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DYNAMIC MODEL OF URBAN TRAFFIC AND OPTIMUM MANAGEMENT OF ITS FLOW AND CONGESTION

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ABSTRACT. In this paper we develop a dynamic model for urban traffic along with physical constraints characteristic of intersections equipped with traffic light. We introduce expressions for throughput and congestion and define an appropriate objective functional. Then we formulate an optimization problem whose solution if implemented is expected to reduce congestion and improve throughput. We use the principle of optimality to construct the optimization algorithm.

Key Words. Dynamic Urban Traffic Model, Nonhomogeneous Poisson Process, Bellman Dynamic Programming, Optimal Control of Congestion and Flow

AMS (MOS) Subject Classification. 39A50, 93E20

1. MOTIVATION

Traffic management is becoming more and more important with the development of modern society. In [1], a basic dynamic model for congestion status of an intersection was reported by Rahman *et al.* without any attempt to optimize traffic flow. In order to fully describe the dynamics of urban traffic, in this paper we propose a stochastic model for urban traffic flow from system's point of view [2], [3]. First we consider one intersection and define the dynamics of traffic flow in eight directions, four directions for cross traffic and four for straight traffic. The incoming traffic to each stream is assumed to be a Poisson random process with variable intensity. Each segment of the road where vehicles line up for service is finite. As soon as the segment is partially occupied radio broadcast is made about developing congestion thereby encouraging drivers to choose alternate routes and thereby avoiding congestion. This is used as a control variable. Another control variable is the fraction of time allocated to each stream. The objective is to maximize throughput and minimize possible congestion, delay and service time.

2. STOCHASTIC TRAFFIC MODEL

Traffic flow is a stochastic process. In other words, the number of vehicles crossing any intersection over any given interval of time is a random variable. A reasonable

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mathematical model can be constructed on the basis of counting process, in particular, the well known Poisson process. Here in this section we present one such model. Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ denote a filtered probability space, where Ω is the sample space, $\mathcal{F}$ is the sigma algebra of Borel subsets of the space Ω and $\mathcal{F}_t \subset \mathcal{F}$ is a family of nondecreasing subsigma algebras and P is the probability measure.

A stochastic process $\{\xi_t \equiv p(0,t), t \geq 0\}$, is called a Poisson random process or a counting process giving the number of events over the time period $[0,t]$. More precisely, for each $\omega \in \Omega$, p_w is a set function defined on the sigma algebra of Borel sets in $R_+ \equiv [0,\infty)$ and taking values from the set of nonnegative integers $\mathcal{N} \equiv \{0,1,2,3,\dots\}$. For example, the number of cars arriving at the intersection during the time interval $(t_1,t_2]$ giving $p_w(t_1,t_2]$ is a random variable.

The process p is called a homogeneous Poisson process if the values of p over disjoint intervals of time are stochastically independent or equivalently the increments of ξ over disjoint intervals of time are statistically independent and there exists a nonnegative constant λ such that the probability that there are exactly n events over the time period $(t_1,t_2]$ is given by

$$P\{p_w(t_1,t_2] = n\} = P\{\xi_{t_2}(\omega) - \xi_{t_1}(\omega) = n\} = e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^n/n! \tag{2.1}$$

The constant λ is called the mean or average number of events per unit time or the frequency of occurrence of events. Indeed the reader can easily verify that the expected value of the random variable $p_\omega((t_1,t_2])$ is given by

$$\mathbf{E}\{p_\omega(t_1,t_2]\} = \sum_{n=0}^{\infty} n e^{-\lambda(t_2-t_1)}(\lambda(t_2-t_1))^n/n! = \lambda(t_2-t_1) \tag{2.2}$$

Throughout the rest of the paper we shall omit the variable ω.

In the study of urban traffic flow, the mean flow rate is not constant. It varies with time. It is large at the rush hour in the morning when people go to work and again in the evening when they return home. These are the two main peak hours and there is also a moderate peak at lunch hour. So we need a nonhomogeneous Poisson process with time varying mean. Let $\xi_t \equiv p(0,t)$ be a nonhomogeneous Poisson process with the mean rate function $\lambda(t), t \geq 0$. Clearly, this is a nonnegative finite real valued function (deterministic). Then the probability that there are exactly n events during the time interval $J \equiv (t_1,t_2]$ is given by

$$P\{p(t_1,t_2] = n\} = P\{\xi_{t_2} - \xi_{t_1} = n\} = \exp\left(-\int_{t_1}^{t_2} \lambda(t)dt\right) \frac{\left(\int_{t_1}^{t_2} \lambda(t)dt\right)^n}{n!} \tag{2.3}$$

In practical applications the function $\lambda(t)$, $t \geq 0$, can be estimated by survey of the traffic at any given major intersection. Let us assume that the function λ is available from midnight to midnight. Suppose this time interval is partitioned into a finite number of intervals like $(t_k, t_{k+1}]$, $k = 0,1,2,\dots,N-1$ so that entire day denoted by $I \equiv (0,T]$ is given by $I = \cup_{k=0}^{N-1}(t_k, t_{k+1}]$. The function λ is then approximated by step functions in the sense that it is constant on each interval and given by

$$\lambda(t) = \lambda(t_k) \equiv \lambda_k, \quad \text{for } t \in (t_k, t_{k+1}], \quad k = 0, 1, 2, \ldots, N-1. \tag{2.4}$$

This is quite reasonable if the length of each time interval $(t_{k+1} - t_k)$ is sufficiently small, say, one minute. Thus the probability that there are exactly n events during the interval $(t_k, t_{k+1}]$ is given by

$$P\{\xi_{t_{k+1}} - \xi_{t_k} = n\} = \exp\left(-\lambda_k(t_{k+1} - t_k)\right) \frac{(\lambda_k(t_{k+1} - t_k))^n}{n!} \tag{2.5}$$

In the modeling process we will denote the corresponding Poisson process by

$$p((t_k, t_{k+1}], \lambda_k)$$

and rewrite equation (2.5) as

$$P\{p((t_k, t_{k+1}], \lambda_k) = n\} = \exp\bigl(-\lambda_k(t_{k+1} - t_k)\bigr) \frac{(\lambda_k(t_{k+1} - t_k))^n}{n!} \tag{2.6}$$

Note that the expected value of $p((t_k, t_{k+1}], \lambda_k)$ is given by

$$\mathbf{E}\bigl[p((t_k, t_{k+1}], \lambda_k)\bigr] = \lambda_k(t_{k+1} - t_k)$$

and the second moment is given by

$$\mathbf{E}\bigl[(p((t_k, t_{k+1}], \lambda_k))^2\bigr] = (\lambda_k(t_{k+1} - t_k))^2 + \lambda_k(t_{k+1} - t_k)$$

and hence the variance is given by

$$\mathbf{E}\bigl[p((t_k, t_{k+1}], \lambda_k) - \lambda_k(t_{k+1} - t_k)\bigr]^2 = \lambda_k(t_{k+1} - t_k)$$

3. DYNAMIC MODEL OF TRAFFIC FLOW

Now we are prepared to develop a dynamic model of traffic flow at an intersection. Let us consider an intersection of any two two-way roads having possibly multiple lanes and let us denote the two roads by $R1$ (road one) and $R2$ (road two). At the intersection there are several streams of traffic flow, total 12 in number. We will disregard the flows that can take right turn from their right lane whenever it is safe to do so. This will eliminate four simple streams. We are left with eight more critical flows. There are 4 streams of straight traffic and 4 streams of cross traffic. It is appropriate to consider discrete time evolution of the traffic status at any intersection. We consider the sequence of time intervals $J_k \equiv (t_k, t_{k+1}]$ starting with $k = 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$. Each interval of time J_k is also considered as a cycle of traffic light sequence. Each interval is given by the union of 4 subintervals

$$J_k = \tau_{c1k} \cup \tau_{s1k} \cup \tau_{c2k} \cup \tau_{s2k}$$

where τ_{c1k} is the period of time allocated for cross traffic from $R1$ to $R2$ during the time segment J_k and τ_{s1k} is the period of time allocated for straight traffic on road

$R1$. Similarly τ_{c2k} and τ_{s2k} denote the time intervals allocated for $R2$ traffic during the time slot J_k. Let $|J_k| \equiv (t_{k+1} - t_k)$ denote the length of one cycle, the time interval J_k, and $|\tau_{c1k}| = \gamma_1(t_k)|J_k|$, $|\tau_{s1k}| = \gamma_2(t_k)|J_k|$, $|\tau_{c2k}| = \gamma_3(t_k)|J_k|$, $|\tau_{s2k}| = \gamma_4(t_k)|J_k|$ where $\{\gamma_i(\cdot)\}$ are positive fractions summing to one, that is $\sum \gamma_i(\cdot) = 1$. Clearly these numbers denote the fraction of time allocated to cross and straight flows at the intersection. The fractions $\{\gamma_i, i = 1, 2, 3, 4\}$ are variables that can be adjusted. We shall consider these as one set of control variables to be chosen to optimize traffic flow.

Throughout the rest of the paper we use the notation I for the indicator function. Let S denote any logical (or mathematical) statement. Then the indicator function of this statement is defined as follow:

$$I(S) = \begin{cases} 1 & \text{if } S \text{ is true,} \\ 0 & \text{otherwise} \end{cases}$$

For different statements such as S_j we use indicators with a subscript such as I_j, $j = 1, 2, 3$. For any two real numbers $\{a, b\}$ we use the notation

$$a \wedge b \equiv \min\{a, b\}$$

to denote the minimum of the two. Now we are prepared to introduce the traffic flow dynamics.

Complementary Cross Traffic from R1 to R2. The complementary cross traffic from $R1$ to $R2$ in two opposite directions is given by the following pair of equations:

$$\begin{aligned} x_{11}(t_{k+1}) = x_{11}(t_k) &+ I_1(0 \le x_{11}(t_k) < \ell_1(t_k))p_1(J_k, \lambda_1(t_k)) \\ &+ I_2(\ell_1(t_k) \le x_{11}(t_k) \le C_1)p_1(J_k, \theta_1\lambda_1(t_k)) - (\beta_1 \wedge x_{11}(t_k))\gamma_1(t_k)|J_k| \end{aligned} \tag{3.1}$$

$$\begin{aligned} x_{12}(t_{k+1}) = x_{12}(t_k) &+ I_1(0 \le x_{12}(t_k) < \ell_2(t_k))p_2(J_k, \lambda_2(t_k)) \\ &+ I_2(\ell_2(t_k) \le x_{12}(t_k) \le C_2)p_2(J_k, \theta_2\lambda_2(t_k)) - (\beta_2 \wedge x_{12}(t_k))\gamma_1(t_k)|J_k| \end{aligned} \tag{3.2}$$

Equation (3.1) represents the cross traffic from $R1$ to $R2$ in one direction and equation (3.2) represents the cross traffic in the opposite direction. The symbol $x_{11}(t_{k+1})$ stands for the number of vehicles accumulated on this stream at time t_{k+1}. This is given by the algebraic sum of 4 terms. The first term on the right gives the number of vehicles that was left on the stream from the previous cycle. The second term gives the random number of vehicles arriving during the time period $J_k \equiv (t_k, t_{k+1}]$. This is denoted by the Poisson random variable $p_1(J_k, \lambda_1(t_k))$ (Poisson random number) with the mean arrival rate $\lambda(t_k)$ provided that $x_{11}(t_k)$ is below the threshold $\ell_1(t_k)$. The number $\ell_1(t_k)$ denotes the warning level explained further in the sequel. The third component on the right represents the number of vehicles arriving after congestion has been broadcast at a reduced rate $\theta_1\lambda(t_k)$ where $0 \le \theta_1 < 1$

provided the lane capacity C_1 is not exceeded. The last term on the right represents the number of vehicles that leaves the stream during the fraction of the time interval τ_{c1k} at the service capacity β_1. The number β_1 represents the service capacity of this stream depending on the number of lanes. Thus the the departure rate is the smaller of the two $\{\beta_1, x_{11}(t_k)\}$ denoted by $\beta_1 \wedge x_{11}(t_k)$.

Equation (3.2) represents the complementary cross traffic from $R1$ to $R2$ in the opposite direction and has identical interpretation as equation (3.1).

Complementary Straight Traffic on R1.

$$x_{13}(t_{k+1}) = x_{13}(t_k) + I_1(0 \leq x_{13}(t_k) < \ell_3(t_k))p_3(J_k, \lambda_3(t_k)) \\ +I_2(\ell_3(t_k) \leq x_{13}(t_k) \leq C_3)p_3(J_k, \theta_3\lambda_3(t_k)) - (\beta_3 \wedge x_{13}(t_k))\gamma_2(t_k)|J_k| \tag{3.3}$$

$$x_{14}(t_{k+1}) = x_{14}(t_k) + I_1(0 \leq x_{14}(t_k) < \ell_4(t_k))p_4(J_k, \lambda_4(t_k)) \\ +I_2(\ell_4(t_k) \leq x_{14}(t_k) \leq C_4)p_4(J_k, \theta_4\lambda_4(t_k)) - (\beta_4 \wedge x_{14}(t_k))\gamma_2(t_k)|J_k| \tag{3.4}$$

Considering the complementary straight traffic on $R1$, equations (3.3) and (3.4) represent streams of straight traffic in opposite directions. Considering equation (3.3), the number of vehicles on this stream at time t_{k+1} denoted by $x_{13}(t_{k+1})$ is given by the sum of 4 terms. The first term represents the residue traffic from the previous cycle ending at time t_k. The second term represents the new arrivals during the time cycle J_k and it is given by the Poisson random variable $p_3(J_k, \lambda_3(t_k))$ corresponding to the mean intensity $\lambda_3(t_k)$ provided the threshold level $\ell_3(t_k)$ has not been reached. The third term represents the number of vehicles which arrive at a reduced rate after the congestion warning has been broadcast given that the road segment capacity C_3 is not reached. The fourth term on the righthand side represents the number of vehicles that left the stream during the fraction of time τ_{s1k} with service capacity β_3. Similar interpretation is valid for the complementary traffic (traffic in the opposite direction).

Similarly the traffic on and from $R2$ is given by another set of 4 equations. They are as follows:

Complementary Cross Traffic from R2 to R1.

$$x_{21}(t_{k+1}) = x_{21}(t_k) + I_1(0 \leq x_{21}(t_k) < \ell_5(t_k))p_5(J_k, \lambda_5(t_k)) \\ +I_2(\ell_5(t_k) \leq x_{21}(t_k) \leq C_5)p_5(J_k, \theta_5\lambda_5(t_k)) - (\beta_5 \wedge x_{21}(t_k))\gamma_3(t_k)|J_k| \tag{3.5}$$

$$x_{22}(t_{k+1}) = x_{22}(t_k) + I_1(0 \leq x_{22}(t_k) < \ell_6(t_k))p_6(J_k, \lambda_6(t_k)) \\ +I_2(\ell_6(t_k) \leq x_{22}(t_k) \leq C_6)p_6(J_k, \theta_6\lambda_6(t_k)) - (\beta_6 \wedge x_{22}(t_k))\gamma_3(t_k)|J_k| \tag{3.6}$$

These equations describe the dynamics of complementary cross traffic from $R2$ to $R1$. They have similar interpretation as those of equations (3.1) and (3.2).

Complementary Straight Traffic on R2.

$$x_{23}(t_{k+1}) = x_{23}(t_k) + I_1(0 \le x_{23}(t_k) < \ell_7(t_k))p_7(J_k, \lambda_7(t_k)) \\ +I_2(\ell_7(t_k) \le x_{23}(t_k) \le C_7)p_7(J_k, \theta_7\lambda_7(t_k)) - (\beta_7 \wedge x_{23}(t_k))\gamma_4(t_k)|J_k| \tag{3.7}$$

$$x_{24}(t_{k+1}) = x_{24}(t_k) + I_1(0 \le x_{24}(t_k) < \ell_8(t_k))p_8(J_k, \lambda_8(t_k)) \\ +I_2(\ell_8(t_k) \le x_{24}(t_k) \le C_8)p_8(J_k, \theta_8\lambda_8(t_k)) - (\beta_8 \wedge x_{24}(t_k))\gamma_4(t_k)|J_k| \tag{3.8}$$

These equations describe the dynamics of complementary straight traffic on $R2$. They have similar interpretation as those of equations (3.3) and (3.4).

4. TRAFFIC FLOW OPTIMIZATION PROBLEM

The objective is to maximize throughput through the intersection and minimize congestion. The lane capacity is fixed. Thus the throughput depends on the lane capacity and the traffic present and more importantly on the distribution of the fraction of time $\{\gamma_i, i = 1, 2, 3, 4\}$ allocated to each of the streams which is a decision variable. Presently we denote this by the four dimensional vector $\gamma \equiv (\gamma_1, \gamma_2, \gamma_3, \gamma_4)' \in R_+^4$, $R_+ = [0, \infty)$. Then the total throughput over one time period $[t_0, t_N]$ is given by the following expression,

$$\begin{aligned} J_{TP}(\gamma) \equiv \mathbf{E}\Bigg\{ \sum_{k=0}^{N-1} \Bigg(&\Bigg\{ \sum_{i=1}^{2} (\beta_i \wedge x_{1i}(t_k)) I_3(t \in \tau_{c1k}) \Bigg\} \gamma_1(t_k)(t_{k+1} - t_k) \\ &+ \Bigg\{ \sum_{i=1}^{2} (\beta_{i+2} \wedge x_{1(i+2)}(t_k) I_3(t \in \tau_{s1k}) \Bigg\} \gamma_2(t_k)(t_{k+1} - t_k) \\ &+ \Bigg\{ \sum_{i=1}^{2} (\beta_{i+4} \wedge x_{2i}(t_k)) I_3(t \in \tau_{c2k}) \Bigg\} \gamma_3(t_k)(t_{k+1} - t_k) \\ &+ \Bigg\{ \sum_{i=1}^{2} (\beta_{i+6} \wedge x_{2(i+2)}(t_k)) I_3(t \in \tau_{s2k}) \Bigg\} \gamma_4(t_k)(t_{k+1} - t_k) \Bigg) \Bigg\} \end{aligned} \tag{4.1}$$

where $\mathbf{E}\{\cdot\}$ stands for the expected value of the random variable within the brace. The first and the second sum within the braces give respectively the number of vehicles served in the complementary cross flow and complementary straight flow from $R1$. The third and the fourth sum represent the number of vehicles served from similar flows with reference to road $R2$.

Now we must introduce a measure of congestion. It is clear that it depends on the size of the lane capacity $\{C_i, i = 1, 2, \ldots, 8\}$, the desired level of threshold for congestion warning and the importance or weight given to the level of congestion. The threshold vector is denoted by $\ell = (\ell_1, \ell_2, \ldots, \ell_8)'$. So a reasonable measure of congestion is given by the following expression,

$$J_c(\ell) \equiv \mathbf{E}\Bigg\{\sum_{k=0}^{N-1}\Bigg(\sum_{i=1}^{4} W_i(x_{1i}(t_k) - \ell_i(t_k))I_2(x_{1i}(t_k) > \ell_i(t_k)) + \sum_{i=1}^{4} W_{i+4}(x_{2i}(t_k) - \ell_{i+4}(t_k))I_2(x_{2i}(t_k) > \ell_{i+4}(t_k))\Bigg)\Bigg\} \tag{4.2}$$

where the first sum within the round bracket gives the weighted level of congestion on $R1$, and the second sum gives the weighted congestion level on $R2$. It is clear that the waiting time for the drivers increases with the increase of threshold setting (warning level). In order to incorporate the measure of waiting time we include a third term to the objective functional given by

$$J_w(\ell) \equiv \mathbf{E}\left\{\sum_{k=0}^{N-1}\left(\sum_{i=1}^{8} V_i \; \ell_i(t_k)\right)\right\} \tag{4.3}$$

The objective is to choose the set of parameters, decision variables (γ, ℓ), that minimizes the following functional

$$J(\gamma, \ell) \equiv J_c(\ell) - J_{TP}(\gamma) + J_w(\ell) \tag{4.4}$$

5. COMPACT FORMULATION OF THE OPTIMIZATION PROBLEM

We are going to use the principle of dynamic programming to solve the optimization problem stated in the preceding section. For this it is convenient to recast the problem using vector notation. For any $k \in \{0, 1, 2, 3, \ldots, N-1\}$ let us denote $t_k \equiv k$ and

$$x_k \equiv (x_{11}(t_k), x_{12}(t_k), x_{13}(t_k), x_{14}(t_k), x_{21}(t_k), x_{22}(t_k), x_{23}(t_k), x_{24}(t_k))' \tag{5.1}$$

$$\hat{\gamma}_k \equiv (\gamma_1(t_k), \gamma_2(t_k), \gamma_3(t_k), \gamma_4(t_k))' \tag{5.2}$$

$$\hat{\ell}_k \equiv (\ell_1(t_k), \ell_2(t_k), \ell_3(t_k), \ell_4(t_k), \ell_5(t_k), \ell_6(t_k), \ell_7(t_k), \ell_8(t_k))' \tag{5.3}$$

These are column vectors in R^8, R^4, and R^8, respectively representing state and controls. The expressions on the righthand side of the equations (3.1)–(3.8) are denoted by the column vector

$$x_k + F(k, x_k, \hat{\gamma}_k, \hat{\ell}_k) \equiv G(k, x_k, \hat{\gamma}_k, \hat{\ell}_k) \in R^8 \tag{5.4}$$

Using the vectors (5.1), (5.2), (5.3) and (5.4) the system of equations (3.1)–(3.8) can be written in the following compact form (as a vector difference equation)

$$x_{k+1} = G(k, x_k, \hat{\gamma}_k, \hat{\ell}_k), \quad k \in \{0, 1, 2, \ldots, N-1\} \tag{5.5}$$

Parametric Constraints. Define the set Γ by

$$\Gamma \equiv \left\{ \gamma \in R^4 : \gamma_i \geq 0, \quad i = 1,2,3,4 \text{ and } \sum_{i=1}^{4} \gamma_i = 1 \right\} \tag{5.6}$$

and the set Λ as

$$\Lambda \equiv \{ \ell \in R^8 : \alpha C_i \leq \ell_i \leq C_i, i = 1,2,\ldots,8 \} \tag{5.7}$$

where the fraction α can be chosen by traffic planner in any desirable range such as $0.8 \leq \alpha \leq 0.95$. The vector ℓ defines the threshold level at which the intersection is declared to have reached congestion. This information is broadcast so that drivers may choose to avoid the particular direction of the intersection or the intersection itself. This can be done by use of microelectronic devices for monitoring the traffic status and transmitting the information to a central radio broadcasting station.

We denote the decision or control set by $U \equiv \Gamma \times \Lambda$ and use $u = (\gamma, \ell)$ to denote any member of the admissible set U. Using this notation the dynamic system given by (5.5) can be written in the standard form as follows:

$$x_{k+1} = G(k, x_k, u_k), \quad x_0 = \xi, \quad k \in \mathcal{Z} \equiv \{0,1,2,\ldots,N-1\} \tag{5.8}$$

where $x_0 = \xi$ is the initial state and $u_k \equiv (\hat{\gamma}_k, \hat{\ell}_k)$ is the control. Note that the vector G at any stage k (time) depends on the Poisson random vector $p \equiv (p_1, p_2, \ldots, p_8)'$ with the intensity (or mean) functions $\lambda \equiv (\lambda_{1,k}, \lambda_{2,k}, \ldots, \lambda_{8,k})'$ at time k and this is an integral part of the model. This is what makes the system stochastic. As mentioned earlier, these functions (data) are available from statistical survey of the arrival history of the intersection under question.

Cost Functional. For convenience of notation, using the expressions inside the parenthesis of the expressions (4.1), (4.2) and (4.3), we define and write

$$\begin{aligned} L(k, x_k, u_k) \equiv \Bigg\{ \Bigg(& \sum_{i=1}^{4} W_i(x_{1i}(t_k) - \ell_i(t_k)) I_2(x_{1i}(t_k) > \ell_i(t_k)) \\ & + \sum_{i=1}^{4} W_{i+4}(x_{2i}(t_k) - \ell_{i+4}(t_k)) I_2(x_{2i}(t_k) > \ell_{i+4}(t_k)) \Bigg) \\ & - \Bigg(\Big\{ \sum_{i=1}^{2} (\beta_i \wedge x_{1i}(t_k)) I_3(t \in \tau_{c1k}) \Big\} \gamma_1(t_k)(t_{k+1} - t_k) \\ & + \Big\{ \sum_{i=1}^{2} (\beta_{i+2} \wedge x_{1(i+2)}(t_k) I_3(t \in \tau_{s1k}) \Big\} \gamma_2(t_k)(t_{k+1} - t_k) \\ & + \Big\{ \sum_{i=1}^{2} (\beta_{i+4} \wedge x_{2i}(t_k)) I_3(t \in \tau_{c2k}) \Big\} \gamma_3(t_k)(t_{k+1} - t_k) \end{aligned}$$

$$+ \{\sum_{i=1}^{2}(\beta_{i+6} \wedge x_{2(i+2)}(t_k))I_3(t \in \tau_{s2k})\}\gamma_4(t_k)(t_{k+1} - t_k)\Big)$$

$$+ \left(\sum_{i=1}^{8} V_i \ell_i(t_k)\right)\Big\} \tag{5.9}$$

Using this expression the objective (cost) functional (4.4) can be written compactly, and also in the canonical form, as

$$J(u) \equiv \mathbf{E}\left\{\sum_{k=0}^{N-1} L(k, x_k, u_k) + \Psi(x_N)\right\} \tag{5.10}$$

where $\Psi(x_N)$ denotes the terminal cost, for example, penalty for deviation from a desired final state and $u \equiv \{u_0, u_1, u_2, \ldots, u_{N-1}\}$ denotes the decision or control policy over the time horizon. Note that there is no impact on the cost functional for any control chosen at the final stage N. The terminal cost $\Psi(x_N)$ depends only on the final state determined by the preceding state and the preceding control.

Optimization Problem. The problem is to minimize the objective functional $J(u)$ given by (5.10) subject to the dynamic constraint (5.8) and the decision or control constraints $U \equiv \Gamma \times \Lambda$.

6. DYNAMIC PROGRAMMING

There are two well known techniques for solving optimal control and decision problems. One is the dynamic programming (DP) technique developed by Bellman [4] and the other is the maximum principle (MP) developed by Pontryagin *et al.* [5]. Maximum principle requires differentiability of the vector field G with respect to the state variable x. It is evident from the model that there are many indicator functions required for its construction and clearly they are not differentiable in the usual sense. However, dynamic programming method does not require this property and therefore DP is the most suitable technique for the optimization problem stated here.

The idea of dynamic programming is based on the simple fact that, given the present, nothing can alter the past but the future can be improved by appropriate actions starting from the present. Suppose $n-1$ stages have passed and the current stage is n so that the available stages are $\{n, n+1, \ldots, N-1\}$ and the decisions at these stages must be taken so as to minimize the "cost to go" to the final stage N. Let $x_n = \xi$ denote the present state and let $\Phi(n, x_n) = \Phi(n, \xi)$ denote the minimum future cost obtained by proper choice of the control policy $\{u_n^o, u_{n+1}^o, \ldots, u_{N-1}^o\}$ over the remaining time horizon. Thus according to this philosophy combined with the Markov property (given the present the future is independent of the past) we obtain the following equations,

$$\Phi(n,\xi) = \inf_{u_n,u_{n+1},\ldots,u_{N-1}} \mathbf{E}\left\{\left(\sum_{k=n}^{N-1} L(k,x_k,u_k) + \Psi(x_N)\right)|_{x_n=\xi}\right\} \tag{6.1}$$

$$= \inf_{u_n} \mathbf{E}\left\{L(n,\xi,u_n) + \Phi(n+1,x_{n+1})\right\} \tag{6.2}$$

The first equation (6.1) gives the conditional expectation of the random variable within the parenthesis given that $x_n = \xi$. The last term in (6.2) is not explicitly dependent on controls but the state x_{n+1} is dependent on u_n through the state equation

$$x_{n+1} = G(n,\xi,u_n)$$

and thus equation (6.2) takes the form

$$\Phi(n,\xi) = \inf_{u_n\in U} \mathbf{E}\left\{L(n,\xi,u_n) + \Phi(n+1,G(n,\xi,u_n))\right\} \tag{6.3}$$

This equation holds for all $n \in \{0,1,2,3,\ldots,N-1\}$ and $\xi \in \mathcal{S} \subset R^8$ where $\mathcal{S}$ denotes the set of admissible states. In practice this may be a proper subset of R^8. According to the above expression, it is clear that the optimal control at the stage n depends on the state at this stage. Denoting the optimal policy by $u_n^o = v(n,\xi)$ for $n \in \mathcal{Z} \equiv \{0,1,2,3,\ldots,N-1\}$ we arrive at the celebrated Bellman's functional equation,

$$\begin{aligned} &\Phi(n,\xi) = \mathbf{E}\left\{L(n,\xi,v(n,\xi)) + \Phi(n+1,G(n,\xi,v(n,\xi)))\right\} \\ &n \in \mathcal{Z} \equiv \{0,1,2,3,\ldots,N-1\} \text{ and } \xi \in \mathcal{S} \subset R^8 \end{aligned} \tag{6.4}$$

where $v(n,\cdot)$ is the optimal decision (control) at the stage n. In other words, the optimal policy is given by

$$v(n,\xi) \equiv \arg\left\{\inf_{u_n\in U} \mathbf{E}\big\{L(n,\xi,u_n) + \Phi(n+1,G(n,\xi,u_n))\big\}\right\} \tag{6.5}$$

which is always expected to be a function of the state, the system is at that stage. Before we discuss the algorithm, we present the following existence result.

Proposition 6.1. *The dynamic programming problem consisting of equations* (6.4) *and* (6.5) *has at least one solution and hence an optimal feedback control law,* $v^o(n,\xi)$*,* $(n,\xi) \in \mathcal{Z}\times\mathcal{S}$*, exists in the class of bounded measurable functions with values in* U*.*

Proof. The proof follows from the simple facts that the set U is compact and that, for every fixed stage $n \in \{0,1,2,\ldots,N-1\}$ and state $\xi \in \mathcal{S}$, the functions $L(n,\xi,\cdot)$ and $G(n,\xi,\cdot)$ are continuous. □

Using the Bellman equation (6.4) and the equation for the optimal policy (6.5) one can develop an algorithm whereby one can compute the optimal policy and the optimal cost. The functional equation (6.4) can be solved only backward because it is the terminal condition $\Psi(\cdot)$ (not the initial condition) that is given. The backward sweep determines the optimal control policy. Once the optimal policy is determined the state equation (5.8) is solved forward in time giving the optimal state. Once we are able to construct the function Φ, all other necessary information such as the optimal feedback control law, the optimal cost for the given initial state and the optimal trajectory can be determined. The task of constructing this function using the Bellman equation (6.4) is computationally intensive, which is the so called curse of dimensionality.

7. SEQUENTIAL OR RECURSIVE ALGORITHM

Backward Sweep. Let $\mathcal{S} \subset R^8$ denote the set of admissible states. Examining the original state equations (3.1)–(3.8) it is clear that this set is bounded (and closed). Similarly the control set $U \equiv \Gamma \times \Lambda \subset R^{12}$ is also a bounded set. By construction of the Bellman function it is meant that for each $n \in \{0, 1, 2, \ldots, N-1\}$ we have the function $\Phi(n, \cdot) = \{\Phi(n, \xi), \xi \in \mathcal{S}\}$. If $\mathcal{S}$ consists of only a finite set of points, the problem is computationally feasible. However if $\mathcal{S}$ is a continuum, the problem is computationally formidable.

In any case the procedure is as described below. Starting at the stage $N-1$ with any state $x_{N-1} \in \mathcal{S}$ we have

$$\begin{aligned}\Phi(N-1, x_{N-1}) &= \inf_{u_{N-1}\in U} \mathbf{E}\left\{L(N-1, x_{N-1}, u_{N-1}) + \Psi(x_N)\right\} \\ &= \inf_{u_{N-1}\in U} \mathbf{E}\left\{L(N-1, x_{N-1}, u_{N-1}) + \Psi(G(N-1, x_{N-1}, u_{N-1}))\right\} \qquad (7.1)\end{aligned}$$

By following (6.5) we have the corresponding optimal decision given by

$$u^o_{N-1} = v(N-1, x_{N-1}) \qquad (7.2)$$

where v is a suitable function of the current stage $N-1$ and current state x_{N-1}. Note that this gives us only a pair of real numbers for (Φ, v) corresponding to the pair $(N-1, x_{N-1}) \in \mathcal{Z} \times \mathcal{S}$, that is, $(\Phi(N-1, x_{N-1}), v(N-1, x_{N-1}))$. Keeping $n = N-1$ fixed one must solve the minimization problem (7.1) for a dense set of values for $x_{N-1} \in \mathcal{S}$. One can choose a finite number ($m < \infty$) of representative points $\{\xi_1, \xi_2, \ldots, \xi_m\}$ from $\mathcal{S}$ and carry out the above procedure to obtain the set

$$\{(\Phi(N-1, \xi_i), v(N-1, \xi_i)),\ i = 1, 2, 3, \ldots, m\}$$

Then one can use interpolation to construct the functions

$$(\Phi(N-1, \cdot), v(N-1, \cdot)) \equiv \{(\Phi(N-1, \xi), v(N-1, \xi)), \xi \in \mathcal{S}\} \qquad (7.3)$$

for the given stage $N-1$. This completes only one stage. To continue we go (backward in time) to the next stage and state. Then we have, for any $x_{N-2} \in \mathcal{S}$,

$$\Phi(N-2, x_{N-2}) = \inf_{u_{N-2}\in U} \mathbf{E}\Big\{L(N-2, x_{N-2}, u_{N-2}) + \Phi(N-1, x_{N-1})\Big\}$$

$$= \inf_{u_{N-2}\in U} \mathbf{E}\Big\{L(N-2, x_{N-2}, u_{N-2}) + \Phi(N-1, G(N-2, x_{N-2}, u_{N-2}))\Big\} \tag{7.4}$$

Again following equation (6.5) we obtain the optimal decision at the stage $N-2$ given by

$$u^o_{N-2} = v(N-2, x_{N-2}) \tag{7.5}$$

Finally following the same procedure as in stage $N-1$, we arrive at the following pair of functions

$$(\Phi(N-2,\cdot), v(N-2,\cdot)) \equiv \{(\Phi(N-2,\xi), v(N-2,\xi)), \xi \in \mathcal{S}\} \tag{7.6}$$

for the stage $N-2$. Continuing this process step by step backward till the initial stage $k=0$ is reached, we arrive at the following expression

$$\Phi(0, x_0) = \inf_{u_0\in U} \mathbf{E}\Big\{L(0, x_0, u_0) + \Phi(1, G(0, x_0, u_o))\Big\} \tag{7.7}$$

Again following equation (6.5) we obtain the optimal decision $u^o_0 = v(0, x_0)$ at stage $k=0$ and state x_0 and finally the pair of functions

$$(\Phi(0,\cdot), v(0,\cdot)) \equiv \{(\Phi(0,\xi), v(0,\xi)), \xi \in \mathcal{S}\} \tag{7.8}$$

Forward Sweep. Using the feedback control law $u^o_k \equiv v(k, x_k)$ determined from the backward sweep, one can solve for the optimal state trajectory using the state dynamics

$$x_{k+1} = G(k, x_k, v(k, x_k)), \quad x_0 = \xi, \quad k = 0, 1, 2, \ldots, N-1. \tag{7.9}$$

This gives the optimal state trajectory $\{x^o_k, k \in \mathcal{Z}\}$.

Backward-Forward Sweep Combined. Combining the results from the backward and forward sweep, we obtain (i): the value function (Bellman function) Φ (ii): the optimal feedback control law v (iii): the optimal cost $J(u^o) = J(v)$ and (iv): the optimal state trajectory as follows:

$$(i) : \Phi(k, x), \quad k \in \mathcal{Z} \equiv \{0, 1, 2, \ldots, N-1\}, \quad x \in \mathcal{S} \tag{7.10}$$

$$(ii) : v(k, x), \quad k \in \mathcal{Z}, \quad x \in \mathcal{S} \subset R^8 \tag{7.11}$$

$$(iii) : J(u^o) = J(v) = \Phi(0, x_0) \tag{7.12}$$

$$(iv) : x^o_{k+1} = G(k, x^o_k, u^o_k) = G(k, x^o_k, v(k, x^o_k)), \quad x^o_0 = x_0, \quad k \in \mathcal{Z} \tag{7.13}$$

Based on the above algorithm, currently we are carrying out extensive numerical computations to determine the optimal controls. We believe that implementation of the optimal controls determined by the technique presented here will significantly improve the overall quality of traffic flow in large cities.

8. CONCLUSION

n this paper we have developed a novel stochastic dynamic model for traffic flow in and around a busy intersection in an urban environment. The traffic is modelled by use of nonhomogeneous Poisson process with variable intensity which is a function of time (midnight to midnight) representing mean flow. We have also constructed an appropriate objective functional that includes throughput, congestion and waiting time. Using the principle of optimality due to Bellman, we have constructed the dynamic programming equation to determine the optimal feedback control policies. The technique developed here should be applied only to some major intersections of the city to improve overall traffic flow, minimize congestion and avoid traffic jams. This technique can be implemented on micro-electronic devices which can be deployed at major intersections based on their corresponding statistics.

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