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FAIR PRICING OF REVERSE MORTGAGE WITHOUT REDEMPTION RIGHT

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ABSTRACT. We derive in this article the pricing of a reverse mortgage without redemption right. In this, the underlying model employs a jump-diffusion process to represent the dynamics of the housing price, the Vasicek model to drive the instantaneous interest rate, and the force mortality model to describe the longevity risk. The said pricing is based on the Principle of Balance between the expected gain and expected payment. We compute the expected gain and the expected payment respectively under the continuous and discrete framework. We also present, with the above model, explicit formulas for the increasing (or decreasing) perpetual annuity and the level perpetual annuity. Furthermore, we discuss the monotonicity property of the annuities, lump sum, and annuity payment factors with respect to the parameters associated with the house price, the interest rate, and the force of mortality model. Finally, some numerical results for the lump sum, the annuity, and the annuity payment factors are presented, and also the sensitivity with respect to the above parameters is discussed. Based on the average change rate, we evaluate all parameters' degree of impact on the annuity, the lump sum, and the annuity payment factors.

Keywords: Reverse mortgage; Fair pricing; Perpetual annuity; Jump-diffusion; Vasicek model; Force of mortality

1. INTRODUCTION

Reverse Mortgage is an inviting financial lending product offered to any senior citizen who owns a house. It is normally categorized by law into two categories, *viz.*, (i) collateral reverse mortgage and (ii) ownership conversion reserve mortgage

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(Ohgaki, 2003). The collateral reverse mortgage can be redeemed and ownership conversion reverse mortgage not. In the case of the *collateral reverse mortgage*, the borrower is able to redeem the reverse mortgage by repaying the loan amount with the accumulated interests through property sale at any time from the mortgage's effective date to due date. Of course, when the reserve mortgage contract is due, the borrower can choose a financial institution to auction off the pledged property to repay loan and due interests. In a collateral reverse mortgage, the elderly householder borrows annuity like periodical installment mortgage on his/her residential house. Home Equity Conversion Mortgage System is a typical collateral reverse mortgage in USA. In the case of the *ownership conversion reverse mortgage*, the borrower enters into a contract with a lending institution to obtain an annuity until his/her death, and at death the pledged property is transferred to the lender. *Rente Viager* is a typical ownership conversion reverse mortgage offered in France (Ohgaki, 2003).

Since the introduction of reverse mortgage, earlier research mainly included the basic principle, operation modes, feasibility, effectiveness, policies, laws, risks, and pricing. The literature on pricing reverse mortgage is not as rich as those on other aspects. The pricing of reverse mortgage mainly refers to how to determine a lump sum and annuity payments that the lender can pay. The main pricing techniques include two areas: (a) *the actuarial pricing technique* and (b) *the option pricing technique*. Generally, the former technique is employed to price the reverse mortgage when the redemption right has not been taken into account, and in the opposite case the latter technique is applied. The main idea of the former is to employ the principle of balance between the expected gain and expected payment under the assumption of perfect competition market. This makes the discounted present value of payment of the lender to be equal to a certain proportion of discounted present value of the mortgaged property, (see DiVenti and Herzog (1990), Tse(1995), Mitchell and Piggott (2004)). The main idea behind the latter is to apply the option pricing technique, which regards the mortgaged property (the pledged property is usually assumed to follow a stochastic process or stochastic series) as the underlying asset, and the loan principal and accumulated interests as the strike price of underlying asset. When the contract expires, the lender or its successor determines whether or not to execute the option (i.e., redeem the pledged property) according to the difference between the price of pledged property and the loan principle and accumulated interests, (see Li et al. (2010), Chen et al. (2010b), Lee et al. (2012), and Tsay et al. (2014)).

The main risks involved with reverse mortgage, as pointed out by Szymanoski (1994), include property value risk, interest rate risk, and longevity risk. In order to rationally price the reverse mortgage, one must build an appropriate model that takes into account the above risks. In general, the risk of housing price is modeled in two ways. The first one is to assume directly that the dynamics of housing price is driven by a forward stochastic differential equation, as in Bardhan *et al.* (2006), Wang *et al.* (2008), Mizrach (2012), Huang *et al.* (2011), Chen *et al.* (2010a), Lee *et al.* (2012), and Tsay *et al.* (2014). The second one is to fit the time series model based on the historical data of the housing price, as discussed by Nothaft *et al.* (1995), Chinloy *et al.* (1997), Chen *et al.* (2010b), and Li *et al.* (2010).

The literature on classical interest rate model includes: the Dothan (1978) model, Vasicek (1977) model, Cox, Ingersoll and Ross (1985) model, Exponential Vasicek model, Hull and White (1990) model, Black and Karasinski (1991) model, Mercurio and Moraleda (2000) model, the CIR++ model, and the Extended Exponential Vasicek model (Brigo and Mercurio, 2006).

There are usually several ways to describe the longevity risk, such as a life table, force of mortality model. The classical force of mortality model can refer to de Moivre (1724), Gompertz (1825), Makeham (1860, 1867), Weibull (1951), Heligman and Pollard (1980), and Lee-Carter (1992).

In the model we study, we use a jump diffusion process to represent the dynamics of the housing price, the Ornstein-Uhlenbeck process is utilized to derive the instantaneous interest rate, and appeal to the force of mortality to describe the longevity risk. With this model we price the reverse mortgage without redemption right.

This article is organized as follows. Section 2 presents the models of risk factors. In Section 3, we first design the reverse mortgage without redemption right with fixed yearly payment until death, and then derive the pricing model for the lump sum and annuity payments by the principle of balance between expected gain and expected payment. In Section 4, we analyze the monotonicity of the lump sum, annuity payments, and annuity payment factors with respect to the parameters involved in housing price, interest rate and force of mortality models. Section 5 provides numerical results to examine how the housing price risk, interest rate risk, and longevity risk impact the lump sum, the annuity payment, and the annuity payment factors. Finally, in Section 6 we draw some conclusions from our findings.

2. RISK FACTORS

In order to obtain a suitable model to value the annuity of reverse mortgage without redemption right, we must first explore how to describe the risk factors that the reverse mortgage enforces. In this section we employ the jump-diffusion model to simulate the dynamics of house price, the Vasicek model to drive the instantaneous interest rate, and a force of mortality model to describe the longevity risk.

2.1. House Price.

Our stochastic quantities are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t\geq 0})$. We assume that the house price $H(t)$, $t \geq 0$, follows the special exponential Lévy process (Lee et al., 2012), namely the generalized Merton jump diffusion model (Merton, 1976). First we set up the notations needed to define the said Merton equation. Let $\{W_h(t), t \geq 0\}$ denote a $\mathcal{P}$-standard Brownian motion capturing the unanticipated instantaneous change of house price, (but, this may not work so well for abnormal shocks); $\{N(t), t \geq 0\}$ be the Poisson process with intensity λ_h, describing the total number of jumps (including the house price sudden rise and drop event) during the time interval of $(0, t]$; $\{J_i, i \geq 0\}$ be a sequence of independent normal random variables modeling the size of the jumps, with mean μ_J and variance σ_J^2; and let $k_h = \exp\left(\mu_J + \frac{1}{2}\sigma_J^2\right) - 1$ with σ_J being some positive constant. With these we model the house price by the generalized Merton jump diffusion process given by

(2.1)
$$h(t) = h(0)\exp\left[\int_0^t \mu_h(s)ds - \left(\frac{1}{2}\sigma_h^2 + \lambda_h k_h\right)t + \sigma_h W_h(t) + \sum_{i=1}^{N(t)} J_i\right], \ h(0) = h_0.$$

Here the standard assumption is that $\{W_h(t), t \geq 0\}$, $\{N(t), t \geq 0\}$ and $\{J_i, i \geq 0\}$ are independent. Note that $\mu_h(t)$ is the annual average return rate function w.r.t time t, and σ_h is the annual volatility of the house price, assuming $\sigma_h > 0$.

2.2. **Interest Rate.** We take the instantaneous short-rate dynamics as the Vasicek model (Vasicek, 1977). Specifically, the *interest rate process* $\{r(t), t \geq 0\}$ is governed by the following stochastic differential equation

$$dr(t) = \alpha_r(\mu_r - r(t))dt + \sigma_r dW_r(t), \quad r(0) = r_0, \tag{2.2}$$

where $\{W_r(t), t \geq 0\}$ is a $\mathcal{P}$-standard Brownian motion, and $r_0, \alpha_r, \mu_r, \sigma_r$ are positive constants. Denote the correlation coefficient between $W_r(t)$ and $W_h(t)$ by ρ_{hr}.

Applying Itô's formula to $e^{\alpha_r u} r(u)$ we obtain

$$r(t) = e^{-\alpha_r t} r(0) + \mu_r(1 - e^{-\alpha_r t}) + \sigma_r \int_0^t e^{-\alpha_r(t-u)} dW_r(u), \quad t \geq 0. \tag{2.3}$$

The *discount factor at time* t is denoted by $d(t)$ and is defined as

$$d(t) := \exp\left(-\int_0^t r(s)ds\right). \tag{2.4}$$

With some trivial computations, we have

(2.5)
$$E\left[d(t)\right] = \exp\left\{\left(\frac{\sigma_r^2}{2\alpha_r^2} - \mu_r\right)t + \frac{1}{\alpha_r}(\mu_r - r_0)(1 - e^{-\alpha_r t}) + \frac{\sigma_r^2}{4\alpha_r^3}\left[1 - (2 - e^{-\alpha_r t})^2\right]\right\}.$$

We refer to Norberg (2004) for the derivation of Equation (2.5) .

2.3. **Longevity.** We designate time $t = 0$ to be the time at which the reverse mortgage without redemption right is signed. Assume that the homeowner's age is x_0 years old at time $t = 0$. Let X represent the life span of the new born infant. Let

$$T(x_0) := X - x_0 \tag{2.6}$$

be the *residual life* of a home owner at his/her age x_0. The *force of mortality at age* x $(x \geq 0)$ is $\lambda(x)$. Then, the density function of $T(x_0)$ is

$$f_T(t) = \lambda(x_0 + t) \cdot \exp\left\{ - \int_0^t \lambda(x_0 + u) du \right\}. \tag{2.7}$$

In our numerical experiment, the force of mortality $\lambda(x)$ will be characterized by the Gompertz-Makeham force of mortality (see Carrière, 1994, Frees et al., 1996, or Huang et al., 2013)

$$\lambda(x) = a + \frac{1}{b} \exp\left(\frac{x - c}{b} \right), \tag{2.8}$$

where $a \geq 0$ denotes the constant hazard rate (independent of age); $b \geq 0$ is the dispersion; and $c \geq 0$ denotes a modal value. Note that the Gompertz-Makeham force of mortality model reduces to a constant force of mortality a as $c \to \infty$.

3. FAIR PRICING OF REVERSE MORTGAGE

In this section, we will first design a reverse mortgage without redemption right with fixed annual payment until the death of the house owner. Then, the pricing model of the reverse mortgage without redemption right are built based on the principle of balance between expected gain and expected payment. Under the two-dimensional Gaussian distribution and independence assumptions, we obtain the explicit pricing formulas for lump sum and annuity payments for the reverse mortgage, particularly, the increasing (decreasing) perpetuity annuity and the level annuity.

3.1. **Reverse Mortgage without Redemption Right.** In this section, we will design a reverse mortgage without redemption right with fixed annual payment paid to the house owner until his/her death. The product that we design has the following basic features:

(I) The lender starts the payment of annuity to the house owner at the end of the year of signing the contract. The annuity payment is terminated upon the death of the house owner. More precisely, had the house owner survived through the k-th year, $(k \geq 1)$, the lender would have paid the annuity payment $A_1, A_2, \ldots, A_k$ to the house owner at the end of the first, second, …, k-th year, respectively.

(II) When the house owner dies, the lender will take over the house-owner's pledged property, sell it in the market, and keep all of the proceeds from the sale of the property.

The essence of the reverse mortgage without redemption right is to exchange the profit from selling the mortgaged house with the house-owner's annuity paid until his/her death. When the house owner dies, the lender will take over the house-owner's mortgaged property and sell it. The cash that is acquired from the sale of the house-owner's house is used to repay loan (including annuities and accumulated interests) that the house owner owes to the lender. Since the reverse mortgage possesses the non-recourse clauses (that is, the lender may not reclaim the loan against the house-owner's other assets or cash income except for his/her pledged property), the lender will suffer a loss when the cash out of selling the mortgaged property is less than the total annuity paid plus the accumulated interest; otherwise the lender will make a profit.

We shall illustrate how the reverse mortgage product functions. Assume that the homeowner dies at the age of $X = 68.7$ and the contract was signed at her age of $x_0 = 65$. This means that the house owner lives in her house and claims three cash payments A_1, A_2, A_3 at the end of the first, second and third year of the contract, respectively. The annuity payment going to the house owner need not be a fixed amount implied by the Feature (I) above; that is, A_1, A_2, A_3 can be unequal amounts. When the house owner dies at the age of 68.7, the lender will take over the pledged house and sell it in the market. Most of the time, it may not be possible to sell the pledged house as soon as the lender take over it. Thus the time at which the pledged house is sold is usually much later than that of taking over the pledged house.

3.2. **Fair Pricing Model.** We assume that we are in the perfectly competitive market. We price the reverse mortgage by the principle of balance between the expected gain and expected payment. That is, the pricing is determined under the principle where the expected discounted present value of future sale of the pledged property balances out the expected discounted present value of annuities paid by the lender.

At time $T(x_0)$, the lender takes over the home-owner's mortgaged property, and sells it at time $T(x_0) + t_0$, where $t_0 \geq 0$ is the delay time between the lender taking over the mortgaged property and the sale of the mortgaged property. We assume that t_0 is fixed and not a random variable. Then the expectation of discounted present value of the sale price of the property (i.e., the lender's expected gain) is

$$E\left[h(T(x_0) + t_0)d(T(x_0) + t_0)\right], \tag{3.1}$$

where recall that $h(t)$ is the value of the mortgaged property at time t given by the Stochastic Differential Equation (2.1), and $d(t)$ is the discount factor at time t given by Equation (2.4).

The expectation of discounted present value of the home-owner's annuities (i.e., the lender's expected payment) is

$$E\left[\mathbf{1}_{\{1 \leq T(x_0) < +\infty\}} \sum_{i=1}^{[T(x_0)]} A_i d(i)\right], \tag{3.2}$$

where the function $[x]$ gives the largest integer not greater than x. Then, the principle of balance between the expected gain and expected payment yields

$$E\left[h(T(x_0)+t_0)d(T(x_0)+t_0)\right] = E\left[\mathbf{1}_{\{1\le T(x_0)<+\infty\}} \sum_{i=1}^{[T(x_0)]} A_i d(i)\right]. \tag{3.3}$$

Though the analytic formula of annuity payment is difficult to obtain from the Equation (3.3), we can obtain the analytic formula under the two-dimension Gauss distribution and independence assumptions. The following Proposition 3.1 presents the analytic formula for the expected discounted present value of the mortgaged property at any time t.

Proposition 3.1. *Define*

$$Y(t) := \int_0^t \left[\sigma_r e^{-\alpha_r s} \int_0^s e^{\alpha_r u} dW_r(u)\right] ds. \tag{3.4}$$

Assume that the dynamics of home price follows the exponential Lévy process given by the Equation (2.1)*, the instantaneous short interest rate is governed by the Equation* (2.2)*, the joint distribution of* $(W_h(t), Y(t))$ *follows the two dimensional normal distribution, and that* $\sigma_h W_h(t) - Y(t)$ *is independent of* $\sum_{i=1}^{N(t)} J_i$*. Then the expectation of discounted present value of the mortgaged property at time t is given by*

$$E\left[h(t)d(t)\right] = G(t)D(t), \tag{3.5}$$

where

$$G(t) = h_0 \exp\left\{\int_0^t \mu(s)ds - \sigma_h \sigma_r \rho_{hr} \frac{1}{\alpha_r}\left(t + \frac{1}{\alpha_r}e^{-\alpha_r t} - \frac{1}{\alpha_r}\right)\right\}, \tag{3.6}$$

and

$$D(t) = \exp\left\{\left(\frac{\sigma_r^2}{2\alpha_r^2} - \mu_r\right)t + \frac{1}{\alpha_r}(\mu_r - r_0)\left(1 - e^{-\alpha_r t}\right) + \frac{\sigma_r^2}{4\alpha_r^3}\left[1 - (2 - e^{-\alpha_r t})^2\right]\right\}. \tag{3.7}$$

Proof. We begin by noting that $Y(t)$ follows the normal distribution with the mean 0 and variance

$$\sigma_y^2(t) = \frac{\sigma_r^2}{\alpha_r^2}t + \frac{\sigma_r^2}{2\alpha_r^3}\left[1 - (2 - e^{-\alpha_r t})^2\right]. \tag{3.8}$$

Recalling that the correlation coefficient between $W_h(t)(t \ge 0)$ and $W_r(t)(t \ge 0)$ is ρ_{hr}, we also note that the covariance between $W_h(t)$ and $Y(t)$ is

$$Cov(W_h(t), Y(t)) = \sigma_r \rho_{hr} \frac{1}{\alpha_r}\left(t + \frac{1}{\alpha_r}e^{-\alpha_r t} - \frac{1}{\alpha_r}\right).$$

Thus the correlation coefficient between $W_h(t)$ and $Y(t)$, denoted by $\rho(t)$, is

$$\rho(t) = \frac{\sigma_r \rho_{hr}}{\alpha_r \sigma_y(t)\sqrt{t}}\left(t + \frac{1}{\alpha_r}e^{-\alpha_r t} - \frac{1}{\alpha_r}\right). \tag{3.9}$$

Since the joint distribution of $(W_h(t), Y(t))$ follows the two dimensional normal distribution with the correlation coefficient $\rho(t)$ obtained above, we have from Equation (3.8) that

$$E\left\{\exp\left[\sigma_h W_h(t) - Y(t)\right]\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(\sigma_h x - y) f(x,y)\,dx\,dy, \tag{3.10}$$

where

$$f(x,y) = \frac{1}{2\pi\sigma_y(t)\sqrt{t(1-\rho^2(t))}} \exp\left\{-\frac{1}{2(1-\rho^2(t))} S_{xy}\right\}, \tag{3.11}$$

and

$$S_{xy} = \left(\frac{x}{\sqrt{t}}\right)^2 - 2\rho(t)\frac{x}{\sqrt{t}}\frac{y}{\sigma_y(t)} + \left(\frac{y}{\sigma_y(t)}\right)^2.$$

Under the substitutions $u = \frac{x}{\sqrt{t}}$ and $v = y - \rho(t)\sigma_y(t)u$ we obtain

$$\begin{aligned} & E\left\{\exp\left[\sigma_h W_h(t) - Y(t)\right]\right\} \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left\{\left[\sigma_h\sqrt{t} - \rho(t)\sigma_y(t)\right]u - v\right\} g(u,v)\,du\,dv \\ &= \exp\left[\frac{1}{2}\sigma_h^2 t - \rho(t)\sigma_y(t)\sigma_h\sqrt{t} + \frac{1}{2}\sigma_y^2(t)\right], \end{aligned} \tag{3.12}$$

where

$$g(u,v) = \frac{1}{2\pi\sigma_y(t)\sqrt{1-\rho^2(t)}} \exp\left\{-\frac{1}{2}\left[u^2 + \frac{v^2}{\sigma_y^2(t)(1-\rho^2(t))}\right]\right\}.$$

Noting that $\{N(t), t \geq 0\}$ and the jumps $\{J_i, i \geq 1\}$ are independent, and that J_i are Gaussian with mean μ_J and variance σ_J^2, we obtain

$$E\left[\exp\left(\sum_{i=1}^{N(t)} J_i\right)\right] = e^{k_h\lambda_h t}, \tag{3.13}$$

where $k_h := \exp\left(\mu_J + \frac{1}{2}\sigma_J^2\right) - 1$, as defined in the interest rate model. Defining

$$m_1 := \int_0^t \mu_h(s)\,ds - \left(\frac{1}{2}\sigma_h^2 + \lambda_h k_h\right)t,$$

$$m_2 := \mu_r t + \frac{1}{\alpha_r}(\mu_r - r_0)\left(e^{-\alpha_r t} - 1\right),$$

it is easy to obtain

$$h(t) = h_0 \exp\left[m_1 + \sigma_h W_h(t) + \sum_{i=1}^{N(t)} J_i\right], \tag{3.14}$$

$$\int_0^t r(u)\,du = m_2 + Y(t). \tag{3.15}$$

Noting that $\sigma_h W_h(t) - Y(t)$ is independent of $\sum_{i=1}^{N(t)} J_i$, it follows from the Equations (3.12)-(3.15) that

$$E[h(t)d(t)] = h_0 e^{m_1 - m_2} E[\exp\{\sigma_h W_h(t) - Y(t)\}] E\left[\exp\left\{\sum_{i=1}^{N(t)} J_i\right\}\right] \qquad (3.16)$$
$$= G(t)D(t),$$

where $G(t)$ and $D(t)$ are respectively defined by Equations (3.6) and (3.7). This concludes the proof of Proposition 3.1. □

The following Proposition 3.2 presents an explicit expressions for the expected lump sum that the house owner can borrow in average at time 0 and the pricing equation that the annuity payments satisfy.

Proposition 3.2. *Assume that $h(t)d(t)$ and $r(t)$, $(t \geq 0)$, are independent of $T(x_0)$, where recall that $h(t)$, $d(t)$, $r(t)$ and $T(x_0)$ are defined by Equations (2.1), (2.4), (2.3) and (2.6), respectively. If the pledged property is sold at time $T(x_0) + t_0$, then:*

(1). *The expectation of the lump sum $\widetilde{G}$ that the householder can borrow, in average, at the time of signing the reverse mortgage contract is given by*

$$\widetilde{G} = \int_0^{+\infty} G(x + t_0)D(x + t_0)f_T(x)dx. \qquad (3.17)$$

and

(2). *The annuity payments A_k $(k = 1, 2, \dots)$ satisfy the following pricing equation*

$$\int_0^{+\infty} G(x + t_0)D(x + t_0)f_T(x)dx = \sum_{k=1}^{+\infty} A_k D(k) \exp\left\{-\int_0^k \lambda(x_0 + u)du\right\}, \qquad (3.18)$$

where $f_T(x)$ is given by the Equation (2.7), and $G(\cdot)$ and $D(\cdot)$ are as in Equations (3.6) and (3.7).

Proof. Since the lender's only gain can result from the proceeds of selling the pledged house, (subject to the principle of balance between expected gain and expected payment), the expectation of lump sum that the house owner can borrow at time 0 of signing the contract is equal to $E[h(T(x_0) + t_0)d(T(x_0) + t_0)]$. Noting that $h(t)d(t)$ $(t \geq 0)$ is independent of $T(x_0)$, we get

$$\widetilde{G} = E[h(T(x_0) + t_0)d(T(x_0) + t_0)] \qquad (3.19)$$
$$= \int_0^{+\infty} E[h(x + t_0)d(x + t_0)] f_T(x)dx$$
$$= \int_0^{+\infty} G(x + t_0)D(x + t_0)f_T(x)dx.$$

From the independence of $r(t)$ and $T(x_0)$, we have

$$E\left[\mathbf{1}_{\{1\le T(x_0)<+\infty\}}\sum_{k=1}^{[T(x_0)]}A_k d(k)\right] = E\left[\sum_{i=1}^{+\infty}\sum_{k=1}^{i}A_k d(k)\mathbf{1}_{\{[T(x_0)]=i\}}\right] = \sum_{i=1}^{+\infty}\sum_{k=1}^{i}A_k E\left[d(k)\right]P([T(x_0)]=i) = \sum_{k=1}^{+\infty}A_k D(k)P(T(x_0)\ge k), \tag{3.20}$$

where $D(k)$ is as in the Equation (3.7). Recalling that the probability density function for $T(x_0)$ is given by the Relation (2.7), we get the Equation (3.18). This proves Proposition 3.2. □

The claims in the following Proposition 3.3 are special cases of the Proposition 3.2, and they present the valuation formulas for the increasing (or decreasing) perpetuity annuity and the level annuity.

Proposition 3.3. *The payments for the increasing (or decreasing) perpetuity annuity are characterized as follows. At the end of k-th period, the annuity payment is* $A_k := A_0 + d\cdot k$, $k = 1, 2, \ldots, n$, *with* A_0 *and* d *positive constants (as the house owner is alive). Here,* A_0 *and* d *are determined by the simultaneous equations*

$$A_0 = \frac{\widetilde{G} - d\cdot\widetilde{F}_2}{\widetilde{F}_1}, \tag{3.21}$$

$$d = \frac{\widetilde{G} - A_0\cdot\widetilde{F}_1}{\widetilde{F}_2}, \tag{3.22}$$

where

$$\widetilde{F}_1 = \sum_{k=1}^{+\infty} D(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\}, \tag{3.23}$$

$$\widetilde{F}_2 = \sum_{k=1}^{+\infty} kD(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\}, \tag{3.24}$$

and $G(x+t_0)$, $D(x+t_0)$, *and* $D(k)$ *are as above.*

For the level annuity, a fixed amount A *of annuity is paid during the entire loan period and is given by*

$$A = \frac{\widetilde{G}}{\widetilde{F}_1}, \tag{3.25}$$

where $\widetilde{G}$ *and* $\widetilde{F}_1$ *are defined by the Equations* (3.17) *and* (3.23), *respectively.*

It is easy to see that $\widetilde{F}_1$ and $\widetilde{F}_2$ can affect the amount of each annuity payment, and therefore we shall hereafter call them the *annuity payment factors*. The $\widetilde{G}$ is equal to the expectation of the discounted present value of the cumulative amount that the lender can lend to the house owner at time 0; we shall henceforth call it the *lump sum* for short.

4. MONOTONICITY PROPERTIES

In this section, we assume that the function $\mu_h(s)$ representing the average rate of return of house price is constant, that is $\mu_h(s) \equiv \mu_h$. The Propositions 4.1A–4.1D proved below analyze the monotonicity of the annuity payment, lump sum, and annuity payment factors with respect to the parameters involved in the house price model, the interest rate model, the force of mortality model, and the delay time of selling the pledged house.

4.1. **Monotonicity w.r.t Parameters of House Price.** The following Proposition 4.1 analyzes the monotonicity of the annuity payment, lump sum and annuity payment factors with respect to the house price model related parameters such as the constant rate of return μ_h, the volatility σ_h, the initial house price h_0, the correlation coefficient between the Brownian motions driving the house price and those driving the interest rate ρ_{hr}, and the delay time of selling the pledged house t_0. Proposition 4.1 is established in four parts via Propositions 4.1A–4.1D.

Proposition 4.1. *With respect to the parameters of the house price, the basic annuity A_0 and the fixed increment d, the annuity payment A, the lump sum $\widetilde{G}$, and the annuity payment factors $\widetilde{F}_i$ $(i = 1, 2)$ have the monotonicity properties presented in the following Propositions* 4.1A–4.1D.

Proposition 4.1A (PARAMETER μ_h): **(i)** *The annuity payment factors $\widetilde{F}_1$ and $\widetilde{F}_2$ are independent of μ_h.*
(ii) *The lump sum $\widetilde{G}$ is an increasing function of the average rate of return μ_h of house price.*
(iii) *The quantities A_0, d and A appearing in Proposition* 3.3 *are increasing functions of μ_h.*

Proof of Proposition 4.1A: From the definitions of the annuity payment factors $\widetilde{F}_1$ and $\widetilde{F}_2$ (see Relations (3.23) and (3.24)), we note that these two annuity payment factors are independent of μ_h.

Note that $D(x + t_0)$ and $f_T(x)$ do not depend on μ_h, and from Relation (3.17) that

$$\frac{\partial \left[G(x + t_0)D(x + t_0)f_T(x)\right]}{\partial \mu_h} = (x + t_0)G(x + t_0)D(x + t_0)f_T(x),$$

Since $G(x+t_0) > 0, D(x+t_0) > 0, f_T(x) \geq 0$, and $x+t_0 \geq 0$, we see from the above that the lump sum $\widetilde{G}$ is increasing function of μ_h.

Furthermore, from the Equations (3.21), (3.22) and (3.25), we have the quantities A_0, d and A appearing in Proposition 3.3 as increasing functions of μ_h. This proves Proposition 4.1A.

Proposition 4.1B (PARAMETER σ_h): **(i)** *The annuity payment factors* $\widetilde{F}_1$ *and* $\widetilde{F}_2$ *are independent of the volatility* σ_h *of the house price.*
(ii) *In case of* $\rho_{hr} > 0, \sigma_r > 0$ *and* $\alpha_r \neq 0$*, the lump sum* $\widetilde{G}$ *is a decreasing function of* σ_h*.*
(iii) *In case of* $\rho_{hr} < 0, \sigma_r > 0$ *and* $\alpha_r \neq 0$*, the lump sum* $\widetilde{G}$ *is an increasing function of* σ_h*.*
(iv) *In case of* $\rho_{hr} > 0, \sigma_r > 0$ *and* $\alpha_r \neq 0$*, the quantities* A_0*,* d *and* A *are decreasing functions of* σ_h*.*
(v) *In case of* $\rho_{hr} < 0, \sigma_r > 0$ *and* $\alpha_r \neq 0$*, the quantities* A_0*,* d *and* A *are increasing functions of* σ_h*.*

Proof of Proposition 4.1B: We note the definitions of the annuity payment factors $\widetilde{F}_1$ and $\widetilde{F}_2$ that these annuity payment factors are independent of σ_h.

Define

$$g_1(z) := \frac{1}{\alpha_r}\left[-z + \frac{1}{\alpha_r}\left(1 - e^{-\alpha_r z}\right)\right]. \tag{4.1}$$

Note that $D(x+t_0)$ and $f_T(x)$ do not depend on σ_h, and that

$$\frac{\partial\left[G(x+t_0)D(x+t_0)f_T(x)\right]}{\partial\sigma_h} = \sigma_r\rho_{hr}G(x+t_0)D(x+t_0)f_T(x)g_1(x+t_0).$$

In the case of $\alpha_r \neq 0$ and $z \geq 0$, we have $g_1(z) \leq 0$ and hence the Proposition 4.1B.

Proposition 4.1C (PARAMETER ρ_{hr}): **(i)** *The annuity payment factors* $\widetilde{F}_1$ *and* $\widetilde{F}_2$ *are independent of* ρ_{hr}*.*
(ii) *In case of* $\sigma_h > 0$*,* $\sigma_r > 0$ *and* $\alpha_r \neq 0$*, the lump sum* $\widetilde{G}$ *is a decreasing function of* ρ_{hr}*.*
(iii) *In case of* $\sigma_h > 0$*,* $\sigma_r > 0$ *and* $\alpha_r \neq 0$*, the quantities* A_0*,* d *and* A *are decreasing functions of* ρ_{hr}*.*

Proof of Proposition 4.1C: Note that $D(x+t_0)$ and $f_T(x)$ are free of ρ_{hr}, and that

$$\frac{\partial\left[G(x+t_0)D(x+t_0)f_T(x)\right]}{\partial\mu_r} = \sigma_h\sigma_r G(x+t_0)D(x+t_0)f_T(x)g_1(x+t_0).$$

It follows from this and the definition of $g_1(z) \leq 0$ (see Equation (4.1)) when $z \geq 0, \alpha_r \neq 0$ that we have the Proposition 4.1C.

From the trend of proof of the Propositions 4.1A–4.1C, the proof of the following Proposition is clear, and hence we omit the proof.

Proposition 4.1D (PARAMETER h_0): **(i)** *The annuity payment factors* $\widetilde{F}_1$ *and* $\widetilde{F}_2$ *are independent of the initial house price* h_0.
(ii) *The lump sum* $\widetilde{G}$ *is an increasing function of* h_0.
(iii) *The quantities* A_0, d *and* A *are increasing functions of* h_0.

Proposition 4.2. *With respect to the delay time* t_0 *between taking over of the pledged property and the sale of that property, the basic annuity* A_0*, the fixed increment* d*, the annuity payment* A*, the lump sum* $\widetilde{G}$*, and the annuity payment factors* $\widetilde{F}_i$ $(i = 1, 2)$ *have the following monotonicity properties:*
(a) *The annuity payment factors* $\widetilde{F}_1$ *and* $\widetilde{F}_2$ *do not depend on* t_0.
(b) *Define*

$$\Delta := \left(\mu_r - r_0 + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r}\right)^2 + \frac{2\sigma_r^2(r_0 - \mu_h)}{\alpha_r^2}, \tag{4.2}$$

$$z_1 := -\frac{\alpha_r^2}{\sigma_r^2}\left[\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} + \sqrt{\Delta}\right], \tag{4.3}$$

and

$$z_2 := -\frac{\alpha_r^2}{\sigma_r^2}\left[\mu_r - r_0 - \frac{\sigma_r^2}{\alpha_r^2} + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} - \sqrt{\Delta}\right]. \tag{4.4}$$

(b-1) *In the case of any one of the following conditions*

$$\Delta \le 0,$$
$$\Delta \ge 0, \alpha_r > 0, z_1 \ge 1,$$
$$\Delta \ge 0, \alpha_r > 0, z_2 \le 0,$$

the lump sum $\widetilde{G}$ *is an increasing function of* t_0. *Also, the quantities* A_0, d *and* A *(appearing in Proposition* 3.3*) are increasing functions of* t_0.

(b-2) *If*

$$\Delta \ge 0, \quad \alpha_r > 0, \quad z_1 \le 0, \quad z_2 \ge 1, \tag{4.5}$$

holds, then the lump sum $\widetilde{G}$ *is a decreasing function of* t_0. *The quantities* A_0, d *and* A *are decreasing functions of* t_0.

Proof. Define

$$g_2(z) := \frac{\sigma_r^2}{2\alpha_r^2} z^2 + \beta_1 z + \beta_0, \quad (-\infty < z < +\infty),$$

where

$$\beta_0 = \mu_h - \mu_r - \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} + \frac{\sigma_r^2}{2\alpha_r^2},$$
$$\beta_1 = \mu_r - r_0 + \frac{\sigma_h \sigma_r \rho_{hr}}{\alpha_r} - \frac{\sigma_r^2}{\alpha_r^2}.$$

It is easy to get that

$$\frac{\partial\left[G(x+t_0)D(x+t_0)f_T(x)\right]}{\partial t_0} = G(x+t_0)D(x+t_0)f_T(x)g_2(e^{-\alpha_r(x+t_0)}).$$

Since the minimum of $g_2(z)$ is $-\frac{\alpha_r^2}{2\sigma_r^2}\Delta$ (Δ given by Equation (4.2)), when the condition $\Delta \le 0$ holds, we then have $g_2(z) \ge 0$. Thus $\widetilde{G}$ is an increasing function of t_0.

Recall the definitions of z_1 and z_2 given above by the Relations (4.3) and (4.4), respectively. Now, if the condition $\Delta \ge 0$ holds, then $g_2(z_i) = 0$, $i = 1, 2$. Moreover, it is obvious that $0 < \exp(-\alpha_r(x+t_0)) \le 1$ in case of $\alpha_r > 0$ and $x + t_0 \ge 0$. Thus the lump sum $\widetilde{G}$ is a decreasing function of t_0 whenever the Condition (4.5) holds. One similarly obtains the rest of the properties, thereby concluding the proof of Proposition 4.2. □

4.2. **Monotonicity w.r.t Parameters of Interest Rate.** The following Proposition 4.3 analyzes how the lump sum and annuity payment factors vary with the parameters involved in the interest rate model, such as the initial interest rate r_0, the mean reversion level μ_r and the volatility σ_r.

Proposition 4.3. *With respect to the parameters of the interest model, the basic annuity A_0, the fixed increment d, the annuity payment A, and the annuity payment factors $\widetilde{F}_i$ $(i = 1, 2)$ have the following properties:*

(1). (Parameter r_0): *If $\alpha_r \neq 0$, then $\widetilde{F}_1, \widetilde{F}_2$ and $\widetilde{G}$ are a decreasing functions of r_0.*

(2). (Parameter μ_r): *If $\alpha_r > 0$, then $\widetilde{F}_1, \widetilde{F}_2$ and $\widetilde{G}$ are a decreasing functions of μ_r. If the opposite case $\alpha_r < 0$ holds, then $\widetilde{F}_1, \widetilde{F}_2$ and $\widetilde{G}$ are an increasing functions of μ_r.*

(3). (Parameter σ_r): **(a)** *In case of $\alpha_r \neq 0, \sigma_r > 0$, $\widetilde{F}_1, \widetilde{F}_2$ are increasing functions of σ_r.*
(b) *In the case of $\alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \ge 0$, $\widetilde{G}$ is a decreasing function of σ_r in the interval $\sigma_r \in (0, \sigma_h\rho_{hr}\alpha_r]$, and that the A_0, d and A are decreasing functions of σ_r.*
(c) *In the case of $\alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \le 0$, $\widetilde{G}$ is an increasing function of σ_r.*

Proof. Note that both $G(x+t_0)$ and $f_T(x)$ are independent of r_0, that the partial derivative of the integrand in the definition of $\widetilde{G}$ is

$$\frac{\partial\left[G(x+t_0)D(x+t_0)f_T(x)\right]}{\partial r_0} = -\frac{1}{\alpha_r}\left[1 - e^{-\alpha_r(x+t_0)}\right]G(x+t_0)D(x+t_0)f_T(x),$$

and that

$$\frac{\partial \widetilde{F}_1}{\partial r_0} = -\frac{1}{\alpha_r}\sum_{k=1}^{+\infty}(1-e^{-\alpha_r k})D(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\},$$

$$\frac{\partial \widetilde{F}_2}{\partial r_0} = -\frac{1}{\alpha_r}\sum_{k=1}^{+\infty}k(1-e^{-\alpha_r k})D(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\}.$$

Now, since $\frac{1}{\alpha_r}(1-e^{-\alpha_r z}) \geq 0$ whenever $\alpha_r \neq 0$ and $z \geq 0$, we obtain Part 1 of the Proposition.

Define now

$$g_3(z) := -z + \frac{1}{\alpha_r}\left(1-e^{-\alpha_r z}\right). \tag{4.6}$$

Since $G(x+t_0)$ and $f_T(x)$ are free of μ_r, we have

$$\frac{\partial\left[G(x+t_0)D(x+t_0)f_T(x)\right]}{\partial \mu_r} = G(x+t_0)D(x+t_0)f_T(x)g_3(x+t_0).$$

Also,

$$\frac{\partial \widetilde{F}_1}{\partial \mu_r} = \sum_{k=1}^{+\infty}D(k)g_3(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\},$$

$$\frac{\partial \widetilde{F}_2}{\partial \mu_r} = \sum_{k=1}^{+\infty}kD(k)g_3(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\}.$$

Since $g_3(z) \leq 0$ in case of $\alpha_r > 0, z \geq 0$, and $g_3(z) \geq 0$ in case of $\alpha_r < 0, z \geq 0$, we obtain Part 2.

For $-\infty < z < +\infty$, define

$$\begin{aligned} g_4(z) &:= -\frac{\sigma_r}{2\alpha_r^3}e^{-2\alpha_r z} + \left(\frac{2\sigma_r}{\alpha_r^3} - \frac{\sigma_h\rho_{hr}}{\alpha_r^2}\right)e^{-\alpha_r z} \\ &\quad + \left(\frac{\sigma_r}{\alpha_r^2} - \frac{\sigma_h\rho_{hr}}{\alpha_r}\right)z + \left(\frac{\sigma_h\rho_{hr}}{\alpha_r^2} - \frac{3\sigma_r}{2\alpha_r^3}\right), \\ g_5(y) &:= \frac{1}{\alpha_r}\left[\frac{\sigma_r}{\alpha_r}y^2 + \left(\sigma_h\rho_{hr} - \frac{2\sigma_r}{\alpha_r}\right)y + \left(\frac{\sigma_r}{\alpha_r} - \sigma_h\rho_{hr}\right)\right]. \end{aligned}$$

We observe the following: (1) If $\frac{\sigma_h\rho_{hr}}{\alpha_r} \geq 0$, then $g_5(y)$ has two zero points $y_2 = 1$ and $y_1 = 1 - \frac{\sigma_h\rho_{hr}\alpha_r}{\sigma_r}$; and (2) If $\frac{\sigma_h\rho_{hr}}{\alpha_r} \leq 0$, then $g_5(y)$ has two zero points $y_1 = 1$ and $y_2 = 1 - \frac{\sigma_h\rho_{hr}\alpha_r}{\sigma_r}$.

Next,

$$\frac{dg_4(z)}{dz} = \frac{1}{\alpha_r}\left[\frac{\sigma_r}{\alpha_r}e^{-2\alpha_r z} + \left(\sigma_h\rho_{hr} - \frac{2\sigma_r}{\alpha_r}\right)e^{-\alpha_r z} + \left(\frac{\sigma_r}{\alpha_r} - \sigma_h\rho_{hr}\right)\right] = g_5(e^{-\alpha_r z}),$$

Note that $0 < e^{-\alpha_r z} \leq 1$ in case of $z \in [0, +\infty), \alpha_r > 0$. If $z \in [0, +\infty), \alpha_r > 0, \sigma_h > 0$ and $\rho_{hr} \geq 0$, we then have $g_5(e^{-\alpha_r z}) \leq 0$ in the interval $\sigma_r \in (0, \sigma_h\rho_{hr}\alpha_r]$. So, $g_4(z) \leq g_4(0) = 0$ in the interval $\sigma_r \in (0, \sigma_h\rho_{hr}\alpha_r]$. In case of $z \in [0, +\infty), \alpha_r >$

$0, \sigma_h > 0$ and $\rho_{hr} \leq 0$, we have $g_5(e^{-\alpha_r z}) \geq 0$, and hence $g_4(z)$ is an increasing function of z and $g_4(z) \geq g_4(0) = 0$.

Noting that $G(x+t_0)$ and $f_T(x)$ do not depend on σ_r, we have

$$\frac{\partial\left[G(x+t_0)D(x+t_0)f_T(x)\right]}{\partial\sigma_r} = G(x+t_0)D(x+t_0)f_T(x)g_4(x+t_0).$$

Now introduce

$$g_6(z) := z + \left[1-(2-e^{-\alpha_r z})^2\right]\frac{1}{2\alpha_r}.$$

Since

$$\frac{\partial\widetilde{F}_1}{\partial\sigma_r} = \frac{\sigma_r}{\alpha_r^2}\sum_{k=1}^{+\infty} D(k)g_6(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\},$$

$$\frac{\partial\widetilde{F}_2}{\partial\sigma_r} = \frac{\sigma_r}{\alpha_r^2}\sum_{k=1}^{+\infty} kD(k)g_6(k)\exp\left\{-\int_0^k \lambda(x_0+u)du\right\},$$

we observe that $g_6(z) \geq 0$ whenever $z \in [0, +\infty)$. This proves Part 3 thereby completing the proof of Proposition 4.3. □

4.3. **Monotonicity w.r.t Parameters of Force of Mortality.** The following Proposition 4.4 analyzes how the annuity payment factors vary with the parameters involved in the force of mortality model (parameters such as the initial age x_0, the constant age-independent hazard rate a, the dispersion coefficient b, and the modal value c.

Proposition 4.4. *With respect to the parameters of the Gompertz-Makeham force of mortality defined by Equation* (2.8)*, the annuity payment factors* $\widetilde{F}_1$, $\widetilde{F}_2$ *have the following monotonicity properties:*

(1). PARAMETER x_0: *In case of* $b > 0$*, the annuity payment factors* $\widetilde{F}_1$ *and* $\widetilde{F}_2$ *are decreasing functions of* x_0*.*

(2). PARAMETER a: *The annuity payment factors* $\widetilde{F}_1$, $\widetilde{F}_2$ *are decreasing functions of* a*.*

(3). PARAMETER b: *If* $x_0 \geq c$*, the annuity payment factors* $\widetilde{F}_1$, $\widetilde{F}_2$ *are increasing functions of* b*.*

(4). PARAMETER c: *The annuity payment factors* $\widetilde{F}_1$, $\widetilde{F}_2$ *are increasing functions of* c*.*

Proof. Define

$$g_7(x) := -\frac{1}{b}\left(e^{\frac{x}{b}}-1\right)\exp\left\{\frac{x_0-c}{b}\right\}\cdot\exp\left\{-\int_0^x \lambda(x_0+u)du\right\},$$

and note that

$$\frac{\partial\widetilde{F}_1}{\partial x_0} = \sum_{k=1}^{+\infty} D(k)g_7(k), \quad \frac{\partial\widetilde{F}_2}{\partial x_0} = \sum_{k=1}^{+\infty} kD(k)g_7(k),$$

and that

$$\frac{\partial \widetilde{F}_1}{\partial a} = -\sum_{k=1}^{+\infty} kD(k) \exp\left\{-\int_0^k \lambda(x_0+u)du\right\},$$
$$\frac{\partial \widetilde{F}_2}{\partial a} = -\sum_{k=1}^{+\infty} k^2D(k) \exp\left\{-\int_0^k \lambda(x_0+u)du\right\}.$$

Define next

$$g_8(x) := \frac{1}{b^2}\left[(x_0-c)\left(e^{\frac{x}{b}}-1\right)+xe^{\frac{x}{b}}\right] \exp\left\{\frac{x_0-c}{b}\right\} \cdot \exp\left\{-\int_0^x \lambda(x_0+u)du\right\},$$

and note that

$$\frac{\partial \widetilde{F}_1}{\partial b} = \sum_{k=1}^{+\infty} D(k)g_8(k), \quad \frac{\partial \widetilde{F}_2}{\partial b} = \sum_{k=1}^{+\infty} kD(k)g_8(k),$$

and

$$\frac{\partial \widetilde{F}_1}{\partial c} = -\sum_{k=1}^{+\infty} D(k)g_7(k), \quad \frac{\partial \widetilde{F}_2}{\partial c} = -\sum_{k=1}^{+\infty} kD(k)g_7(k).$$

Now the assumptions made for the Proposition 4.4 prove the proposition. □

Remark 4.1. Obviously, the following properties are also true: The A_0 in Proposition 3.3 is a decreasing function of d, and in turn, d is a decreasing function of A_0.

5. NUMERICAL EXPERIMENT

We shall test in this section the monotonicity conclusions established in Section 4 through the following numerical analysis. This sections illustrates the impacts of risks involved in the house price, the interest rate, and the longevity on the annuity payment, the lump sum, and the annuity payment factors. We take the parameters involved in the models of house price, interest rate and longevity with the following values as the standard case. Here, the values of parameters a, b, and c come from the Gompertz-Makeham force of mortality in Huang et al., 2013. With these parameters

TABLE 1. Parameters of the standard case

Para	μ_h	σ_h	ρ_{hr}	h_0	t_0	r_0	μ_r	σ_r	α_r	x_0	d	a	b	c
Value	0.04	0.07	0.025	100	0	0.04	0.06	0.01	0.25	65	0	0	9.5	86.3

of standard case, we obtain the level annuity $A = 7.138$; the lump sum $\widetilde{G} = 75.796$; the annuity payment factors $\widetilde{F}_1 = 10.618$ and $\widetilde{F}_2 = 92.651$.

TABLE 2. Impacts of the house price

μ_h	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16
A	5.121	7.138	10.246	15.139	23.005	35.907	57.473	94.174
$\widetilde{G}$	54.377	75.796	108.795	160.750	244.277	381.273	610.266	999.965
σ_h	0.02	0.07	0.12	0.17	0.22	0.27	0.32	0.37
A	7.143	7.138	7.133	7.129	7.124	7.119	7.114	7.109
$\widetilde{G}$	75.848	75.796	75.744	75.693	75.641	75.589	75.538	75.486
ρ_{hr}	-1	-0.9	-0.6	-0.3	0	0.3	0.6	1
A	7.426	7.397	7.312	7.228	7.145	7.064	6.984	6.879
$\widetilde{G}$	78.850	78.545	77.639	76.747	75.869	75.004	74.153	73.038
h_0	100	200	300	400	500	600	700	800
A	7.138	14.277	21.415	28.553	35.692	42.830	49.968	57.106
$\widetilde{G}$	75.796	151.593	227.389	303.185	378.981	454.778	530.574	606.370

5.1. **Sensitivity Analysis for the House Price.** We start the numerical analysis of how the parameters of the house price model impacts the annuity, lump sum, and annuity factors, while we keep fixed the other parametric values given above.

Table 2 shows the following:

(a) Parameter μ_h: The higher the mean return of the home price, greater the lump sum and annuity are; however, the annuity payment factors remain constant, $\widetilde{F}_1 = 10.618$ and $\widetilde{F}_2 = 92.651$ (as they are not affected by μ_h), which coincides with the conclusions in Proposition 4.1A. Since any higher mean return of the home price contributes to increased profit from the sale of the mortgaged house in future, the aforementioned phenomenon is sensible.

(b) Parameter σ_h: The greater the volatility of the home price, lesser the lump sum and annuity are; however, the annuity payment factors remain constant, $\widetilde{F}_1 = 10.618$ and $\widetilde{F}_2 = 92.651$ (as they are not affected by σ_h). These coincide with the conclusions in Proposition 4.1B. Since any higher volatility of the home price implies the greater market risk, the lender have to decrease the annuities and the lump sum in order to attenuate the market risk.

(c) Parameter ρ_{hr}: The bigger the correlation coefficient between the house price and the interest rate, smaller the lump sum and annuity are; however, the annuity payment factors remain constant, $\widetilde{F}_1 = 10.618$ and $\widetilde{F}_2 = 92.651$ (as they are not affected by ρ_{hr}). These are supported in theory by the conclusions in Proposition 4.1C.

(d) Parameter h_0: The larger the initial house price, larger the lump sum and annuity are; however, the annuity payment factors remain constant, $\widetilde{F}_1 = 10.618$ and $\widetilde{F}_2 = 92.651$ (as they are not affected by h_0). These are supported in theory by the conclusions in Proposition 4.1D. Since a greater initial house price implies a greater profit from the sale of the pledged house in future, the house owner will obtain larger annuities and lump sum subject to the principle of balance between expected gain and expected payment.

Next we vary the time delay t_0 (while keeping other parameters fixed as above) and analyze how it affects the annuity, lump sum, and annuity payment factors.

Table 3. Impacts of the delay time of selling house

t_0	0	0.5	1	1.5	2	2.5	3	3.5
A	7.138	7.075	7.012	6.948	6.885	6.822	6.760	6.697
$\widetilde{G}$	75.796	75.124	74.452	73.781	73.110	72.441	71.775	71.111

(e) Parameter t_0: Table 3 shows that, the larger the delay time in selling the pledged house, smaller the lump sum and annuity, (while the annuity payment factors remain constant $\widetilde{F}_1 = 10.618$ and $\widetilde{F}_2 = 92.651$ (not influenced by t_0)). This is in accord with Proposition 4.2(b-2) in the case of $\Delta = 0.000403 \geq 0, \alpha_r = 0.25 > 0, z_1 = -24.0875 \leq 0$ and $z_2 = 1 \geq 1$. However, it should be noted that if we changed some parameters, the lump sum might also increase with an increase in t_0 (refer to Proposition 4.2(b-1)). It implies that the lender may choose the right time to sell the pledged house according to the parameters in the house price model and the interest rate model.

5.2. **Sensitivity Analysis for the Interest Rate.** This subsection provides the numerical analysis of how the interest rate impacts the annuity value A, the lump sum $\widetilde{G}$, and the annuity factors $\widetilde{F}_i$, $i = 1, 2$. Again, when we select one parameter to vary, we keep the remaining parametric values fixed.

Table 4. Impacts of the initial interest rate

r_0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16
A	7.236	7.138	7.040	6.942	6.844	6.746	6.648	6.550
$\widetilde{G}$	81.574	75.796	70.440	65.476	60.873	56.605	52.648	48.978
$\widetilde{F}_1$	11.273	10.618	10.005	9.431	8.894	8.391	7.919	7.477
$\widetilde{F}_2$	99.542	92.651	86.250	80.302	74.776	69.641	64.869	60.433

(a) Parameter r_0: We note the following from the Table 4. The lump sum $\widetilde{G}$ and annuity payment factors $\widetilde{F}_i, i = 1, 2$, are decreasing as the initial interest rate r_0 increases. This agrees with our conclusions in Proposition 4.3. The annuity A is also decreasing. With the explicit solution of interest rate in Equation (2.3), we know that a higher initial interest rate means an increase in the average interest rate. This contributes to a decreased average discounted factor of interest rate, and that in turn results in the lower lump sum and annuity payment factors.

(b) Parameter μ_r: Table 5 provides the numerical values resulting from the impact of the average reversion level μ_r of the interest rate. Here we note that the

TABLE 5. Impacts of the average reversion level of interest rate

μ_r	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16
A	10.042	8.418	7.138	6.133	5.345	4.724	4.235	3.846
$\widetilde{G}$	138.084	100.954	75.796	58.421	46.187	37.401	30.969	26.168
$\widetilde{F}_1$	13.751	11.993	10.618	9.525	8.642	7.917	7.313	6.804
$\widetilde{F}_2$	143.213	114.121	92.651	76.535	64.238	54.705	47.203	41.213

lump sum $\widetilde{G}$ and annuity payment factors $\widetilde{F}_i, i = 1, 2$, decrease with the increase of average reversion level μ_r of interest rate. This conclusion is theoretically supported by our Proposition 4.3. The annuities are also decreasing with the increase of μ_r.

TABLE 6. Impacts of the volatility of interest rate

σ_r	0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040
A	7.110	7.138	7.188	7.260	7.356	7.476	7.623	7.798
$\widetilde{G}$	75.292	75.796	76.671	77.936	79.617	81.751	84.385	87.582
$\widetilde{F}_1$	10.590	10.618	10.666	10.735	10.824	10.935	11.070	11.231
$\widetilde{F}_2$	92.190	92.651	93.428	94.532	95.982	97.801	100.021	102.681
A	6.995	6.984	6.973	6.964	6.955	6.947	6.940	6.933
$\widetilde{G}$	70.718	70.614	70.526	70.452	70.393	70.349	70.319	70.303
$\widetilde{F}_1$	10.110	10.111	10.113	10.117	10.121	10.127	10.133	10.140
$\widetilde{F}_2$	86.916	86.936	86.971	87.019	87.081	87.157	87.246	87.350

(c) Parameter σ_r: Table 6 reveals that the annuity payment factors $\widetilde{F}_i, i = 1, 2$, increase as the volatility of interest rate σ_r increases, and this is consistent with the property 3(a) of Proposition 4.3. This is reasonable since the higher volatility rate of interest rate contributes to the higher average level of the discounted factor.

The first part of Table 6 shows that the annuity and the lump sum are increasing with the increase of the volatility of interest rate σ_r. However, this does not seem reasonable from the perspective of risk aversion. We also note that the payment of annuity and lump sum are increasing faster with the increase in the volatility of interest rate σ_r. On one hand, when the volatility of interest rate is at a higher level, a slight increase in the volatility will greatly increase the annuity and lump sum payments. On the other hand, when the volatility is at a lower level, the increase in volatility only make the annuity and lump sum amounts increase slightly. Thus, our pricing models can be grudgingly applied to pricing the annuity and lump sum in the lower volatility case, and they are unsuitable to price the annuity and lump sum in the higher volatility case.

In the second part of Table 6, we assign $\sigma_h = 0.12$, $\rho_{hr} = 0.25$, and $\alpha_r = 1.4$ (while keeping the other parameters as the standard case). We note that the lump sum $\widetilde{G}$ and annuity A are both decreasing as σ_r increases from 0.005 to 0.04, which is consistent with the Property 3(b) of Proposition 4.3 in case of $\alpha_r = 1.4 > 0$, $\sigma_h = 0.12 > 0$, $\rho_{hr} = 0.25 \geq 0$ and $\sigma_r \in (0, \sigma_h\rho_{hr}\alpha_r] = (0, 0.042]$. If the volatility σ_r of interest rate can be controlled by the product $\sigma_h\rho_{hr}\alpha_r$, then the Property 3(b) of Proposition 4.3 implies that the annuities will decrease with the increase in σ_r . In this case we note that the annuity and lump sum pricing formulas are quite reasonable. In particular, the annuity and lump sum pricing formulas are still applicable in the higher volatility case as long as the volatility can be controlled by $\sigma_h\rho_{hr}\alpha_r$.

TABLE 7. Impacts of the reversion speed of interest rate

α_r	0.05	0.25	0.5	0.75	1	1.25	1.5	1.75
A	8.021	7.138	7.039	7.018	7.011	7.008	7.007	7.006
$\widetilde{G}$	92.884	75.796	72.755	71.756	71.265	70.974	70.781	70.645
$\widetilde{F}_1$	11.580	10.618	10.336	10.224	10.164	10.127	10.102	10.084
$\widetilde{F}_2$	107.486	92.651	89.246	88.049	87.449	87.090	86.851	86.682

(d) Parameter α_r: From Table 7, it is clear that the lump sum and the annuity payment factors decrease with the increasing of the reversion speed α_r of interest rate as other parameters take the standard values. The decreasing speed of the annuity, the lump sum and the annuity payment factors become slower and slower with the increase of α_r.

5.3. **Sensitivity Analysis for the Initial Age.** In this subsection we discuss the impact made by the initial age on A, $\widetilde{G}$, and $\widetilde{F}_i$, $i = 1, 2$.

TABLE 8. Impacts of the initial age

x_0	50	55	60	65	70	75	80	85
A	4.267	4.979	5.903	7.138	8.845	11.288	14.927	20.598
$\widetilde{G}$	59.712	64.974	70.382	75.796	81.033	85.875	90.105	93.547
$\widetilde{F}_1$	13.995	13.051	11.924	10.618	9.162	7.608	6.036	4.542
$\widetilde{F}_2$	164.831	141.353	116.949	92.651	69.689	49.302	32.490	19.762

Table 8 illustrates that as the age x_0 of the home owner as she signs the contract increases, the lump sum $\widetilde{G}$ and annuity A are increasing, while annuity payment factors $\widetilde{F}_i, i = 1, 2$, show a decreasing trend, and this is supported in theory by our Proposition 4.4. As the house owner enters into the contract at a later age, the resulting lower expected residual life time of the owner provides increased annuity payment.

5.4. **Sensitivity Analysis for the Increasing (or Decreasing) Annuity.** For the increasing (or decreasing) annuity in Proposition 3.3, Table 9 shows that the increment d decreases as A_0 increases, and that A_0 decreases as d increases. This

TABLE 9. Impacts of the incremental creep

A_0	1	2	3	4	5	6	7	8
d	0.703	0.589	0.474	0.360	0.245	0.130	0.016	-0.099
d	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
A_0	7.138	6.266	5.393	4.521	3.648	2.775	1.903	1.030

conforms with our Remark 4.1. The lump sum and the annuity payment factors remain constant, $\widetilde{G} = 75.796$, $\widetilde{F}_1 = 10.618$ and $\widetilde{F}_2 = 92.651$.

The annuity with varying payment may turn out to be decreasing or increasing depending on the value of the basic annuity A_0. If the basic annuity payment A_0 is determined at a higher level ($A_0 \geq 8$ in our example), the annuity with varying payment would become the decreasing annuity, (that is, the lender pays less and less annuity payments to the house owner and the decrement of each period is about 0.099 when the basic annuity A_0 is fixed as 8). In our example, the annuity with varying payment becomes an increasing annuity if the basic annuity A_0 is fixed at a level less than or equal to 7.

TABLE 10. Average Change Rate

$Para$	μ_h	σ_h	ρ_{hr}	h_0	t_0	r_0	μ_r	σ_r	α_r	x_0
$ACR1$	636.093	0.097	0.274	0.071	0.126	4.900	44.257	19.658	0.597	0.467
$ACR2$	6754.199	1.034	2.906	0.758	1.339	232.827	799.396	351.122	13.082	0.967
$ACR3$	0	0	0	0	0	27.112	49.618	18.331	0.880	0.270
$ACR4$	0	0	0	0	0	279.349	728.567	299.727	12.238	4.145

Note: ACR1-the average change rate of annuity; ACR2-the average change rate of lump sum; ACR3-the average change rate of annuity payment factor 1; ACR4-the average change rate of annuity payment factor 2. The average change rate of annuity is defined as the $\frac{A_M - A_m}{I_R - I_L}$, A_M and A_m respectively means the maximum annuity payment and the minimum annuity payment; I_L and I_R respectively means the maximum and the minimum of the parameter. For example, $\frac{94.174-5.121}{0.16-0.02} = 636.093$. $ACR2, ACR3$ and $ACR4$ can be similarly obtained.

5.5. **Comparison of All Parameters.** Compared with other parameters of the home price and interest rate model, the mean return of house price μ_h has a dominating influence on both the annuity and the lump sum payments. The average reversion level μ_r, the volatility σ_r and the initial interest rate r_0 of interest rate respectively exert the second, third and fourth strongest impact on both the annuity and the lump sum. The remaining parameters have a slight effect on both the annuity and the lump sum. Table 9 shows that μ_r exert the most strongest influence on the annuity payment factors, followed by r_0, and σ_r. The parameters α_r and x_0 slightly affect the annuity payment factors. The annuity payment factors are not affected by $\mu_h, \sigma_h, \rho_{hr}, h_0$ and t_0.

6. CONCLUSION

This paper builds a pricing model for the lifetime annuity of the reverse mortgage without redemption right, and derives the explicit pricing formula for the increasing (or decreasing) perpetuity annuity and the level annuity. We then discuss the monotonicity of the lump sum, annuity, and annuity payment factors with respect to the parameters associated with the home price, the interest rate, and the force of mortality model. Furthermore, we present some numerical results of the annuity, the lump sum, and the annuity payment factors, and analyze their sensitivity to the said parameters. Finally, based on the average change rate, we compare the impact of various parameters on the annuity, the lump sum, and the annuity payment factors. The results show that the average return of home price exerts a dominating influence on both the annuity and the lump sum. Next to the average return of home price, the mean reversion level of interest rate, the volatility of interest rate and the initial interest rate make the second, third and fourth strongest impact on both the annuity and the lump sum. Otherwise, the remaining parameters slightly affect both the annuity and the lump sum.

However, it should be noted that the average change rate depends on the range of the parameter. Once the ranges of parameters change, they will change the evaluation results for the parameter. Thus, the right range should be chosen in order to more properly evaluate the importance of the parameters. Moreover, the model selection of the house price, interest rate and force of mortality will directly affect the final pricing results. Therefore, it is suggested to collect the data of house price, interest rate and population data of the particular area that the reverse mortgage product covers, and model the special house price, interest rate and force of mortality model based on the collected historical data. This will be propitious to better price the reverse mortgage product.

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