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LOCAL RISK MINIMIZING OPTION IN A REGIME-SWITCHING DOUBLE HESTON MODEL

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ABSTRACT. We address risk minimizing option pricing in a regime switching double Heston model with three jumps when the underlying asset price follows a general state-dependent regime-switching jump-diffusion process. Using minimal martingale measure, an optimal hedging strategy is obtained by the local risk minimization.

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1. Introduction

The Markov regime switching markets contain dramatic change in macroeconomic by incorporating a continuous-time Markov chain. In fact the rare events information reflect on stock price in those frame work. As known the regime switching markets are incomplete. So the pricing of regime switching risk gets an important issue. Option pricing is one of the most important concept in modern finance. Black and Scholes developed the methodology of option valuation. A major challenge in the Black-Scholes model is that interest rate and the volatility rate are assumed to be constants which are not consistent with reality [3].

To get more realistic models, many extensions to the Black-Scholes model have been presented. Among those the regime-switching models provide more realistic description for asset price dynamics. In these models the parameters are functions of a finite-state Markov chain [5, 6, 9, 13].

Because of several previous studies and the display of the dates, we added two stochastic volatility with three jumps. An excellent contribution of the proposed model is developing the model of stochastic volatility. In fact, in this study, we model the stock price process by the Markov-modulated jump diffusion model with double stochastic volatility with three jumps. So our model better corresponds with reality than the another one.

A unique equivalent martingale measure by minimizing the quadratic utility of the losses is identified by Föllmer and Sondermann. Then the minimal martingale

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measure and risk-minimizing hedging were further developed by several researchers [1, 4, 8, 10, 11, 12, 15, 16, 17].

As it's well known, equivalent martingale measure is not unique in the incomplete market [14]. In this paper, Firstly, we investigate the minimal martingale measure. Then we address risk minimizing option pricing under our proposed model.

The rest of the paper is organized as follows. In Section 2, we present the notation, assumptions, and model for the underlying market. In Section 3, we investigate an explicit representation of the density process of the minimal martingale measure. In Section 4, a PDE of the option pricing is driven. The locally risk-minimizing strategy is studied in Section 5.

2. **Preliminaries**

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ be the complete probability space. Suppose the states of an economy are modeled by a finite state continuous-time Markov chain $\{X_t : t \geq 0\}$. Without loss of generality, we can identify the state space of $\{X_t : t \geq 0\}$ with a finite set of unit vectors $\chi := \{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^N$, whose transition probabilities satisfy

$$P(X_{t+\delta t} = j \mid X_t = i) = q_{ij}\delta t + o(\delta t), \quad i \neq j;$$
$$P(X_{t+\delta t} = i \mid X_t = i) = 1 + q_{ii}\delta t + o(\delta t),$$

when $\delta \to 0$, where $q_{ij} \geq 0$, $i \neq j$; $q_{ii} = -\sum_{j=1}^{N} q_{ij}$. Let $Q = [q_{ij}]$ denote the generating Q-matrix of the Markov chain. The financial market itself is consisting of a riskless asset $(B_t)_{t\in[0,T]}$ and a risky asset $(S_t)_{t\in[0,T]}$ which S_t is square integrable and $S_0 > 0$ is a constant, dynamics of $(B_t)_{t\in[0,T]}$ and $(S_t)_{t\in[0,T]}$ are as follows:

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t, \quad B_0 = 1.$$

$$\mathrm{d}S_t = \mu_t S_{t\_}\mathrm{d}t + \sqrt{V_t^{(1)}} S_{t\_}\mathrm{d}W_t^1 + \sqrt{V_t^{(2)}} S_{t\_}\mathrm{d}W_t^3 + \int_{-1}^{\infty} S_{t\_}y(N(\mathrm{d}y, \mathrm{d}t) - v(\mathrm{d}y)\mathrm{d}t),$$

$$\mathrm{d}V_t^{(1)} = k_1(\theta_1 - V_t^{(1)})\mathrm{d}t + \sigma_{v_1}\sqrt{V_t^{(1)}}\mathrm{d}W_t^2 + Z_1\mathrm{d}N_t,$$

$$\mathrm{d}V_t^{(2)} = k_2(\theta_2 - V_t^{(2)})\mathrm{d}t + \sigma_{v_2}\sqrt{V_t^{(2)}}\mathrm{d}W_t^4 + Z_2\mathrm{d}N_t,$$

where W_t^1, W_t^2, W_t^3, and W_t^4 are standard Brownian motions, that

$$\mathrm{d}W_t^1 \cdot \mathrm{d}W_t^2 = \rho_1 \mathrm{d}t,$$
$$\mathrm{d}W_t^3 \cdot \mathrm{d}W_t^4 = \rho_2 \mathrm{d}t.$$

θ_1 and θ_2 are the long-run average of $V_t^{(1)}$ and $V_t^{(2)}$, respertively, k_1 and k_2 are the rates of mean reversion, σ_{v_1} and σ_{v_2} are the variance of $V_t^{(1)}$ and $V_t^{(2)}$, respectively, Z_1

and Z_2 two exponential stochastic processes with parameters μ_{v_1} and μ_{v_2}, $\rho_i \in (-1,1)$ for $i = 1,2$ are given constants, and process $N(\mathrm{d}y, \mathrm{d}t)$ is a Poisson random measure with P-compensator $v(\mathrm{d}y)\mathrm{d}t = \lambda f(y)\mathrm{d}y\mathrm{d}t$. Let $\widetilde{N}(\mathrm{d}y, \mathrm{d}t) = N(\mathrm{d}y, \mathrm{d}t) - v(\mathrm{d}y)\mathrm{d}t$ be the compensated Poisson random measure. Moreover, we assume that $\int_{-1}^{\infty} y^2 v(\mathrm{d}y) < \infty$. In this setting, the locally risk-free floating interest rate r_t and the appreciation rate μ_t of the stock price evolve over time depending on the state of the market X_t, therefore $r_t = r(X_t)$ and $\mu_t = \mu(X_t)$ be two functions of X_t; that is, $r_t = r(i) = r_i$ and $\mu_t = \mu(i) = \mu_i$ when the state of X_t is $i, i \in \chi$.

Following the description of [2], for $i, j \in \chi$, $i \neq j$, let Δ_{ij} be consecutive left closed right open intervals of the real line, each having length q_{ij}. By embedding χ in $\mathbb{R}^N$ by identifying i with $e_i \in \mathbb{R}^N$ define a function $h : \chi \times \mathbb{R} \to \mathbb{R}^N$ by

$$h(i,z) = \begin{cases} j - i & z \in \Delta_{ij} \\ 0 & \text{o.w.} \end{cases}$$

Then

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} h(X_{u_-}, z) P(\mathrm{d}z, \mathrm{d}u)$$

where the integration is over the interval $(0, T]$ and $P(\mathrm{d}z, \mathrm{d}t)$ is a Poisson random measure with intensity $m(\mathrm{d}z)\mathrm{d}t$; where $m(\mathrm{d}z)$ is the Lebesgue measure on $\mathbb{R}$. $P(\mathrm{d}z, \mathrm{d}t)$, $N(\mathrm{d}y, \mathrm{d}t)$, and X_t are mutually independent, and independent of W_t^1, W_t^2, W_t^3, and W_t^4.

The semimartingale $\widetilde{S}_t = e^{-\int_0^t r_s \mathrm{d}s} S_t$ has the following decomposition

$$\widetilde{S}_t = \widetilde{S}_0 + M_t + A_t$$

with M_t a square-integrable martingale for which $M_0 = 0$, and with A_t is a predictable process of finite variation, where

$$M_t = \int_0^t \widetilde{S}_{u_-} \sqrt{V_u^{(1)}} \mathrm{d}W_u^1 + \int_0^t \widetilde{S}_{u_-} \sqrt{V_u^{(2)}} \mathrm{d}W_u^3 + \int_0^t \int_{-1}^{\infty} \widetilde{S}_{u_-} y \widetilde{N}(\mathrm{d}y, \mathrm{d}u), \tag{2.1}$$

and

$$A_t = \int_0^t \widetilde{S}_{u_-} (\mu_u - r_u) \mathrm{d}u. \tag{2.2}$$

3. **The Minimal Martingale Measure**

Noting that our proposed market is incomplete. More than, one martingale measure exists. In this section, we investigate the minimal martingale measure for presented market.

Definition 3.1. A martingale measure $\hat{P} \approx P$ will be called minimal if

$$\hat{P} = P \quad \text{on } \mathcal{F}_0.$$

and if any square-integrable P-martingale which is orthogonal to M under P remains a martingale under $\hat{P}$.

From [7], for some predictable process $\alpha = (\alpha_t)_{0 \le t \le T}$ we have

$$A_t = \int_0^t \alpha_u \mathrm{d}\langle M \rangle_u.$$

Theorem 3.2. *$\hat{P}$ exists if and only if*

$$G_t = \exp\left(-\int_0^t \alpha_s \mathrm{d}M_s - \frac{1}{2}\int_0^t {\alpha_s}^2 \mathrm{d}\langle M \rangle_u \right) \quad 0 \le t \le T$$

is a square-integrable martingale under P; in that case, $\hat{P}$ is given by $\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} = G_T$.

Let $\{\mathcal{F}^S\}_{t\in[0,T]}$, $\{\mathcal{F}^{V^{(1)}}\}_{t\in[0,T]}$, $\{\mathcal{F}^{V^{(2)}}\}_{t\in[0,T]}$ and $\{\mathcal{F}^X\}_{t\in[0,T]}$ denote the P-augmentation of the natural filtrations generated by S, $V^{(1)}$, $V^{(2)}$ and X, respectively. For each $t \in [0,T]$, set $\mathcal{G}_t = {\mathcal{F}_t}^X \vee ({\mathcal{F}_t}^{V^{(1)}} \vee {\mathcal{F}_t}^{V^{(2)}})$ and $\mathcal{A}_t = {\mathcal{F}_t}^S \vee \mathcal{G}_T$. Given $\mathcal{G}_T$, to avoid the possibility that the minimal martingale measure becomes a signed measure, we need the following condition.

$$\frac{(\mu_t - r_t)y}{V_t^{(1)} + V_t^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} < 1, \quad \text{a.s. for } \ t \in [0,T] \text{ and } y > -1. \tag{3.1}$$

From theorem (3.2) we have

$$\begin{aligned} Z_t = \exp\Bigg\{ &\int_0^t \frac{-(\mu_s - r_s)\sqrt{V_s^{(1)}}}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \mathrm{d}W_s^1 \\ &- \int_0^t \frac{(\mu_s - r_s)\sqrt{V_s^{(2)}}}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \mathrm{d}W_s^3 \\ &- \frac{1}{2}\int_0^t \frac{(\mu_s - r_s)^2 V_s^{(1)}}{(V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y))^2} \mathrm{d}s \\ &- \frac{1}{2}\int_0^t \frac{(\mu_s - r_s)^2 V_s^{(2)}}{(V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y))^2} \mathrm{d}s \\ &- \int_0^t \int_{-1}^{\infty} \frac{(\mu_s - r_s)y}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} N(\mathrm{d}y, \mathrm{d}s) \\ &+ \int_0^t \int_{-1}^{\infty} \frac{(\mu_s - r_s)y}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} v(\mathrm{d}y)\mathrm{d}s \\ &- \frac{1}{2}\int_0^t \int_{-1}^{\infty} \frac{(\mu_s - r_s)^2 y^2}{(V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y))^2} v(\mathrm{d}y)\mathrm{d}s \Bigg\}. \end{aligned}$$

Hence

$$\mathbb{E}\exp\left\{ \int_0^t \frac{(\mu_s - r_s)^2}{V_s^{(1)} + V_s^{(2)}} \mathrm{d}s + \int_0^t \int_{-1}^{\infty} \frac{(\mu_s - r_s)^2 y^2}{(V_s^{(1)} + V_s^{(2)})^2} \mathrm{d}s \right\} < \infty, \tag{3.2}$$

for all $t \in [0,T]$. Now we will show that Z_t is a square-integrable martingale under P and the measure $\hat{P}$ defined by $\frac{\mathrm{d}\hat{P}}{\mathrm{d}p}|_{\mathcal{A}_t} = Z_t$ satisfies the definition of minimal martingale measure(see Definition 3.1).

Assume that there exists a minimal martingale measure, and let us denote it by P^*. Define Z_t by

$$Z_t = \mathbb{E}\left[\frac{\mathrm{d}P^*}{\mathrm{d}P}\middle|\mathcal{A}_t\right].$$

Under P^*, the Doob-Meyer decomposition of M is given by

$$M_t = \widetilde{S}_t - \widetilde{S}_0 + (-A_t).$$

But the theory of the Girsanov transformation shows that the predictable process of bounded variation can also be computed in terms of P^*

$$-A_t = \int_0^t \frac{1}{Z_{s_-}} \mathrm{d}\left\langle M, Z\right\rangle_s.$$

By Kunita-Watanabe decomposition, we have

$$Z_t = 1 + \int_0^t \beta_s \mathrm{d}M_s + L_t,$$

where L is a square-integrable martingale under P orthogonal to M, and $\beta = (\beta_t)_{0\le t\le T}$ is a predictable process with

$$E\left[\int_0^T \beta_s^2 \mathrm{d}\left\langle M\right\rangle\right] < \infty.$$

Since P^* is a minimal martingale measure, we can easily obtain that L is P^* martingale and that LZ is a P martingale. Then we have

$$\langle L, L\rangle = \langle L, Z\rangle = 0,$$

hence $L \equiv 0$, $Z_t = 1 + \int_0^t \beta_s \mathrm{d}M_s$, and $\mathrm{d}A_t = -\frac{\beta_t}{Z_{t_-}} \mathrm{d}\left\langle M, M\right\rangle$, so

$$Z_t = 1 - \int_0^t Z_{s_-} \frac{\mathrm{d}A_s}{\mathrm{d}\left\langle M\right\rangle_s} \mathrm{d}M_s.$$

Let $\mathrm{d}Y_s = -\frac{\mathrm{d}A_s}{\mathrm{d}\langle M\rangle_s}\mathrm{d}M_s$. From (2.1) and (2.2), we get

$$Y_t = \frac{-(\mu_t - r_t)\left(\sqrt{V_t^{(1)}}\mathrm{d}W_t^1 + \sqrt{V_t^{(2)}}\mathrm{d}W_t^3 + \int_{-1}^{\infty} y\widetilde{N}(dy,dt)\right)}{V_t^{(1)} + V_t^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)},$$

$$Z_t = 1 + \int_0^t Z_{s_-} \mathrm{d}Y_s. \tag{3.3}$$

Noting that there is a unique solution of (3.3), the minimal martingale measure is unique if it exists. We can get

$$Z_t = e^{Y_t^c - \frac{1}{2}\langle Y^c, Y^c\rangle} \prod_{u\le t} (1 + \varDelta Y_u),$$

from the formula of the Doleans-Dade exponential. Under conditions (3.1) and (3.2), Z is a square-integrable P martingale.

First, we can see that $\hat{P}$ is an equivalent martingale measure to P. Next, let L' be a P martingale and let it be orthogonal to M; that is, $\langle L', M\rangle = 0$.

$$\langle L', Z\rangle_t = \int_0^t Z_{S_-} \mathrm{d}\langle L', Y\rangle_s = -\int_0^t Z_{s_-} \frac{\mathrm{d}A_s}{\mathrm{d}\langle M\rangle_s} \mathrm{d}\langle L', M\rangle_s = 0.$$

By the Girsanov-Meyer theorem, L' is a $\hat{P}$-martingale. Hence, $\hat{P}$ is the unique minimal martingale measure of S.

From the Girsanov theorem we have

$$\widehat{W}_t^1 = W_t^1 + \int_0^t \frac{(\mu_s - r_s)\sqrt{V_s^{(1)}}}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \mathrm{d}s,$$

$$\widehat{W}_t^2 = W_t^2 + \rho_1 \int_0^t \frac{(\mu_s - r_s)\sqrt{V_s^{(1)}}}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \mathrm{d}s,$$

$$\widehat{W}_t^3 = W_t^3 + \int_0^t \frac{(\mu_s - r_s)\sqrt{V_s^{(2)}}}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \mathrm{d}s,$$

$$\widehat{W}_t^4 = W_t^4 + \rho_2 \int_0^t \frac{(\mu_s - r_s)\sqrt{V_s^{(2)}}}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \mathrm{d}s,$$

are standard $\hat{P}$-Brownian motions.

Remark 3.3. Given $\mathcal{G}_T$, under $\hat{P}$, the compensator of $N(\mathrm{d}y, \mathrm{d}t)$ is

$$\tilde{v}(\mathrm{d}y)\mathrm{d}u = \left(1 - \frac{(\mu_u - r_u)y}{V_u^{(1)} + V_u^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\right) v(\mathrm{d}y)\mathrm{d}u.$$

4. **Option pricing**

In this section, we derive the options pricing by Local risk minimization method. The price at time t of the European call option with strike price K and time to expiration T is given by

$$V(t, T) = E^{\hat{P}}[e^{-\int_t^T r_s \mathrm{d}s}(S_T - K)^+ \mid \mathcal{A}_t],$$

We set $V_t^{(1)} = \alpha_t$ and $V_t^{(2)} = \alpha_t'$, and let

$$C(t, S_t, \alpha_t, \alpha_t', X_t) = e^{-\int_0^t r_s \mathrm{d}s} V(t, S_t, \alpha_t, \alpha_t', X_t).$$

In the sequel, we apply Itô's formula for $C(t, S_t, \alpha_t, \alpha_t', X_t)$ and find its dynamics.

$$\mathrm{d}C(t, S_t, \alpha_t, \alpha_t', X_t) = -r_t e^{-\int_0^t r_s \mathrm{d}s} V(t, S_{t_-}, \alpha_{t_-}, \alpha_{t_-}', X_{t_-})\mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s} \frac{\partial V}{\partial t} \mathrm{d}t$$

$$
\begin{aligned}
&+ e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial S}\mathrm{d}S^c + e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial \alpha}\mathrm{d}\alpha^c + e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial \alpha'}\mathrm{d}\alpha'^c \\
&+ \frac{1}{2}e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial S^2}\mathrm{d}\left\langle S^c, S^c\right\rangle + \frac{1}{2}e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial \alpha^2}\mathrm{d}\left\langle \alpha^c, \alpha^c\right\rangle \\
&+ \frac{1}{2}e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial \alpha'^2}\mathrm{d}\left\langle \alpha'^c, \alpha'^c\right\rangle + e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial S\partial \alpha}\mathrm{d}\left\langle S^c, \alpha^c\right\rangle \\
&+ e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial S\partial \alpha'}\mathrm{d}\left\langle S^c, \alpha'^c\right\rangle + e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial \alpha\partial \alpha'}\mathrm{d}\left\langle \alpha^c, \alpha'^c\right\rangle \\
&+ e^{-\int_0^t r_s \mathrm{d}s}\sum_{u\le t}(V(u, S_u, \alpha_u, \alpha'_u, X_u) - V(u, S_{u_-}, \alpha_{u_-}, \alpha'_{u_-}, X_{u_-})) \\
= &-r_t e^{-\int_0^t r_s \mathrm{d}s}V(t, S_{t_-}, \alpha_{t_-}, \alpha'_{t_-}, X_{t_-})\mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial t}\mathrm{d}t \\
&+ e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial S}\left[\mu_t S_{t_-}\mathrm{d}t + \sqrt{\alpha_t}S_{t_-}\mathrm{d}\hat{W}_t^1 + \sqrt{\alpha'_t}S_{t_-}\mathrm{d}\hat{W}_t^3 - \int_{-1}^{\infty} S_{t_-} y v(\mathrm{d}y)\mathrm{d}t\right] \\
&+ e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial \alpha}\left[k_1(\theta_1 - \alpha_t)\mathrm{d}t + \sigma_{v_1}\sqrt{\alpha_t}\mathrm{d}\hat{W}_t^2\right] \\
&+ e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial \alpha'}\left[k_2(\theta_2 - \alpha'_t)\mathrm{d}t + \sigma_{v_2}\sqrt{\alpha'_t}\mathrm{d}\hat{W}_t^4\right] \\
&+ \frac{1}{2}e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial S^2}S_t^2\left(\alpha_t + \alpha'_t\right)\mathrm{d}t + \frac{1}{2}e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial \alpha_t^2}\sigma_{v_1}^2\alpha_t\mathrm{d}t \\
&+ \frac{1}{2}e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial {\alpha'_t}^2}\sigma_{v_2}^2\alpha'_t\mathrm{d}t + e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial S\partial \alpha_t}\rho_1\sigma_{v_1}\alpha_t S_t\mathrm{d}t \\
&+ e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial^2 V}{\partial S\partial \alpha'_t}\rho_2\sigma_{v_2}\alpha'_t S_t\mathrm{d}t - e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial S}S_{t_-}\frac{(\mu_t - r_t)\alpha_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\mathrm{d}t \\
&- e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial S}S_{t_-}\frac{(\mu_t - r_t)\alpha'_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\mathrm{d}t \\
&- e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial \alpha_t}\rho_1\sigma_{v_1}\frac{(\mu_t - r_t)\alpha_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\mathrm{d}t \\
&- e^{-\int_0^t r_s \mathrm{d}s}\frac{\partial V}{\partial \alpha'_t}\rho_2\sigma_{v_2}\frac{(\mu_t - r_t)\alpha'_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\mathrm{d}t \\
&+ \int_{-1}^{\infty} e^{-\int_0^t r_s \mathrm{d}s}(V(t, S_{t_-}(1+y), \alpha_t, \alpha'_t, X_{t_-}) \\
&- V(t, S_{t_-}, \alpha_{t_-}, \alpha'_{t_-}, X_{t_-}))\tilde{v}(\mathrm{d}y)\mathrm{d}t \\
&+ \int_{-1}^{\infty} e^{-\int_0^t r_s \mathrm{d}s}(V(t, S_{t_-}(1+y), \alpha_t, \alpha'_t, X_{t_-}) \\
&- V(t, S_{t_-}, \alpha_{t_-}, \alpha'_{t_-}, X_{t_-}))\hat{N}(\mathrm{d}y, \mathrm{d}t) \\
&+ \int_{\mathbb{R}} e^{-\int_0^t r_s \mathrm{d}s}(V(t, S_{t_-}, \alpha_t, \alpha'_t, X_{t_-} + h(X_{t_-}, z)) \\
&- V(t, S_{t_-}, \alpha_{t_-}, \alpha'_{t_-}, X_{t_-}))\tilde{P}(\mathrm{d}z, \mathrm{d}t)
\end{aligned}
$$

$$+\int_{\mathbb{R}} e^{-\int_0^t r_s \mathrm{d}s} \sum_j V(t, S_{t_-}, \alpha_{t_-}, \alpha'_{t_-}, j) q_{X_{u_-},j} \mathrm{d}t,$$

where $\tilde{P}(\mathrm{d}z, \mathrm{d}t) = P(\mathrm{d}y, \mathrm{d}t) - m(\mathrm{d}z)\mathrm{d}t$ is the compensated Poisson random measure and $\hat{N}(\mathrm{d}y, \mathrm{d}t) = N(\mathrm{d}y, \mathrm{d}t) - \tilde{v}(\mathrm{d}y)\mathrm{d}t$. Since $C(t, S_t, \alpha_t, \alpha'_t, X_t)$ is a $\hat{P}$ martingale, the drift term must be identical to zero. Hence, we have

$$\begin{aligned}
&-r_t V(t, S_{t_-}, \alpha_{t_-}, \alpha'_{t_-}, X_{t_-}) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} S_{t_-} \\
&\left(r_t - \frac{(\mu_t - r_t)\alpha_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} - \frac{(\mu_t - r_t)\alpha'_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} - \int_{-1}^{\infty} y v(\mathrm{d}y) \right) \\
&+ \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S_{t_-}^2 (\alpha_t + \alpha'_t) + \frac{\partial V}{\partial \alpha_t} \left(k_1(\theta_1 - \alpha_t) - \frac{(\mu_t - r_t)\alpha_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \rho_1 \sigma_{v_1} \right) \\
&+ \frac{1}{2} \frac{\partial^2 V}{\partial \alpha_t^2} \sigma_{v_1}^2 \alpha_t + \frac{\partial V}{\partial \alpha'_t} \left(k_2(\theta_2 - \alpha'_t) - \frac{(\mu_t - r_t)\alpha'_t}{\alpha_t + \alpha'_t + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)} \rho_2 \sigma_{v_2} \right) \\
&+ \frac{1}{2} \frac{\partial^2 V}{\partial {\alpha'_t}^2} \sigma_{v_2}^2 \alpha'_t + \frac{\partial^2 V}{\partial S \partial \alpha_t} \rho_1 \sigma_{v_1} \alpha_t S_{t_-} + \frac{\partial^2 V}{\partial S \partial \alpha'_t} \rho_2 \sigma_{v_2} \alpha'_t S_{t_-} \\
&+ \int_{-1}^{\infty} (v(t, S_{t_-}(1+y), \alpha_t, \alpha'_t, X_{t_-}) - v(t, S_{t_-}, \alpha_{t_-}, \alpha'_{t_-}, X_{t_-})) \tilde{v}(\mathrm{d}y) \\
&+ \sum_{j=1}^{N} V(t, S_{t_-}, \alpha_t, \alpha'_t, j) q_{X_{t_-},j} = 0,
\end{aligned}$$

with the terminal condition $V(T, S_T, \sigma_T, \sigma'_T, X_T) = (S_T - K)^+$.

5. **Locally risk-minimizing strategies**

In this section we obtain an optimal hedging strategy in terms of local risk minimization.

Let H be the contingent claim with $H \in L^2(\Omega, \mathcal{A}, P)$ at time T and $\varphi = (\theta, \alpha)$ be a portfolio, where $\theta = (\theta_t)_{0 \le t \le T}$ is the amount of risky asset and $\alpha = (\alpha_t)_{0 \le t \le T}$ the amount of risk less asset. The discounted portfolio valuation at time t is

$$V_t = \theta_t \widetilde{S}_t + \alpha_t.$$

Suppose α_t adapted process with $\mathbb{E}(\alpha^2) < \infty$, θ is predictable process and

$$\mathbb{E}\left[\int_0^t \theta_u^2 \mathrm{d} \langle M \rangle_u + \left(\int_0^t |\theta_u \mathrm{d} A_u| \right)^2 \right] < \infty. \tag{5.1}$$

Our market is incomplete, so we find an admissible portfolio φ which minimizes, at each time t, the residual risk, given by

$$R_t(\varphi) = \mathbb{E}\left[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{A}_t \right], \quad t \le T$$

over all admissible portfolio. $C_t(\varphi) = V_t(\theta) - \int_0^t \theta_s \mathrm{d}\tilde{S}_s$ is the discounted cost accumulated up to time t.

We have the following definitions from [16].

Definition 5.1 (Small Perturbation). A trading strategy $\Delta = (\delta, \varepsilon)$ is called a small perturbation if it satisfies the following conditions:

1. δ is bounded,
2. $\int_0^T |\delta_u \mathrm{d}A_u|$ is bounded,
3. $\delta T = \varepsilon T = 0$.

Definition 5.2 (Locally Risk Minimizing). For a trading strategy φ, a small perturbation Δ, and a partition τ of $[0, T]$, the risk quotient $r^\tau[\varphi, \Delta]$ is defined as

$$r^\tau[\varphi, \Delta] := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}(\varphi + \Delta \mid_{(t_i, t_{i+1})}) - R_{t_i}(\varphi)}{\mathbb{E}\left[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}\right]} I_{(t_i, t_{i+1}]}.$$

A trading strategy φ is called locally risk minimizing if

$$\liminf_{n \to \infty} r^\tau[\varphi, \Delta] \geq 0 \text{ P-a.e. on } \Omega \times [0, T],$$

for every small perturbation Δ and every increasing sequence (τ_n) of the partition of $[0, T]$ tending to the identity.

Definition 5.3 (Pseudo Locally Risk-Minimizing Hedging Strategy). A strategy is called pseudo locally risk minimizing, or equivalently pseudo optimal risk minimizing, if the associated cost process $C(\varphi)$ is a martingale under P and orthogonal to M_t.

Definition 5.4 (Föllmer-Schweizer Decomposition).

$$\widetilde{H} = \widetilde{H}_0 + \int_0^T \theta_s^H \mathrm{d}\widetilde{S}_s + L_T^H,$$

is the Föllmer-Schweizer Decomposition of the discounted contingent claim $\widetilde{H} = e^{\int_0^t r_s \mathrm{d}s} H$, if θ^H satisfies formula (5.1) and if L_t^H is a square-integrable P-martingale orthogonal to M_t, with $L_0^H = 0$. The associated optimal strategy given by $\varphi_t = (\theta^H, \widetilde{H}_0 + \int_0^t \theta_s^H \mathrm{d}\widetilde{S}_s + L_t^H - \theta_t^H \widetilde{S}_t)$ is locally risk minimizing.

We also need the following assumptions in [16]:

(1) For P-almost all ω, the measure on $[0, T]$ induced by $\langle M \rangle(\omega)$ has the whole interval $[0, T]$ as its support. This means that $\langle M \rangle$ should be P-almost surely strictly increasing on the whole interval $[0, T]$.
(2) A is continuous.
(3) A is absolutely continuous with respect to $\langle M \rangle$ with density α satisfying $\mathbb{E}\left[|\alpha \ln^+ |\alpha||\right] < \infty$. A sufficient condition is that $\mathbb{E}\left[\left\langle \int \alpha \mathrm{d}M \right\rangle\right] < \infty$.

By [16], the pseudo locally risk-minimizing hedging strategy is the locally risk-minimizing strategy if assumptions (1)–(3) are satisfied. So, for $\widetilde{S}_t$,we check conditions (1)–(3).

$$\begin{aligned}\langle M\rangle_t &= \left\langle \int_0^t \widetilde{S}_{u-}\sqrt{V_u^{(1)}}\mathrm{d}W_u^1 + \int_0^t \widetilde{S}_{u-}\sqrt{V_u^{(2)}}\mathrm{d}W_u^3 + \int_0^t\int_{-1}^{\infty}\widetilde{S}_{u-}y\widetilde{N}(\mathrm{d}y,\mathrm{d}u)\right\rangle \\ &= \int_0^t \widetilde{S}_{u-}^2 V_u^{(1)}\mathrm{d}u + \int_0^t \widetilde{S}_{u-}^2 V_u^{(2)}\mathrm{d}u + \int_0^t\int_{-1}^{\infty}\widetilde{S}_{u-}^2 y^2 v(\mathrm{d}y)\mathrm{d}u \\ &= \int_0^t \widetilde{S}_{u-}^2\left(V_u^{(1)} + V_u^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)\right)\mathrm{d}u.\end{aligned}$$

$\widetilde{S}_{u-}^2\left(V_u^{(1)} + V_u^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)\right)\mathrm{d}u > 0$, $\langle M\rangle_t$ is strictly increasing for every $t \in [0,T]$. Assumption (1) is verified. Note that

$$A_t = \int_0^t \widetilde{S}_{u-}(\mu_u - r_u)\mathrm{d}u,$$

is continuous, assumption (2) is satisfied. Also we have

$$\frac{\mathrm{d}A_s}{\mathrm{d}\langle M\rangle_s} = \frac{\mu_s - r_s}{\widetilde{S}_s(V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y))},$$

and

$$\begin{aligned}\mathbb{E}\left[\left\langle\int \frac{\mathrm{d}A_s}{\mathrm{d}\langle M\rangle_s}\mathrm{d}M_u\right\rangle\right] &= \mathbb{E}\left[\int \frac{(\mu_s - r_s)^2}{\widetilde{S}_s^2(V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y))^2}\mathrm{d}\langle M\rangle_u\right] \\ &= \mathbb{E}\left[\int \frac{(\mu_s - r_s)^2}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\mathrm{d}u\right].\end{aligned} \tag{5.2}$$

Since

$$\mathbb{E}\left[\int \frac{(\mu_s - r_s)^2}{V_s^{(1)} + V_s^{(2)} + \int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\mathrm{d}u\right] < \mathbb{E}\left[\exp\int \frac{(\mu_s - r_s)^2}{V_s^{(1)} + V_s^{(2)}}\right] < \infty,$$

then (5.2) is finite. So, assumption (3) is satisfied.

Now we derive the locally risk-minimizing strategy for the associated discounted portfolio. The Föllmer-Schweizer decomposition of the associated discounted portfolio is

$$V(\varphi) = V_0(\varphi) + \int_0^t \phi(s,u)\mathrm{d}\widetilde{S}_u + L_t, \tag{5.3}$$

So, we have

$$\begin{aligned}L_t &= V_t - V_0 - \int_0^t \phi(s,u)\mathrm{d}\widetilde{S}_u \\ &= \int_0^t e^{-\int_0^u r_s\mathrm{d}s}\frac{\partial V}{\partial S}S_u\sqrt{V_u^{(1)}}\mathrm{d}\hat{W}_u^1 + \int_0^t e^{-\int_0^u r_s\mathrm{d}s}\frac{\partial V}{\partial S}S_u\sqrt{V_u^{(2)}}\mathrm{d}\hat{W}_u^3 \\ &\quad + \int_0^t e^{-\int_0^u r_s\mathrm{d}s}\frac{\partial V}{\partial V^{(1)}}\left[\sigma_{v_1}\sqrt{V_u^{(1)}}\mathrm{d}\hat{W}_u^2\right] + \int_0^t e^{-\int_0^u r_s\mathrm{d}s}\frac{\partial V}{\partial V^{(2)}}\left[\sigma_{v_2}\sqrt{V_u^{(2)}}\mathrm{d}\hat{W}_u^4\right]\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{-1}^{\infty} e^{-\int_0^t r_s \mathrm{d}s}(V(u, S_{u_-}(1+y), V_u^{(1)}, V_u^{(2)}, X_{t_-})\\
&- V(u, s_{u_-}, V_{u_-}^{(1)}, V_{u_-}^{(2)}, X_{u_-}))\hat{N}(\mathrm{d}y, \mathrm{d}u)\\
&+\int_0^t\int_{\mathbb{R}} e^{-\int_0^t r_s \mathrm{d}s}(V(u, S_{u_-}, V_u^{(1)}, V_u^{(2)}, X_{u_-}+h(X_{u_-}, z))\\
&- V(u_-, S_{u_-}, V_{u_-}^{(1)}, V_{u_-}^{(2)}, X_{u_-}))\widetilde{P}(\mathrm{d}z, \mathrm{d}u)\\
&-\int_0^t \phi(s,u)\mathrm{d}\widetilde{S}_u.
\end{aligned}
$$

Since L_t is a P martingale, the integrands with respect to du on the right-hand side should vanish. This gives us the following equation:

$$
\begin{aligned}
&\frac{\partial V}{\partial S}\widetilde{S}_u\frac{(\mu_u - r_u)V_u^{(1)}}{V_u^{(1)}+V_u^{(2)}+\int_{-1}^{\infty} y^2 v(\mathrm{d}y)} + \frac{\partial V}{\partial S}\widetilde{S}_u\frac{(\mu_u - r_u)V_u^{(2)}}{V_u^{(1)}+V_u^{(2)}+\int_{-1}^{\infty} y^2 v(\mathrm{d}y)}\\
&\quad+\frac{\partial V}{\partial V^{(1)}}\rho_1\sigma_{v_1}\frac{(\mu_u - r_u)V_u^{(1)}}{V_u^{(1)}+V_u^{(2)}+\int_{-1}^{\infty} y^2 v(\mathrm{d}y)}e^{-\int_0^t r_s \mathrm{d}s}\\
&\quad+\frac{\partial V}{\partial V^{(2)}}\rho_2\sigma_{v_2}\frac{(\mu_u - r_u)V_u^{(2)}}{V_u^{(1)}+V_u^{(2)}+\int_{-1}^{\infty} y^2 v(\mathrm{d}y)}e^{-\int_0^t r_s \mathrm{d}s} - \phi(s,u)(\mu_t - r_t)\widetilde{S}_u\\
&\quad+\int_{-1}^{\infty} e^{-\int_0^t r_s \mathrm{d}s}(V(u, S_{u_-}(1+y), V_u^{(1)}, V_u^{(2)}, X_{u_-})\\
&\quad- V(u, S_{u_-}, V_{u_-}^{(1)}, V_{u_-}^{(2)}, X_{u_-}))(v-\tilde{v})(\mathrm{d}y)\mathrm{d}u = 0,
\end{aligned}
$$

a.s. for $u \in [0, T]$. We can derive

$$
\begin{aligned}
\phi(s,u) &= \frac{\frac{\partial V}{\partial S}\widetilde{S}_u\left(V_u^{(1)}+V_u^{(2)}\right) + \rho_1\sigma_{v_1}\frac{\partial V}{\partial V^{(1)}}e^{-\int_0^t r_s \mathrm{d}s}V_u^{(1)} + \rho_2\sigma_{v_2}\frac{\partial V}{\partial V^{(2)}}e^{-\int_0^t r_s \mathrm{d}s}V_u^{(2)}}{\widetilde{S}_u(V_u^{(1)}+V_u^{(2)}+\int_{-1}^{\infty} y^2 v(\mathrm{d}y))}\\
&+\frac{\int_{-1}^{\infty} e^{-\int_0^t r_s \mathrm{d}s}(V(u, S_{u_-}(1+y), V_u^{(1)}, V_u^{(2)}, X_{t_-}) - V(u, s_{u_-}, V_{u_-}^{(1)}, V_{u_-}^{(2)}, X_{u_-}))yv(\mathrm{d}y)}{\widetilde{S}_u(V_u^{(1)}+V_u^{(2)}+\int_{-1}^{\infty} y^2 v(\mathrm{d}y))},
\end{aligned}
$$

and $\alpha(s,u) = V(\varphi) - \phi(s,u)\widetilde{S}_u$.

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