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KRASNOSELSKII COINCIDENCE TYPE RESULTS FOR GENERAL CLASSES OF MAPS

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ABSTRACT. In this paper coincidence type results for general classes of maps are presented. A variety of different situations are discussed.

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1. INTRODUCTION

This paper presents Krasnoselskii compression type theorems for general classes of maps. The approach is elementary and relies on the fact that in an infinite dimensional normed linear space there exists a retraction from the unit ball to the unit sphere (note also in a normed linear space there exists a retraction from the unit ball (in a cone) to the unit sphere (in a cone)). Under appropriate conditions a Krasnoselskii type theorem guarantees the existence of a fixed point in a particular annulus for continuous, compact single valued maps. In this paper we extend this further by establishing the existence of coincidence points in an annulus for maps (which may be multivalued) in a general class.

2. MAIN RESULTS

Let E be a topological space and U an open subset of E.

We consider classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of maps.

Definition 2.1. We say $F \in D(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E and $\overline{U}$ denotes the closure of U in E.

Definition 2.2. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$, $F \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F.

Remark 2.3. Note Ψ is a selection of F (in Definition 2.2) if $\Psi(x) \subseteq F(x)$ for $x \in \overline{U}$.

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Definition 2.4. We say $F \in M(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$.

Definition 2.5. We say $F \in D(E, E)$ (respectively $F \in M(E, E)$) if $F : E \to 2^E$ and $F \in \mathbf{D}(E, E)$ (respectively $F \in \mathbf{A}(E, E)$).

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$.

Our first main result is a Krasnoselskii type theorem for A maps. Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space. For $\rho > 0$, $r > 0$, $R > r$ let

$$B_\rho = \{x \in E : \|x\| < \rho\}, \quad \overline{B_\rho} = \{x \in E : \|x\| \leq \rho\}, \quad S_\rho = \{x \in E : \|x\| = \rho\},$$

$$EB_\rho = \{x \in E : \|x\| \geq \rho\} \quad \text{and} \quad B_{r,R} = \{x \in E : r \leq \|x\| \leq R\}.$$

Theorem 2.6. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space, and r, R constants with $0 < r < R$. Let $F \in A(\overline{B_R}, E)$, $\Phi \in B(\overline{B_R}, E)$ fixed, and assume the following conditions hold:*

$$\tag{2.1} \begin{cases} \textit{for any selection } \Lambda \in D(\overline{B_R}, E) \textit{ of } F \textit{ and any} \\ \textit{continuous map } \eta : E \to \overline{B_R} \textit{ the map } \Lambda\eta \in D(E, E) \end{cases}$$

$$\tag{2.2} \textit{for any map } T \in D(E, E) \textit{ there exists } x \in E \textit{ with } \Phi(x) \cap T(x) \neq \emptyset$$

$$\tag{2.3} \begin{cases} \textit{for any selection } \Psi \in D(\overline{B_R}, E) \textit{ of } F \textit{ we have} \\ \Phi(x) \cap \Psi(y) = \emptyset \textit{ for } x \in B_r \textit{ and } y \in S_r \end{cases}$$

and

$$\tag{2.4} \begin{cases} \textit{for any selection } \Psi \in D(\overline{B_R}, E) \textit{ of } F \textit{ we have} \\ \Phi(\lambda y) \cap \Psi(y) = \emptyset \textit{ for } y \in S_R \textit{ and } \lambda > 1. \end{cases}$$

Then there exists $x \in B_{r,R}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. Let $r_0 : \overline{B_r} \to S_r$ be a continuous retraction (see [2]) and let

$$g(x) = \begin{cases} r_0(x), & x \in \overline{B_r} \\ x, & x \in B_{r,R} \\ R\frac{x}{\|x\|}, & x \in EB_R. \end{cases}$$

Note $g : E \to \overline{B_R}$ is a continuous map. Now since $F \in A(\overline{B_R}, E)$ there exists a selection $\Psi \in D(\overline{B_R}, E)$ of F and from (2.1), (2.2) there exists an $x \in E$ with $\Phi(x) \cap \Psi(g(x)) \neq \emptyset$. If $x \in B_r$ then $\Phi(x) \cap \Psi(r_0(x)) \neq \emptyset$ and this contradicts (2.3) (note $r_0(x) \in S_r$ since $x \in B_r$). If $\|x\| > R$ then $\Phi(x) \cap \Psi\left(R\frac{x}{\|x\|}\right) \neq \emptyset$ so if $y = R\frac{x}{\|x\|}$ (note $\|y\| = R$) then $\Phi\left(\frac{\|x\|}{R}y\right) \cap \Psi(y) \neq \emptyset$, and this contradicts (2.4). Thus $x \in B_{r,R}$ and $\Phi(x) \cap \Psi(x) \neq \emptyset$. Now since $\Psi(x) \subseteq F(x)$ we have $\Phi(x) \cap F(x) \neq \emptyset$. $\square$

Remark 2.7. In the proof of Theorem 2.6 notice (2.3) can be replaced by

$$\Phi(x) \cap F(y) = \emptyset \quad \text{for } x \in B_r \text{ and } y \in S_r, \tag{2.5}$$

and (2.4) can be replaced by

$$\Phi(\lambda y) \cap F(y) = \emptyset \quad \text{for } y \in S_R \text{ and } \lambda > 1. \tag{2.6}$$

Of course one could replace (2.1), (2.2), (2.3) and (2.4) in Theorem 2.6 with more abstract formulations. For example we could replace (2.1) with

$$\text{for any selection } \Lambda \in D(\overline{B_R}, E) \text{ of } F \text{ the map } \Lambda g \in D(E, E),$$

and we could replace (2.2) with

$$\begin{cases} \text{for any selection } \Lambda \in D(\overline{B_R}, E) \text{ of } F \text{ there} \\ \text{exists } x \in E \text{ with } \Phi(x) \cap \Lambda(g(x)) \neq \emptyset, \end{cases}$$

and we could replace (2.3) with

$$\begin{cases} \text{for any selection } \Psi \in D(\overline{B_R}, E) \text{ of } F \text{ we have} \\ \Phi(x) \cap \Psi(r_0(x)) = \emptyset \text{ for } x \in B_r. \end{cases}$$

Remark 2.8. Let $E = (E, \|\cdot\|)$ be a normed linear space and $C \subseteq E$ a cone (i.e. C is a closed, convex, invariant under multiplication by non-negative real numbers, and $C \cap (-C) = \{0\}$). For $\rho > 0$ let

$$B_\rho = \{x \in C : \|x\| < \rho\}, \quad \overline{B_\rho} = \{x \in C : \|x\| \leq \rho\},$$

$$S_\rho = \{x \in C : \|x\| = \rho\}, \quad \text{and} \quad EB_\rho = \{x \in C : \|x\| \geq \rho\}.$$

Let r, R be constants with $0 < r < R$. Let $F \in A(\overline{B_R}, C)$, $\Phi \in B(\overline{B_R}, C)$ fixed, and assume the following conditions hold:

$$\begin{cases} \text{for any selection } \Lambda \in D(\overline{B_R}, C) \text{ of } F \text{ and any} \\ \text{continuous map } \eta : C \to \overline{B_R} \text{ the map } \Lambda\eta \in D(C, C) \end{cases}$$

$$\text{for any map } T \in D(C, C) \text{ there exists } x \in C \text{ with } \Phi(x) \cap T(x) \neq \emptyset$$

$$\begin{cases} \text{for any selection } \Psi \in D(\overline{B_R}, C) \text{ of } F \text{ we have} \\ \Phi(x) \cap \Psi(y) = \emptyset \text{ for } x \in B_r \text{ and } y \in S_r \end{cases}$$

and

$$\begin{cases} \text{for any selection } \Psi \in D(\overline{B_R}, C) \text{ of } F \text{ we have} \\ \Phi(\lambda y) \cap \Psi(y) = \emptyset \text{ for } y \in S_R \text{ and } \lambda > 1. \end{cases}$$

Then there exists $x \in B_{r,R} = \{x \in C : r \leq \|x\| \leq R\}$ with $\Phi(x) \cap F(x) \neq \emptyset$. The proof is similar to that in Theorem 2.6 once one notes that there exists a continuous retraction $r_1 : \overline{B_r} \to S_r$ (see [7]).

One can easily generalize Theorem 2.6 to open convex sets.

Theorem 2.9. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space, and U_1 and U_2 are open convex subsets of E with $0 \in U_1$ and $\overline{U_1} \subset U_2$. Let $F \in A(\overline{U_2}, E)$, $\Phi \in B(\overline{U_2}, E)$ fixed, and assume* (2.2) *and the following conditions hold:*

$$\begin{cases} \textit{for any selection } \Lambda \in D(\overline{U_2}, E) \textit{ of } F \textit{ and any} \\ \textit{continuous map } \eta : E \to \overline{U_2} \textit{ the map } \Lambda\eta \in D(E, E) \end{cases} \tag{2.7}$$

$$\begin{cases} \textit{for any selection } \Psi \in D(\overline{U_2}, E) \textit{ of } F \textit{ we have} \\ \Phi(x) \cap \Psi(y) = \emptyset \textit{ for } x \in U_1 \textit{ and } y \in \partial U_1 \end{cases} \tag{2.8}$$

and

$$\begin{cases} \textit{for any selection } \Psi \in D(\overline{U_2}, E) \textit{ of } F \textit{ we have} \\ \Phi(\lambda y) \cap \Psi(y) = \emptyset \textit{ for } y \in \partial U_2 \textit{ and } \lambda > 1. \end{cases} \tag{2.9}$$

Then there exists $x \in \overline{U_2} \backslash U_1$ with $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. It is easy to see [1] that there exists a continuous retraction $r_2 : \overline{U_1} \to \partial U_1$. Let

$$g(x) = \begin{cases} r_2(x), & x \in \overline{U_1} \\ x, & x \in \overline{U_2} \backslash U_1 \\ \frac{x}{\mu(x)}, & x \in E \backslash U_2 \end{cases}$$

where μ is the Minkowski functional on $\overline{U_2}$. Note $g : E \to \overline{U_2}$ is a continuous map. Now since $F \in A(\overline{U_2}, E)$ there exists a selection $\Psi \in D(\overline{U_2}, E)$ of F and (2.2), (2.7) guarantees that there exists an $x \in E$ with $\Phi(x) \cap \Psi(g(x)) \neq \emptyset$. If $x \in U_1$ then $\Phi(x) \cap \Psi(r_2(x)) \neq \emptyset$ and this contradicts (2.8) (note $r_2(x) \in \partial U_1$). If $x \in E \backslash \overline{U_2}$ then $\Phi(x) \cap \Psi\left(\frac{x}{\mu(x)}\right) \neq \emptyset$ so if $y = \frac{x}{\mu(x)}$ (note $\mu(y) = 1$ so $y \in \partial U_2$) then $\Phi\left(\mu(x) y\right) \cap \Psi(y) \neq \emptyset$, and this contradicts (2.9) (note $x \in E \backslash \overline{U_2}$ so $\mu(x) > 1$). Thus $x \in \overline{U_2} \backslash U_1$ and $\Phi(x) \cap \Psi(x) \neq \emptyset$, so $\Phi(x) \cap F(x) \neq \emptyset$. □

Remark 2.10. In the proof of Theorem 2.9 notice (2.8) can be replaced by

$$\Phi(x) \cap F(y) = \emptyset \quad \text{for } x \in U_1 \text{ and } y \in \partial U_1, \tag{2.10}$$

and (2.9) can be replaced by

$$\Phi(\lambda y) \cap F(y) = \emptyset \quad \text{for } y \in \partial U_2 \text{ and } \lambda > 1. \tag{2.11}$$

Theorem 2.11. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space, and r, R constants with $0 < r < R$. Let $F \in M(\overline{B_R}, E)$, $\Phi \in B(\overline{B_R}, E)$ fixed, and assume the following conditions hold:*

$$\textit{for any continuous map } \eta : E \to \overline{B_R} \textit{ the map } F\eta \in M(E, E) \tag{2.12}$$

and

$$\textit{for any map } T \in M(E, E) \textit{ there exists } x \in E \textit{ with } \Phi(x) \cap T(x) \neq \emptyset. \tag{2.13}$$

Finally assume (2.5) *and* (2.6) *hold. Then there exists* $x \in B_{r,R}$ *with* $\Phi(x) \cap F(x) \neq \emptyset$.

Proof. Let r_0 and g be as in Theorem 2.6. Now $F \in M(\overline{B_R}, E)$ so (2.12), (2.13) guarantee that there exists $x \in E$ with $\Phi(x) \cap F(g(x)) \neq \emptyset$. As in Theorem 2.6 we obtain $x \in B_{r,R}$. □

Remark 2.12. There is an obvious analogue of Remark 2.8 and Theorem 2.9 with A replaced by M. Also in Theorem 2.11 one could replace (2.12) with the assumption that $Fg \in M(E, E)$.

We now present Corollaries of Theorem 2.6 and Theorem 2.11 when $\Phi = I$ (the identity map) (there are also analogues of Remark 2.8 and Theorem 2.9 with A replaced by M).

Corollary 2.13. *Let* $E = (E, \|\cdot\|)$ *be an infinite dimensional normed linear space, and* r, R *constants with* $0 < r < R$. *Let* $F \in A(\overline{B_R}, E)$ *and suppose* (2.1) *holds. In addition assume the following conditions hold:*

$$\text{for any map } T \in D(E,E) \text{ there exists } x \in E \text{ with } x \in T(x) \tag{2.14}$$

$$\begin{cases} \text{for any selection } \Psi \in D(\overline{B_R}, E) \text{ of } F \text{ we have} \\ x \notin \Psi(y) \text{ for } x \in B_r \text{ and } y \in S_r \end{cases} \tag{2.15}$$

and

$$\begin{cases} \text{for any selection } \Psi \in D(\overline{B_R}, E) \text{ of } F \text{ we have} \\ y \notin \mu\Psi(y) \text{ for } y \in S_R \text{ and } \mu \in (0,1). \end{cases} \tag{2.16}$$

Then there exists $x \in B_{r,R}$ *with* $x \in F(x)$.

Remark 2.14. Note (2.15) can be replaced by

$$x \notin F(y) \quad \text{for } x \in B_r \text{ and } y \in S_r, \tag{2.17}$$

and (2.16) can be replaced by

$$y \notin \mu F(y) \quad \text{for } y \in S_R \text{ and } \mu \in (0,1). \tag{2.18}$$

Proof. The result follows from Theorem 2.6 with $\Phi = I$. We note that if for any selection $\Psi \in D(\overline{B_R}, E)$ of F and $y \in S_R$, $\lambda > 1$ we had $\lambda y \in \Psi(y)$ then $y \in \mu\Psi(y)$ with $\mu = \frac{1}{\lambda} \in (0,1)$, and this contradicts (2.16). □

Corollary 2.15. *Let* $E = (E, \|\cdot\|)$ *be an infinite dimensional normed linear space, and* r, R *constants with* $0 < r < R$. *Let* $F \in M(\overline{B_R}, E)$ *and suppose* (2.12), (2.17) *and* (2.18) *hold. In addition assume*

$$\text{for any map } T \in M(E,E) \text{ there exists } x \in E \text{ with } x \in T(x). \tag{2.19}$$

Then there exists $x \in B_{r,R}$ *with* $x \in F(x)$.

Now we consider a special case of Corollary 2.13. We first recall the PK maps from the literature. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in PK(Z, W)$ if W is convex and there exists a map $S : Z \to W$ with $Z = \cup\{int\ S^{-1}(w) : w \in W\}$, $co(S(x)) \subseteq F(x)$ for $x \in Z$ and $S(x) \neq \emptyset$ for each $x \in Z$; here $S^{-1}(w) = \{z : w \in S(z)\}$.

Corollary 2.16. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space, and r, R constants with $0 < r < R$. Let $F \in PK(\overline{B_R}, E)$ be a compact map and assume* (2.17) *and* (2.18) *hold. Then there exists $x \in B_{r,R}$ with $x \in F(x)$.*

Proof. In this case we let $D = \mathbf{D}$ and $A = \mathbf{A}$. We say $Q \in D(\overline{B_R}, E)$ if $Q : \overline{B_R} \to E$ is a continuous compact map. We say $G \in A(\overline{B_R}, E)$ if $G \in PK(\overline{B_R}, E)$ and G is a compact map (the existence of a continuous selection Ψ of G is guaranteed from [Theorem 1.3, 6] and note Ψ is compact since Ψ is a selection of G and G is compact). Note (2.1) and (2.14) (Schauder's fixed point theorem) hold. The result follows from Corollary 2.13 (and Remark 2.14). $\square$

Next we consider a special case of Corollary 2.15. We first recall the $\mathbf{U}_c^\kappa$ maps from the literature. Suppose X and Y are Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : Fix\ F \neq \emptyset \text{ for all } F \in \mathbf{X}(Z, Z)\},$$

where $Fix\ F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i). $\mathbf{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;
(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and
(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, \dots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$.

We say $F \in \mathbf{U}_c^\kappa(X, Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Examples of $\mathbf{U}_c^\kappa(X, Y)$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, the maps admissible in the sense of Gorniewicz and the permissible maps; see [4]. Recall $\mathbf{U}_c^k$ is closed under compositions [8].

Corollary 2.17. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space, and r, R constants with $0 < r < R$. Let $F \in \mathbf{U}_c^\kappa(\overline{B_R}, E)$ be a compact map and assume* (2.17) *and* (2.18) *hold. Then there exists $x \in B_{r,R}$ with $x \in F(x)$.*

Proof. In this case we let $M = \mathbf{A}$ and say $F \in M(\overline{B_R}, E)$ if $F \in \mathbf{U}_c^\kappa(\overline{B_R}, E)$ is a compact map. Note (2.12) is immediate since $\mathbf{U}_c^k$ is closed under compositions and $F\eta : E \to 2^E$ is compact (here $\eta : E \to \overline{B_R}$ is a continuous map). Finally we note that (2.19) holds (see [8, 9]). The result follows from Corollary 2.15. $\square$

We now show that the ideas in this section can be applied to other natural situations. Let E be a topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\ L \subseteq E \to Y$ be a linear single valued map; here $dom\ L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L+T : dom\ L \to Y$ a bijection; for convenience we say $T \in H_L(E,Y)$.

Definition 2.18. We say $F \in D(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}(F+T) \in D(\overline{U}, E)$ (respectively $(L+T)^{-1}(F+T) \in B(\overline{U}, E)$).

Definition 2.19. We say $F \in A(\overline{U}, Y; L, T)$ if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}(F+T) \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, Y; L, T)$ of F.

Definition 2.20. We say $F \in D(E, Y; L, T)$ if $F : E \to 2^Y$ and $(L+T)^{-1}(F+T) \in D(E, E)$.

Remark 2.21. One could also define the class $M(\overline{U}, Y; L, T)$ (i.e. $F \in M(\overline{U}, Y; L, T)$ if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}(F+T) \in \mathbf{A}(\overline{U}, E)$) and $M(E, Y; L, T)$.

In our next result we <u>fix</u> a $\Phi \in B(\overline{U}, Y; L, T)$.

We obtain an analogue of Theorem 2.6 in this setting (it is also easy to obtain an analogue of Theorem 2.11 using the class M in this setting).

Theorem 2.22. *Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space, Y a topological vector space, $L : dom\ L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E,Y)$, and r, R constants with $0 < r < R$. Let $F \in A(\overline{B_R}, Y; L, T)$, $\Phi \in B(\overline{B_R}, Y; L, T)$ fixed, and assume the following conditions hold:*

$$\text{(2.20)} \qquad \begin{cases} \text{for any selection } \Lambda \in D(\overline{B_R}, Y; L, T) \text{ of } F \text{ and any} \\ \text{continuous map } \eta : E \to \overline{B_R} \text{ the map } \Lambda\eta \in D(E, Y; L, T) \end{cases}$$

$$\text{(2.21)} \qquad \begin{cases} \text{for any map } \Omega \in D(E, Y; L, T) \text{ there exists } x \in E \text{ with} \\ (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(\Omega+T)(x) \neq \emptyset \end{cases}$$

$$\text{(2.22)} \qquad \begin{cases} \text{for any selection } \Psi \in D(\overline{B_R}, Y; L, T) \text{ of } F \text{ we have} \\ (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(\Psi(y)+T(x)) = \emptyset \\ \text{for } x \in B_r \text{ and } y \in S_r \end{cases}$$

and

$$\text{(2.23)} \qquad \begin{cases} \text{for any selection } \Psi \in D(\overline{B_R}, Y; L, T) \text{ of } F \text{ we have} \\ (L+T)^{-1}(\Phi+T)(\lambda y) \cap (L+T)^{-1}(\Psi(y)+T(\lambda y)) = \emptyset \\ \text{for } y \in S_R \text{ and } \lambda > 1. \end{cases}$$

Then there exists $x \in B_{r,R}$ with $(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(F+T)(x) \neq \emptyset$.

Proof. Let r_0 and g be as in Theorem 2.6. Now since $F \in A(\overline{B_R}, Y; L, T)$ there exists a selection $\Psi \in D(\overline{B_R}, Y; L, T)$ of F and from (2.20), (2.21) there exists $x \in E$ with

$$(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(\Psi \circ g + T)(x) \neq \emptyset.$$

If $x \in B_r$ then

$$(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(\Psi \circ r_0 + T)(x) \neq \emptyset,$$

and this contradicts (2.22). If $\|x\| > R$ then

$$(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}\left(\Psi\left(\frac{R}{\|x\|}x\right) + T(x)\right) \neq \emptyset,$$

so if $y = R\frac{x}{\|x\|}$ then

$$(L+T)^{-1}(\Phi+T)\left(\frac{\|x\|}{R}y\right) \cap (L+T)^{-1}\left(\Psi(y) + T\left(\frac{\|x\|}{R}y)\right)\right) \neq \emptyset,$$

and this contradicts (2.23). Thus $x \in B_{r,R}$ with $(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(\Psi + T)(x) \neq \emptyset$. $\square$

Remark 2.23. There are analogues of Remark 2.7, Remark 2.8, Theorem 2.9 and Theorem 2.11 in this setting (we leave the obvious statements to the reader).

Finally in this paper we discuss briefly a different strategy in Theorem 2.11 (a similar strategy can be applied in Theorem 2.6 and Theorem 2.22). Let $E = (E, \|\cdot\|)$ be an infinite dimensional normed linear space, and r, R constants with $0 < r < R$. Let $F \in M(\overline{B_R}, E)$ and $\Phi \in B(\overline{B_R}, E)$ fixed. Assume

$$F(S_r) \subseteq \overline{B_r} \quad \text{and} \quad F(S_R) \subseteq EB_R. \tag{2.24}$$

Let r_0 and g be as in Theorem 2.6. Note if $x \in \overline{B_r}$ then $F(g(x)) = F(r_0(x)) \subseteq F(S_r) \subseteq \overline{B_r}$ and for $x \in EB_R$ then $F(g(x)) = F\left(\frac{R}{\|x\|}x\right) \subseteq F(S_R) \subseteq EB_R$. Thus $Fg : \overline{B_r} \to 2^{\overline{B_r}}$ and $Fg : EB_R \to 2^{EB_R}$. Let $\Omega = \overline{B_r} \cup EB_R$ and note $Fg : \Omega \to 2^\Omega$. Next assume

$$\begin{cases} Fg \in M(\overline{B_r}, \overline{B_r}), Fg \in M(EB_R, EB_R), Fg \in M(\Omega, \Omega) \\ \text{and there exists } x \in B_{r,R} \text{ with } \Phi(x) \cap Fg(x) \neq \emptyset. \end{cases} \tag{2.25}$$

If (2.25) is true then automatically $\Phi(x) \cap F(x) \neq \emptyset$ for the x in (2.25).

Note Ω is an ANR and Ω is the disjoint union of two contractible components $\overline{B_r}$ and EB_R. Condition (2.25) arises naturally in applications. For example if $\Phi = I$ (the identity map) and one has an index theory for the M maps (with appropriate properties) then one can deduce immediately the existence of a fixed point of Fg in $B_{r,R}$ (of course one needs the usual Bowszyc theorem [5] for the class M). If M denotes the maps admissible in the sense of Gorniewicz [4] or the permissible maps [3, 4] then one can deduce immediately (see [5] or [section 57, 4]) that there exists

$x \in B_{r,R}$ with $x \in Fg(x)$ i.e. (2.25) holds (thus the strategy to establish (2.25) here is to obtain an analogue of a theorem of C. Bowszyc for the class of maps considered).

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