frac{2\sigma_r}{\alpha_r} \right) y + \left( \frac{\sigma_r}{\alpha_r} - \sigma_h \rho_h \right) \right].
\]

We observe the following: (1) If \(\frac{\sigma_h \rho_h}{\sigma_r} \geq 1\), then \(g_5(y)\) has two zero points \(y_2 = 1\) and \(y_1 = 1 - \frac{\sigma_h \rho_h}{\sigma_r}\); and (2) If \(\frac{\sigma_h \rho_h}{\sigma_r} \leq 0\), then \(g_5(y)\) has two zero points \(y_1 = 1\) and \(y_2 = 1 - \frac{\sigma_h \rho_h}{\sigma_r} \).

Next,
\[
\frac{dg_4(z)}{dz} = \frac{1}{\alpha_r} \left[ \frac{\sigma_r}{\alpha_r} e^{-2\alpha_r z} + \left( \sigma_h \rho_h - \frac{2\sigma_r}{\alpha_r} \right) e^{-\alpha_r z} + \left( \frac{\sigma_r}{\alpha_r} - \sigma_h \rho_h \right) \right] = g_5(e^{-\alpha_r z}),
\]

Note that \(0 < e^{-\alpha_r z} \leq 1\) in case of \(z \in [0, +\infty), \alpha_r > 0\). If \(z \in [0, +\infty), \alpha_r > 0, \sigma_h > 0\) and \(\rho_h \geq 0\), we then have \(g_5(e^{-\alpha_r z}) \leq 0\) in the interval \(\sigma_r \in (0, \sigma_h \rho_h \alpha_r]\). So, \(g_4(z) \leq g_4(0) = 0\) in the interval \(\sigma_r \in (0, \sigma_h \rho_h \alpha_r]\). In case of \(z \in [0, +\infty), \alpha_r > 0, \sigma_h > 0\) and \(\rho_h < 0\), we then have \(g_5(e^{-\alpha_r z}) \geq 0\) in the interval \(\sigma_r \in (\sigma_h \rho_h \alpha_r, 0]\). So, \(g_4(z) \geq g_4(0) = 0\) in the interval \(\sigma_r \in (\sigma_h \rho_h \alpha_r, 0]\).
$0, \sigma_h > 0$ and $\rho_{hr} \leq 0$, we have $g_5(e^{-\alpha_r z}) \geq 0$, and hence $g_4(z)$ is an increasing function of $z$ and $g_4(z) \geq g_4(0) = 0$.

Noting that $G(x + t_0)$ and $f_T(x)$ do not depend on $\sigma_r$, we have
\[
\frac{\partial [G(x + t_0)D(x + t_0)f_T(x)]}{\partial \sigma_r} = G(x + t_0)D(x + t_0)f_T(x)g_4(x + t_0).
\]

Now introduce
\[
g_6(z) := z + \left[1 - \frac{(2 - e^{-\alpha_r z})^2}{2\alpha_r}ight].
\]

Since
\[
\frac{\partial \tilde{F}_1}{\partial \sigma_r} = \frac{\sigma_r}{\alpha_r^2} \sum_{k=1}^{+\infty} D(k)g_6(k) \exp \left\{ - \int_0^k \lambda(x_0 + u)du \right\},
\]
\[
\frac{\partial \tilde{F}_2}{\partial \sigma_r} = \frac{\sigma_r}{\alpha_r^2} \sum_{k=1}^{+\infty} kD(k)g_6(k) \exp \left\{ - \int_0^k \lambda(x_0 + u)du \right\},
\]
we observe that $g_6(z) \geq 0$ whenever $z \in [0, +\infty)$. This proves Part 3 thereby completing the proof of Proposition 4.3.

4.3. Monotonicity w.r.t Parameters of Force of Mortality. The following Proposition 4.4 analyzes how the annuity payment factors vary with the parameters involved in the force of mortality model (parameters such as the initial age $x_0$, the constant age-independent hazard rate $a$, the dispersion coefficient $b$, and the modal value $c$.

**Proposition 4.4.** With respect to the parameters of the Gompertz-Makeham force of mortality defined by Equation (2.8), the annuity payment factors $\tilde{F}_1$, $\tilde{F}_2$ have the following monotonicity properties:

1. **Parameter $x_0$:** In case of $b > 0$, the annuity payment factors $\tilde{F}_1$ and $\tilde{F}_2$ are decreasing functions of $x_0$.

2. **Parameter $a$:** The annuity payment factors $\tilde{F}_1$, $\tilde{F}_2$ are decreasing functions of $a$.

3. **Parameter $b$:** If $x_0 \geq c$, the annuity payment factors $\tilde{F}_1$, $\tilde{F}_2$ are increasing functions of $b$.

4. **Parameter $c$:** The annuity payment factors $\tilde{F}_1$, $\tilde{F}_2$ are increasing functions of $c$.

**Proof.** Define
\[
g_7(x) := -\frac{1}{b} \left(e^{\frac{x}{b}} - 1\right) \exp \left\{ \frac{x_0 - c}{b} \right\} \cdot \exp \left\{ - \int_0^x \lambda(x_0 + u)du \right\},
\]
and note that
\[
\frac{\partial \tilde{F}_1}{\partial x_0} = \sum_{k=1}^{+\infty} D(k)g_7(k), \quad \frac{\partial \tilde{F}_2}{\partial x_0} = \sum_{k=1}^{+\infty} kD(k)g_7(k),
\]
and that
\[
\frac{\partial \tilde{F}_1}{\partial a} = -\sum_{k=1}^{+\infty} kD(k) \exp \left\{ -\int_0^k \lambda(x_0 + u) du \right\},
\]
\[
\frac{\partial \tilde{F}_2}{\partial a} = -\sum_{k=1}^{+\infty} k^2 D(k) \exp \left\{ -\int_0^k \lambda(x_0 + u) du \right\}.
\]

Define next
\[
g_8(x) := \frac{1}{b^2} [(x_0 - c) (e^{\frac{x}{b}} - 1) + xe^{\frac{x}{b}}] \exp \left\{ \frac{x_0 - c}{b} \right\} \cdot \exp \left\{ -\int_0^x \lambda(x_0 + u) du \right\},
\]
and note that
\[
\frac{\partial \tilde{F}_1}{\partial b} = \sum_{k=1}^{+\infty} D(k)g_8(k), \quad \frac{\partial \tilde{F}_2}{\partial b} = \sum_{k=1}^{+\infty} kD(k)g_8(k),
\]
and
\[
\frac{\partial \tilde{F}_1}{\partial c} = -\sum_{k=1}^{+\infty} D(k)g_7(k), \quad \frac{\partial \tilde{F}_2}{\partial c} = -\sum_{k=1}^{+\infty} kD(k)g_7(k).
\]
Now the assumptions made for the Proposition 4.4 prove the proposition.

\[\square\]

Remark 4.1. Obviously, the following properties are also true: The \( A_0 \) in Proposition 3.3 is a decreasing function of \( d \), and in turn, \( d \) is a decreasing function of \( A_0 \).

5. NUMERICAL EXPERIMENT

We shall test in this section the monotonicity conclusions established in Section 4 through the following numerical analysis. This sections illustrates the impacts of risks involved in the house price, the interest rate, and the longevity on the annuity payment, the lump sum, and the annuity payment factors. We take the parameters involved in the models of house price, interest rate and longevity with the following values as the standard case. Here, the values of parameters \( a, b, \) and \( c \) come from the Gompertz-Makeham force of mortality in Huang et al., 2013. With these parameters

| Table 1. Parameters of the standard case |
|----|----|----|----|----|----|----|----|----|----|----|
| Para | \( \mu_h \) | \( \sigma_h \) | \( \rho_{hc} \) | \( h_0 \) | \( t_0 \) | \( r_0 \) | \( \mu_r \) | \( \sigma_r \) | \( \alpha_r \) | \( x_0 \) | \( d \) | \( a \) | \( b \) | \( c \) |
| Value | 0.04 | 0.07 | 0.025 | 100 | 0 | 0.04 | 0.06 | 0.01 | 0.25 | 65 | 0 | 9.5 | 86.3 |

of standard case, we obtain the level annuity \( A = 7.138 \); the lump sum \( \tilde{G} = 75.796 \); the annuity payment factors \( \tilde{F}_1 = 10.618 \) and \( \tilde{F}_2 = 92.651 \).
Table 2. Impacts of the house price

<table>
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<th>$\mu_h$</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.1</th>
<th>0.12</th>
<th>0.14</th>
<th>0.16</th>
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<tr>
<td>$A$</td>
<td>5.121</td>
<td>7.138</td>
<td>10.246</td>
<td>15.139</td>
<td>23.005</td>
<td>35.907</td>
<td>57.473</td>
<td>94.174</td>
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<tr>
<td>$\bar{G}$</td>
<td>54.377</td>
<td>75.796</td>
<td>108.795</td>
<td>160.750</td>
<td>244.277</td>
<td>381.273</td>
<td>610.266</td>
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<tr>
<td>$\sigma_h$</td>
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<td>0.07</td>
<td>0.12</td>
<td>0.17</td>
<td>0.22</td>
<td>0.27</td>
<td>0.32</td>
<td>0.37</td>
</tr>
<tr>
<td>$\bar{G}$</td>
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<td>75.796</td>
<td>75.744</td>
<td>75.693</td>
<td>75.641</td>
<td>75.589</td>
<td>75.538</td>
<td>75.486</td>
</tr>
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<td>-0.9</td>
<td>-0.6</td>
<td>-0.3</td>
<td>0</td>
<td>0.3</td>
<td>0.6</td>
<td>1</td>
</tr>
<tr>
<td>$A$</td>
<td>7.426</td>
<td>7.397</td>
<td>7.312</td>
<td>7.228</td>
<td>7.154</td>
<td>7.064</td>
<td>6.984</td>
<td>6.879</td>
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<td>75.869</td>
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<td>$h_0$</td>
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<td>400</td>
<td>500</td>
<td>600</td>
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<td>800</td>
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5.1. Sensitivity Analysis for the House Price. We start the numerical analysis of how the parameters of the house price model impacts the annuity, lump sum, and annuity factors, while we keep fixed the other parametric values given above.

Table 2 shows the following:

(a) **Parameter $\mu_h$:** The higher the mean return of the home price, greater the lump sum and annuity are; however, the annuity payment factors remain constant, $F_1 = 10.618$ and $F_2 = 92.651$ (as they are not affected by $\mu_h$), which coincides with the conclusions in Proposition 4.1A. Since any higher mean return of the home price contributes to increased profit from the sale of the mortgaged house in future, the aforementioned phenomenon is sensible.

(b) **Parameter $\sigma_h$:** The greater the volatility of the home price, lesser the lump sum and annuity are; however, the annuity payment factors remain constant, $F_1 = 10.618$ and $F_2 = 92.651$ (as they are not affected by $\sigma_h$). These coincide with the conclusions in Proposition 4.1B. Since any higher volatility of the home price implies the greater market risk, the lender have to decrease the annuities and the lump sum in order to attenuate the market risk.

(c) **Parameter $\rho_{hr}$:** The bigger the correlation coefficient between the house price and the interest rate, smaller the lump sum and annuity are; however, the annuity payment factors remain constant, $F_1 = 10.618$ and $F_2 = 92.651$ (as they are not affected by $\rho_{hr}$). These are supported in theory by the conclusions in Proposition 4.1C.

(d) **Parameter $h_0$:** The larger the initial house price, larger the lump sum and annuity are; however, the annuity payment factors remain constant, $F_1 = 10.618$ and $F_2 = 92.651$ (as they are not affected by $h_0$). These are supported in theory by the conclusions in Proposition 4.1D. Since a greater initial house price implies a greater profit from the sale of the pledged house in future, the house owner will obtain larger...
annuities and lump sum subject to the principle of balance between expected gain and expected payment.

Next we vary the time delay $t_0$ (while keeping other parameters fixed as above) and analyze how it affects the annuity, lump sum, and annuity payment factors.

**Table 3.** Impacts of the delay time of selling house

<table>
<thead>
<tr>
<th>$t_0$</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{G}$</td>
<td>75.796</td>
<td>75.124</td>
<td>74.452</td>
<td>73.781</td>
<td>73.110</td>
<td>72.441</td>
<td>71.775</td>
<td>71.111</td>
</tr>
</tbody>
</table>

(e) **Parameter $t_0$:** Table 3 shows that, the larger the delay time in selling the pledged house, smaller the lump sum and annuity, (while the annuity payment factors remain constant $\tilde{F}_1 = 10.618$ and $\tilde{F}_2 = 92.651$ (not influenced by $t_0$)). This is in accord with Proposition 4.2(b-2) in the case of $\Delta = 0.000403 \geq 0, \alpha_c = 0.25 > 0, z_1 = -24.0875 \leq 0$ and $z_2 = 1 \geq 1$. However, it should be noted that if we changed some parameters, the lump sum might also increase with an increase in $t_0$ (refer to Proposition 4.2(b-1)). It implies that the lender may choose the right time to sell the pledged house according to the parameters in the house price model and the interest rate model.

5.2. **Sensitivity Analysis for the Interest Rate.** This subsection provides the numerical analysis of how the interest rate impacts the annuity value $A$, the lump sum $\tilde{G}$, and the annuity factors $\tilde{F}_i, i = 1, 2$. Again, when we select one parameter to vary, we keep the remaining parametric values fixed.

**Table 4.** Impacts of the initial interest rate

<table>
<thead>
<tr>
<th>$r_0$</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.1</th>
<th>0.12</th>
<th>0.14</th>
<th>0.16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{G}$</td>
<td>81.574</td>
<td>75.796</td>
<td>70.440</td>
<td>65.476</td>
<td>60.873</td>
<td>56.605</td>
<td>52.648</td>
<td>48.978</td>
</tr>
<tr>
<td>$\tilde{F}_1$</td>
<td>11.273</td>
<td>10.618</td>
<td>10.005</td>
<td>9.431</td>
<td>8.894</td>
<td>8.391</td>
<td>7.919</td>
<td>7.477</td>
</tr>
<tr>
<td>$\tilde{F}_2$</td>
<td>99.542</td>
<td>92.651</td>
<td>86.250</td>
<td>80.302</td>
<td>74.776</td>
<td>69.641</td>
<td>64.869</td>
<td>60.433</td>
</tr>
</tbody>
</table>

(a) **Parameter $r_0$:** We note the following from the Table 4. The lump sum $\tilde{G}$ and annuity payment factors $\tilde{F}_i, i = 1, 2$, are decreasing as the initial interest rate $r_0$ increases. This agrees with our conclusions in Proposition 4.3. The annuity $A$ is also decreasing. With the explicit solution of interest rate in Equation (2.3), we know that a higher initial interest rate means an increase in the average interest rate. This contributes to a decreased average discounted factor of interest rate, and that in turn results in the lower lump sum and annuity payment factors.

(b) **Parameter $\mu_r$:** Table 5 provides the numerical values resulting from the impact of the average reversion level $\mu_r$ of the interest rate. Here we note that the
Table 5. Impacts of the average reversion level of interest rate

<table>
<thead>
<tr>
<th>$\mu_r$</th>
<th>0.02</th>
<th>0.04</th>
<th>0.06</th>
<th>0.08</th>
<th>0.1</th>
<th>0.12</th>
<th>0.14</th>
<th>0.16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{G}$</td>
<td>138.084</td>
<td>100.954</td>
<td>75.796</td>
<td>58.421</td>
<td>46.187</td>
<td>37.401</td>
<td>30.969</td>
<td>26.168</td>
</tr>
<tr>
<td>$\tilde{F}_2$</td>
<td>143.213</td>
<td>114.121</td>
<td>92.651</td>
<td>76.535</td>
<td>64.238</td>
<td>54.705</td>
<td>47.203</td>
<td>41.213</td>
</tr>
</tbody>
</table>

Lump sum $\tilde{G}$ and annuity payment factors $\tilde{F}_i, i = 1, 2$, decrease with the increase of average reversion level $\mu_r$ of interest rate. This conclusion is theoretically supported by our Proposition 4.3. The annuities are also decreasing with the increase of $\mu_r$.

Table 6. Impacts of the volatility of interest rate

<table>
<thead>
<tr>
<th>$\sigma_r$</th>
<th>0.005</th>
<th>0.010</th>
<th>0.015</th>
<th>0.020</th>
<th>0.025</th>
<th>0.030</th>
<th>0.035</th>
<th>0.040</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{G}$</td>
<td>75.292</td>
<td>75.796</td>
<td>76.671</td>
<td>77.936</td>
<td>79.617</td>
<td>81.751</td>
<td>84.385</td>
<td>87.582</td>
</tr>
<tr>
<td>$\tilde{F}_1$</td>
<td>10.590</td>
<td>10.618</td>
<td>10.666</td>
<td>10.735</td>
<td>10.824</td>
<td>10.935</td>
<td>11.070</td>
<td>11.231</td>
</tr>
<tr>
<td>$\tilde{F}_2$</td>
<td>92.190</td>
<td>92.651</td>
<td>93.428</td>
<td>94.532</td>
<td>95.982</td>
<td>97.801</td>
<td>100.021</td>
<td>102.681</td>
</tr>
<tr>
<td>$\tilde{G}$</td>
<td>70.718</td>
<td>70.614</td>
<td>70.526</td>
<td>70.452</td>
<td>70.393</td>
<td>70.349</td>
<td>70.319</td>
<td>70.303</td>
</tr>
<tr>
<td>$\tilde{F}_1$</td>
<td>10.110</td>
<td>10.111</td>
<td>10.113</td>
<td>10.117</td>
<td>10.121</td>
<td>10.127</td>
<td>10.133</td>
<td>10.140</td>
</tr>
<tr>
<td>$\tilde{F}_2$</td>
<td>86.916</td>
<td>86.936</td>
<td>86.971</td>
<td>87.019</td>
<td>87.081</td>
<td>87.157</td>
<td>87.246</td>
<td>87.350</td>
</tr>
</tbody>
</table>

(c) Parameter $\sigma_r$: Table 6 reveals that the annuity payment factors $\tilde{F}_i, i = 1, 2$, increase as the volatility of interest rate $\sigma_r$ increases, and this is consistent with the property 3(a) of Proposition 4.3. This is reasonable since the higher volatility rate of interest rate contributes to the higher average level of the discounted factor.

The first part of Table 6 shows that the annuity and the lump sum are increasing with the increase of the volatility of interest rate $\sigma_r$. However, this does not seem reasonable from the perspective of risk aversion. We also note that the payment of annuity and lump sum are increasing faster with the increase in the volatility of interest rate $\sigma_r$. On one hand, when the volatility of interest rate is at a higher level, a slight increase in the volatility will greatly increase the annuity and lump sum payments. On the other hand, when the volatility is at a lower level, the increase in volatility only make the annuity and lump sum amounts increase slightly. Thus, our pricing models can be grudgingly applied to pricing the annuity and lump sum in the lower volatility case, and they are unsuitable to price the annuity and lump sum in the higher volatility case.

In the second part of Table 6, we assign $\sigma_h = 0.12$, $\rho_{hr} = 0.25$, and $\alpha_r = 1.4$ (while keeping the other parameters as the standard case). We note that the lump sum $\tilde{G}$ and annuity $A$ are both decreasing as $\sigma_r$ increases from 0.005 to 0.04, which
is consistent with the Property 3(b) of Proposition 4.3 in case of $\alpha_r = 1.4 > 0$, $\sigma_h = 0.12 > 0$, $\rho_{hr} = 0.25 \geq 0$ and $\sigma_r \in (0, \sigma_h \rho_{hr} \alpha_r] = (0, 0.042]$. If the volatility $\sigma_r$ of interest rate can be controlled by the product $\sigma_h \rho_{hr} \alpha_r$, then the Property 3(b) of Proposition 4.3 implies that the annuities will decrease with the increase in $\sigma_r$. In this case we note that the annuity and lump sum pricing formulas are quite reasonable. In particular, the annuity and lump sum pricing formulas are still applicable in the higher volatility case as long as the volatility can be controlled by $\sigma_h \rho_{hr} \alpha_r$.

### Table 7. Impacts of the reversion speed of interest rate

<table>
<thead>
<tr>
<th>$\alpha_r$</th>
<th>0.05</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.5</th>
<th>1.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>8.021</td>
<td>7.138</td>
<td>7.039</td>
<td>7.018</td>
<td>7.011</td>
<td>7.008</td>
<td>7.007</td>
<td>7.006</td>
</tr>
<tr>
<td>$G$</td>
<td>92.884</td>
<td>75.796</td>
<td>72.755</td>
<td>71.756</td>
<td>71.265</td>
<td>70.974</td>
<td>70.781</td>
<td>70.645</td>
</tr>
<tr>
<td>$F_2$</td>
<td>107.486</td>
<td>92.651</td>
<td>89.246</td>
<td>88.049</td>
<td>87.449</td>
<td>87.090</td>
<td>86.851</td>
<td>86.682</td>
</tr>
</tbody>
</table>

(d) Parameter $\alpha_r$: From Table 7, it is clear that the lump sum and the annuity payment factors decrease with the increasing of the reversion speed $\alpha_r$ of interest rate as other parameters take the standard values. The decreasing speed of the annuity, the lump sum and the annuity payment factors become slower and slower with the increase of $\alpha_r$.

### 5.3. Sensitivity Analysis for the Initial Age.

In this subsection we discuss the impact made by the initial age on $A$, $\tilde{G}$, and $\tilde{F}_i$, $i = 1, 2$.

### Table 8. Impacts of the initial age

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>50</th>
<th>55</th>
<th>60</th>
<th>65</th>
<th>70</th>
<th>75</th>
<th>80</th>
<th>85</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>59.712</td>
<td>64.974</td>
<td>70.382</td>
<td>75.796</td>
<td>81.033</td>
<td>85.875</td>
<td>90.105</td>
<td>93.547</td>
</tr>
<tr>
<td>$F_2$</td>
<td>164.831</td>
<td>141.353</td>
<td>116.949</td>
<td>92.651</td>
<td>69.689</td>
<td>49.302</td>
<td>32.490</td>
<td>19.762</td>
</tr>
</tbody>
</table>

Table 8 illustrates that as the age $x_0$ of the home owner as she signs the contract increases, the lump sum $\tilde{G}$ and annuity $A$ are increasing, while annuity payment factors $\tilde{F}_i$, $i = 1, 2$, show a decreasing trend, and this is supported in theory by our Proposition 4.4. As the house owner enters into the contract at a later age, the resulting lower expected residual life time of the owner provides increased annuity payment.

### 5.4. Sensitivity Analysis for the Increasing (or Decreasing) Annuity.

For the increasing (or decreasing) annuity in Proposition 3.3, Table 9 shows that the increment $d$ decreases as $A_0$ increases, and that $A_0$ decreases as $d$ increases. This
Table 9. Impacts of the incremental creep

<table>
<thead>
<tr>
<th>$A_0$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>0.703</td>
<td>0.589</td>
<td>0.474</td>
<td>0.360</td>
<td>0.245</td>
<td>0.130</td>
<td>0.016</td>
<td>-0.099</td>
</tr>
<tr>
<td>$d$</td>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>$A_0$</td>
<td>7.138</td>
<td>6.266</td>
<td>5.393</td>
<td>4.521</td>
<td>3.648</td>
<td>2.775</td>
<td>1.903</td>
<td>1.030</td>
</tr>
</tbody>
</table>

conforms with our Remark 4.1. The lump sum and the annuity payment factors remain constant, $\tilde{G} = 75.796$, $\tilde{F}_1 = 10.618$ and $\tilde{F}_2 = 92.651$.

The annuity with varying payment may turn out to be decreasing or increasing depending on the value of the basic annuity $A_0$. If the basic annuity payment $A_0$ is determined at a higher level ($A_0 \geq 8$ in our example), the annuity with varying payment would become the decreasing annuity, (that is, the lender pays less and less annuity payments to the house owner and the decrement of each period is about 0.099 when the basic annuity $A_0$ is fixed as 8). In our example, the annuity with varying payment becomes an increasing annuity if the basic annuity $A_0$ is fixed at a level less than or equal to 7.

Table 10. Average Change Rate

<table>
<thead>
<tr>
<th>Para</th>
<th>$\mu_h$</th>
<th>$\sigma_h$</th>
<th>$\rho_{hr}$</th>
<th>$h_0$</th>
<th>$t_0$</th>
<th>$r_0$</th>
<th>$\mu_r$</th>
<th>$\sigma_r$</th>
<th>$\alpha_r$</th>
<th>$x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACR1</td>
<td>636.093</td>
<td>0.097</td>
<td>0.274</td>
<td>0.071</td>
<td>0.126</td>
<td>4.900</td>
<td>44.257</td>
<td>19.658</td>
<td>0.597</td>
<td>0.467</td>
</tr>
<tr>
<td>ACR2</td>
<td>6754.199</td>
<td>1.034</td>
<td>2.906</td>
<td>0.758</td>
<td>1.339</td>
<td>232.827</td>
<td>799.396</td>
<td>351.122</td>
<td>13.082</td>
<td>0.967</td>
</tr>
<tr>
<td>ACR3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>27.112</td>
<td>49.618</td>
<td>18.331</td>
<td>0.880</td>
<td>0.270</td>
</tr>
<tr>
<td>ACR4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>279.349</td>
<td>728.567</td>
<td>299.727</td>
<td>12.238</td>
<td>4.145</td>
</tr>
</tbody>
</table>

Note: ACR1-the average change rate of annuity; ACR2-the average change rate of lump sum; ACR3-the average change rate of annuity payment factor 1; ACR4-the average change rate of annuity payment factor 2. The average change rate of annuity is defined as the $\frac{A_M - A_m}{I_R - I_L}$. $A_M$ and $A_m$ respectively means the maximum annuity payment and the minimum annuity payment; $I_L$ and $I_R$ respectively means the maximum and the minimum of the parameter. For example, $^{94.174-5.121}_{0.16-0.02} = 636.093$. ACR2, ACR3 and ACR4 can be similarly obtained.

5.5. Comparison of All Parameters. Compared with other parameters of the home price and interest rate model, the mean return of house price $\mu_h$ has a dominating influence on both the annuity and the lump sum payments. The average reversion level $\mu_r$, the volatility $\sigma_r$ and the initial interest rate $r_0$ of interest rate respectively exert the second, third and fourth strongest impact on both the annuity and the lump sum. The remaining parameters have a slight effect on both the annuity and the lump sum. Table 9 shows that $\mu_r$ exert the most strongest influence on the annuity payment factors, followed by $r_0$, and $\sigma_r$. The parameters $\alpha_r$ and $x_0$ slightly affect the annuity payment factors. The annuity payment factors are not affected by $\mu_h$, $\sigma_h$, $\rho_{hr}$, $h_0$ and $t_0$. 
6. CONCLUSION

This paper builds a pricing model for the lifetime annuity of the reverse mortgage without redemption right, and derives the explicit pricing formula for the increasing (or decreasing) perpetuity annuity and the level annuity. We then discuss the monotonicity of the lump sum, annuity, and annuity payment factors with respect to the parameters associated with the home price, the interest rate, and the force of mortality model. Furthermore, we present some numerical results of the annuity, the lump sum, and the annuity payment factors, and analyze their sensitivity to the said parameters. Finally, based on the average change rate, we compare the impact of various parameters on the annuity, the lump sum, and the annuity payment factors. The results show that the average return of home price exerts a dominating influence on both the annuity and the lump sum. Next to the average return of home price, the mean reversion level of interest rate, the volatility of interest rate and the initial interest rate make the second, third and fourth strongest impact on both the annuity and the lump sum. Otherwise, the remaining parameters slightly affect both the annuity and the lump sum.

However, it should be noted that the average change rate depends on the range of the parameter. Once the ranges of parameters change, they will change the evaluation results for the parameter. Thus, the right range should be chosen in order to more properly evaluate the importance of the parameters. Moreover, the model selection of the house price, interest rate and force of mortality will directly affect the final pricing results. Therefore, it is suggested to collect the data of house price, interest rate and population data of the particular area that the reverse mortgage product covers, and model the special house price, interest rate and force of mortality model based on the collected historical data. This will be propitious to better price the reverse mortgage product.

REFERENCES


