TOPOLOGICAL STRUCTURE OF THE COINCIDENCE SET FOR
ABSTRACT CLASSES OF MAPS

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ABSTRACT. The existence of a coincidence point is discussed in an abstract setting. In addition we consider the case when the coincidence set contains a continuum intersecting a given set.

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1. INTRODUCTION

In this paper we present general results for homotopies \( H \) for which the maps \( H_t \) may be defined in different domains. Conditions are put not only to guarantee the existence of a coincidence point but also to guarantee that the coincidence set contains a continuum (i.e. a compact connected set) which intersects a given set. Our theory is based on the notion of \( \Phi \)-essentiality (see [1, 10] and the references therein). The results in this paper were motivated in part by results in [1, 3, 4, 8, 9, 11].

2. PRELIMINARIES

We recall some results from the literature. Let \( X \) be a completely regular topological space and \( V \) an open subset of \( X \).

We consider classes \( A \) and \( B \) of maps.

Definition 2.1. We say \( F \in MA(\overline{V}, X) \) (respectively \( F \in B(\overline{V}, X) \)) if \( F : \overline{V} \to 2^X \) and \( F \in A(\overline{V}, X) \) (respectively \( F \in B(\overline{V}, X) \)); here \( 2^X \) denotes the family of nonempty subsets of \( X \) and \( \overline{V} \) denotes the closure of \( V \) in \( X \).

Fix a \( \Psi \in B(\overline{V}, X) \).

Definition 2.2. We say \( F \in MA_{\partial V}(\overline{V}, X) \) if \( F \in MA(\overline{V}, X) \) with \( F(x) \cap \Psi(x) = \emptyset \) for \( x \in \partial V \); here \( \partial V \) denotes the boundary of \( V \) in \( X \).
Definition 2.3. Let $F \in MA_{\partial V}(\nabla, X)$. We say $F : \nabla \to 2^X$ is $\Psi$-essential in $MA_{\partial V}(\nabla, X)$ if for every map $J \in MA_{\partial V}(\nabla, X)$ with $J|_{\partial V} = F|_{\partial V}$ there exists $x \in V$ with $J(x) \cap \Psi(x) \neq \emptyset$.

The following result was established in [5, 10].

Theorem 2.4. Let $X$ be a completely regular (respectively normal) topological space, $V$ an open subset of $X$ and let $F \in MA_{\partial V}(\nabla, X)$ be $\Psi$-essential in $MA_{\partial V}(\nabla, X)$. Suppose there exists a map $H : \nabla \times [0, 1] \to 2^X$ with $H(\cdot, \eta(\cdot)) \in MA(\nabla, X)$ for any continuous function $\eta : \nabla \to [0, 1]$ with $\eta(\partial V) = 0$, $\Psi(x) \cap H_0(x) = \emptyset$ for any $x \in \partial V$ and $t \in [0, 1]$, $H_0 = F$ and $\{x \in \nabla : \Psi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}$ is compact (respectively closed). Then there exists $x \in V$ with $\Psi(x) \cap H_1(x) \neq \emptyset$; here $H_t(x) = H(x, t)$.

Next we present a homotopy result for $\Psi$-essential maps. To achieve this we need to change Definition 2.3 (see Definition 2.6 below).

Definition 2.5. Let $X$ be a completely regular (respectively normal) topological space, and $V$ an open subset of $X$. Let $F, G \in MA_{\partial V}(\nabla, X)$. We say $F \cong G$ in $MA_{\partial V}(\nabla, X)$ if there exists a map $H : \nabla \times [0, 1] \to 2^X$ with $H(\cdot, \eta(\cdot)) \in MA(\nabla, X)$ for any continuous function $\eta : \nabla \to [0, 1]$ with $\eta(\partial V) = 0$, $H_t(x) \cap \Psi(x) = \emptyset$ for any $x \in \partial V$ and $t \in [0, 1]$, $H_1 = F$, $H_0 = G$ and

$$\{x \in \nabla : \Psi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

is compact (respectively closed); here $H_t(x) = H(x, t)$.

The following conditions will be assumed in our next result:

(2.1) $\cong$ is an equivalence relation in $MA_{\partial V}(\nabla, X)$,

and for any map $\Lambda \in MA_{\partial V}(\nabla, X)$ we have

\[
\begin{align*}
\text{if there exists a map } J \in MA_{\partial V}(\nabla, X) \text{ with } J \cong \Lambda \\
in MA_{\partial V}(\nabla, X) \text{ and } J(x) \cap \Psi(x) = \emptyset \text{ for all } x \in \nabla
\end{align*}
\]

and if $H : \nabla \times [0, 1]$ is a map with $H(\cdot, \eta(\cdot)) \in MA(\nabla, X)$ for any continuous function $\eta : \nabla \to [0, 1]$ with $\eta(\partial V) = 0$,

\[
\begin{align*}
H_t(x) \cap \Psi(x) = \emptyset \text{ for any } x \in \partial V \text{ and } t \in [0, 1], H_1 = \Lambda, H_0 = J \\
\text{and } \{x \in \nabla : \Psi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}
\end{align*}
\]

is compact (respectively closed) and if $\mu : \nabla \to [0, 1]$ is a continuous map with $\mu(\partial V) = 0$, then

\[
\begin{align*}
\{x \in \nabla : \emptyset \neq \Psi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0, 1]\}
\end{align*}
\]

is closed.

Definition 2.6. Let $F \in MA_{\partial V}(\nabla, X)$. We say $F : \nabla \to 2^X$ is $\Psi$-essential in $MA_{\partial V}(\nabla, X)$ if for every map $J \in MA_{\partial V}(\nabla, X)$ with $J|_{\partial V} = F|_{\partial V}$ and $J \cong F$ in $MA_{\partial V}(\nabla, X)$ there exists $x \in V$ with $J(x) \cap \Psi(x) \neq \emptyset$. 
The following result was established in [10].

**Theorem 2.7.** Let $X$ be a completely regular (respectively normal) topological space, $V$ an open subset of $X$ and assume (2.1) and (2.2) hold. Suppose $F$ and $G$ are two maps in $MA_{\partial V}(\nabla, X)$ with $F \cong G$ in $MA_{\partial V}(\nabla, X)$. Then $F$ is $\Psi$-essential in $MA_{\partial V}(\nabla, X)$ if and only if $G$ is $\Psi$-essential in $MA_{\partial V}(\nabla, X)$.

**Remark 2.8.** Another homotopy result without conditions (2.1) and (2.2) can be found in [6]: Let $X$ be a completely regular (respectively normal) topological space, $V$ an open subset of $X$, $F \in MA_{\partial V}(\nabla, X)$ and let $G \in MA_{\partial V}(\nabla, X)$ be $\Psi$-essential in $MA_{\partial V}(\nabla, X)$ (Definition 2.3). For any map $R \in MA_{\partial V}(\nabla, X)$ with $R|_{\partial V} = F|_{\partial V}$ assume there exists a map $H^R : \nabla \times [0, 1] \to 2^X$ with $H^R(\cdot, \eta(\cdot)) \in MA(\nabla, X)$ for any continuous function $\eta : \nabla \to [0, 1]$ with $\eta(\nabla) = 0$, $\Psi(x) \cap H^R_t(x) = \emptyset$ for any $x \in \partial V$ and $t \in (0, 1)$ and \{ $x \in \nabla : \Psi(x) \cap H^R(x, t) \neq \emptyset$ for some $t \in [0, 1]$ \} is compact (respectively closed) and $H^R_0 = G$, $H^R_1 = R$; here $H^R_t(x) = H^R(x, t)$. Then $F$ is $\Psi$-essential in $MA_{\partial V}(\nabla, X)$. There is an analogue result also in [6] if $G \in MA_{\partial V}(\nabla, X)$ is $\Psi$-essential in $MA_{\partial V}(\nabla, X)$ (Definition 2.3) is changed to $G \in MA_{\partial V}(\nabla, X)$ is $\Psi$-essential in $MA_{\partial V}(\nabla, X)$ (Definition 2.6).

**Remark 2.9.** The ideas presented in this paper could be applied to other natural situations. Let $X$ be a Hausdorff topological vector space, $Y$ a topological vector space, and $V$ an open subset of $X$. Also let $L : dom L \subseteq X \to Y$ be a linear (not necessarily continuous) single valued map; here $dom L$ is a vector subspace of $X$. Finally $T : X \to Y$ will be a linear single valued map with $L + T : dom L \to Y$ a bijection; for convenience we say $T \in H_L(X, Y)$. We say $F \in MA(\nabla, Y; L, T)$ (respectively $F \in B(\nabla, Y; L, T)$) if $F : \nabla \to 2^Y$ and $(L + T)^{-1}(F + T) \in MA(\nabla, X)$ (respectively $(L + T)^{-1}(F + T) \in B(\nabla, X)$). Fix a $\Phi \in B(\nabla, Y; L, T)$. We say $F \in MA_{\partial V}(\nabla, Y; L, T)$ if $F \in MA(\nabla, Y; L, T)$ with $(L + T)^{-1}(F + T)(x) \cap (L + T)^{-1}(\Psi + T)(x) = \emptyset$ for $x \in \partial V$. Now $F \in MA_{\partial V}(\nabla, Y; L, T)$ is $(L, T)\Psi$-essential in $MA_{\partial V}(\nabla, Y; L, T)$ if for every map $J \in MA_{\partial V}(\nabla, Y; L, T)$ with $J|_{\partial V} = F|_{\partial V}$ there exists $x \in V$ with $(L + T)^{-1}(J + T)(x) \cap (L + T)^{-1}(\Psi + T)(x) \neq \emptyset$ (this is the analogue of Definition 2.3). There are analogues of Theorem 2.4, Theorem 2.7 and Remark 2.8 in this situation; see [5, 6, 10] where the results and proofs are presented. For example the analogue of Theorem 2.4 in this situation is: Let $X$ be a topological vector space (so automatically completely regular), $Y$ a topological vector space, $V$ an open subset of $X$, $L : dom L \subseteq X \to Y$ a linear single valued map and $T \in H_L(X, Y)$. Let $F \in MA_{\partial V}(\nabla, Y; L, T)$ be $(L, T)\Psi$-essential in $MA_{\partial V}(\nabla, Y; L, T)$. Suppose there exists a map $H : \nabla \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in MA(\nabla, X)$ for any continuous function $\eta : \nabla \to [0, 1]$ with $\eta(\nabla) = 0$, $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Psi + T)(x) = \emptyset$ for any $x \in \partial V$ and $t \in (0, 1]$, $H_0 = F$ (here $H_t(x) = H(x, t)$) and $D = \{ x \in \nabla : (L + T)^{-1}(\Psi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset$ for some $t \in [0, 1] \}$. is
compact. Then there exists \( x \in V \) with \((L+T)^{-1}(H_1+T)(x) \cap (L+T)^{-1}(\Psi+T)(x) \neq \emptyset\).

If \( X \) is a normal topological vector space then the assumption that \( D \) is compact, can be replaced by \( D \) is closed. It is easy to state and prove analogues of the results in Section 3 in this situation; we leave the details to the reader.

**Remark 2.10.** It is of interest also to note other general classes of maps in the literature. Consider classes \( A, B \) and \( D \) of maps. We say \( F \in D(\overline{V}, X) \) (respectively \( F \in B(\overline{V}, X) \)) if \( F : \overline{V} \to 2^X \) and \( F \in D(\overline{V}, X) \) (respectively \( F \in B(\overline{V}, X) \)). We say \( F \in A(\overline{V}, X) \) if \( F : \overline{V} \to 2^X \) and \( F \in A(\overline{V}, X) \) and there exists a selection \( \theta \in D(\overline{V}, X) \) of \( F \). There are analogues of Theorem 2.4 and Remark 2.8 for these maps; see for example [7].

Recall a compact connected set is called a continuum. Whyburn’s lemma from topology can be stated as follows.

**Theorem 2.11.** Let \( A \) and \( B \) be disjoint closed subsets of a compact Hausdorff topological space \( K \) such that no connected component of \( K \) intersects both \( A \) and \( B \). Then there exists a partition \( K = K_1 \cup K_2 \) where \( K_1 \) and \( K_2 \) are disjoint compact sets containing \( A \) and \( B \) respectively.

An easy consequence of Theorem 2.11 is the following (see [3]).

**Theorem 2.12.** Let \( X \) be a metric space and \( K \) a compact subset of \( X \). Assume that \( A \) and \( B \) are two disjoint closed subsets of \( K \) such that no connected component of \( K \) intersects both. Then there exists an open bounded set \( U \) such that

\[
A \subset U, \quad \overline{U} \cap B = \emptyset \quad \text{and} \quad \partial U \cap K = \emptyset.
\]

### 3. MAIN RESULTS

In many applications results are needed for homotopies \( H \) for which the maps \( H_t \) may be defined on different domains. The idea is to reduce the study of this family to that of a new family (of course depending on the old one) defined on the same domain. For notational purposes let \( Z \) be a topological space and \( \Omega \) a subset of \( Z \times [0,1] \). We write \( \Omega_\lambda = \{ x \in Z : (x, \lambda) \in \Omega \} \) to denote the section of \( \Omega \) at \( \lambda \).

In our next results we assume \( E \) is a completely regular topological space and \( U \) an open subset of \( E \times [0,1] \) (note \( E \times [0,1] \) is a completely regular topological space). We begin by presenting some results which guarantee the existence of a coincidence point.

**Theorem 3.1.** Let \( E \) be a completely regular topological space and \( U \) an open subset of \( E \times [0,1] \). Suppose \( N \in MA(\overline{U}, E) \) and fix \( \Phi : \overline{U} \to 2^E \) with \( \Phi^* \in B(\overline{U}, E \times [0,1]) \); here \( \Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda) \) for \( (x, \lambda) \in \overline{U} \). Let \( H : \overline{U} \times [0,1] \to 2^{E \times [0,1]} \) be given
by $H(x, \lambda, \mu) = (N(x, \lambda), \mu)$ for $(x, \lambda) \in \overline{U}$ and $\mu \in [0, 1]$ and assume $H(\cdot, \cdot, \eta(\cdot)) \in MA(U, E \times [0, 1])$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$. Also suppose the following conditions are satisfied:

\begin{equation}
D = \left\{ (x, \lambda) \in \overline{U} : \Phi^*(x, \lambda) \cap H(x, \lambda, \mu) \neq \emptyset \text{ for some } \mu \in [0, 1] \right\}
\end{equation}

\begin{equation}
\text{is compact}
\end{equation}

\begin{equation}
H_0 \text{ is } \Phi^*-\text{essential in } MA_{OU}(\overline{U}, E \times [0, 1]) \text{ (Definition 2.3); here } H_0(x, \lambda) = H(x, \lambda, 0) = (N(x, \lambda), 0) \text{ for } (x, \lambda) \in \overline{U}
\end{equation}

and

\begin{equation}
\Phi(x, \lambda) \cap N(x, \lambda) = \emptyset \text{ for } (x, \lambda) \in \partial U.
\end{equation}

Then there exists $x \in U_1 = \{ y \in E : (y, 1) \in U \}$ with $\Phi(x, 1) \cap N(x, 1) \neq \emptyset$.

**Proof.** Suppose there exists $(x_0, \lambda_0) \in \partial U$ and $\mu_0 \in [0, 1]$ with $\Phi^*(x_0, \lambda_0) \cap H(x_0, \lambda_0, \mu_0) \neq \emptyset$ i.e. $(\Phi(x_0, \lambda_0), \lambda_0) \cap (N(x_0, \lambda_0), \mu_0) \neq \emptyset$. Then $\mu_0 = \lambda_0$ and $\Phi(x_0, \lambda_0) \cap N(x_0, \lambda_0) \neq \emptyset$, which contradicts (3.3). Thus

\[ \Phi^*(x, \lambda) \cap H(x, \lambda, \mu) = \emptyset \text{ for } (x, \lambda) \in \partial U \text{ and } \mu \in [0, 1]. \]

Now Theorem 2.4 (with $X = E \times [0, 1]$, $V = U$ and $\Psi = \Phi^*$) guarantees that there exists $(x, \lambda) \in U$ with $\Phi^*(x, \lambda) \cap H(x, \lambda, 1) \neq \emptyset$ i.e. $\Phi(x, \lambda) \cap (N(x, \lambda), 1) \neq \emptyset$ i.e. $\Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset$ and $\lambda = 1$ i.e. $x \in U_1$ and $\Phi(x, 1) \cap N(x, 1) \neq \emptyset$. \qed

**Remark 3.2.** If $E \times [0, 1]$ is a normal topological space then (3.1) can be changed to: $D$ is closed.

**Theorem 3.3.** Let $E$ be a completely regular topological space and $U$ an open subset of $E \times [0, 1]$. Suppose $N \in MA(\overline{U}, E)$ and fix $\Phi : \overline{U} \to 2^E$ with $\Phi^* \in B(\overline{U}, E \times [0, 1])$; here $\Phi^*(x, \lambda) = (\Phi(x), \lambda)$ for $(x, \lambda) \in \overline{U}$. Also suppose (2.1) and (2.2) hold with $X = E \times [0, 1]$, $V = U$ and $\Psi = \Phi^*$. Let $H : \overline{U} \times [0, 1] \to 2^{E \times [0, 1]}$ be given by $H(x, \lambda, \mu) = (N(x, \lambda), \mu)$ for $(x, \lambda) \in \overline{U}$ and $\mu \in [0, 1]$ and assume $H(\cdot, \cdot, \eta(\cdot)) \in MA(\overline{U}, E \times [0, 1])$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$. In addition assume (3.1), (3.2) (with Definition 2.6) and (3.3) hold. Then $H_1$ is $\Phi$-essential in $MA_{OU}(\overline{U}, E \times [0, 1])$ (in particular there exists $x \in U_1$ with $\Phi(x, 1) \cap N(x, 1) \neq \emptyset$); here $H_1(x, \lambda) = H(x, \lambda, 1) = (N(x, 1), 1)$ for $(x, \lambda) \in \overline{U}$.

**Proof.** As in Theorem 3.1 note

\[ \Phi^*(x, \lambda) \cap H(x, \lambda, \mu) = \emptyset \text{ for } (x, \lambda) \in \partial U \text{ and } \mu \in [0, 1]. \]

Also the conditions in the statement of Theorem 3.3 guarantees that $H_0 \cong H_1$ in $MA_{OU}(\overline{U}, E \times [0, 1])$. Theorem 2.7 guarantees that $H_1$ is $\Phi$-essential in $MA_{OU}(\overline{U}, E \times [0, 1])$. \qed
Remark 3.4. If \( E \times [0, 1] \) is a normal topological space then (3.1) can be changed to: \( D \) is closed.

Remark 3.5. We now consider the situation in Remark 2.9. Let \( E \) be a Hausdorff topological vector space, \( Y \) a topological vector space, and \( U \) an open subset of \( E \times [0, 1] \). Also let \( L : dom L \subseteq E \to Y \) be a linear (not necessarily continuous) single valued map. Now let \( L : dom L = dom L \times [0, 1] \to Y \times [0, 1] \) be given by \( L(y, \lambda) = (Ly, \lambda) \). Let \( T : E \to Y \) be a linear single valued map with \( L + T : dom L \to Y \) a bijection and let \( T : E \times [0, 1] \to Y \times [0, 1] \) be given by \( T(y, \lambda) = (Ty, 0) \). Notice \((L + T)^{-1}(y, \lambda) = ((L + T)^{-1}y, \lambda) \) for \((y, \lambda) \in Y \times [0, 1] \). There are analogues of Theorem 3.1 and Theorem 3.3 in this situation. For example the analogue of Theorem 3.1 is: Suppose \( N \in MA(U, Y; L, T) \) and fix \( \Phi : U \to 2^Y \) with \( \Phi^* \in B(U, Y \times [0, 1]; L, T) \); here \( \Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda) \) for \((x, \lambda) \in U \). Let \( H : U \times [0, 1] \to 2^{Y \times [0, 1]} \) be a map given by \( H(x, \lambda, \mu) = (N(x, \lambda), \mu) \) for \((x, \lambda) \in U \) and \( \mu \in [0, 1] \) and assume \((L + T)^{-1}(H(\cdot, \cdot, \eta(\cdot)) + T) \in MA(U, E \times [0, 1]) \) for any continuous function \( \eta : U \to [0, 1] \) with \( \eta(\partial U) = 0 \). Also suppose the following conditions are satisfied:

\[
\begin{align*}
D &= \{(x, \lambda) \in U : (L + T)^{-1}(\Phi^* + T)(x, \lambda) \cap (L + T)^{-1}(H_\mu + T)(x, \lambda) \neq \emptyset \text{ for some } \mu \in [0, 1]\} \\
&\text{is compact}
\end{align*}
\]  

(3.4)

(3.5)

\[
\begin{align*}
H_0 \text{ is } (L, T)\Phi^*-essential \text{ in } MA_{\Phi^*}(U, Y \times [0, 1]; L, T); \text{ here} \\
H_0(x, \lambda) = H(x, \lambda, 0) = (N(x, \lambda), 0) \text{ for } (x, \lambda) \in U
\end{align*}
\]

and

(3.6)

\[
\begin{align*}
(L + T)^{-1}(\Phi + T)(x, \lambda) \cap (L + T)^{-1}(N + T)(x, \lambda) = \emptyset \\
\text{for } (x, \lambda) \in \partial U; \text{ here } (N + T)(x, \lambda) = N(x, \lambda) + T(x).
\end{align*}
\]

Then there exists \( x \in U_1 = \{y \in E : (y, 1) \in U \} \) with \((L + T)^{-1}(\Phi + T)(x, 1) \cap (L + T)^{-1}(N + T)(x, 1) \neq \emptyset \). If \( E \times [0, 1] \) is a normal topological vector space then \( D \) compact above can be changed to \( D \) closed.

Next we discuss the topological structure of the coincidence set.

Theorem 3.6. Let \( E \) be a completely regular topological space and \( U \) an open subset of \( E \times [0, 1] \). Suppose \( N \in MA(U, E) \) and fix \( \Phi : U \to 2^E \) with \( \Phi^* \in B(U, E \times [0, 1]) \); here \( \Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda) \) for \((x, \lambda) \in U \). Let \( H : U \times [0, 1] \to 2^{E \times [0, 1]} \) be given by \( H(x, \lambda, \mu) = (N(x, \lambda), \mu) \) for \((x, \lambda) \in U \) and \( \mu \in [0, 1] \) and assume (3.2) (with Definition 2.3) and (3.3) hold. For any continuous map \( \mu : U \to [0, 1] \) assume \( \Lambda \in MA(U, E \times [0, 1]) \) where

\[
\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \text{ for } (x, \lambda) \in U.
\]
Also suppose

\[(3.7) \quad \Omega = \{(x, \lambda) \in \overline{U} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\}\]

is compact and \(\Omega_1 \neq \emptyset\);

here \(\Omega_t = \{x \in E : (x,t) \in \Omega\}\) for \(t \in [0,1]\). Then \(\Omega\) contains a continuum intersecting \(\Omega_0 \times \{0\}\) and \(\Omega_1 \times \{1\}\).

**Proof.** Note \(A = \Omega_0 \times \{0\} \subseteq \Omega\) and \(B = \Omega_1 \times \{1\} \subseteq \Omega\) are closed and compact. If there is no continuum intersecting \(A\) and \(B\) then from Theorem 2.11, \(\Omega\) can be represented as \(\Omega = \Omega^* \cup \Omega^{**}\) where \(\Omega^*\) and \(\Omega^{**}\) are disjoint compact sets with \(A \subseteq \Omega^*\) and \(B \subseteq \Omega^{**}\). Notice \(\Omega^*\) and \(\Omega^{**} \cup \partial U\) are closed and disjoint (note \(\Omega^* \cap \partial U = \emptyset\) since if there exists a \((x, \lambda) \in \Omega^*\) then \((\Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\) which contradicts (3.3)). Now there exists a continuous map \(\mu : \overline{U} \to [0,1]\) with \(\mu(\Omega^{**} \cup \partial U) = 0\) and \(\mu(\Omega^*) = 1\). Let

\[\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \quad \text{for} \quad (x, \lambda) \in \overline{U}.\]

From the statement of Theorem 3.6 note \(\Lambda \in MA(\overline{U}, E \times [0,1])\) and in fact \(\Lambda \in MA_{\partial U}(\overline{U}, E \times [0,1])\) since if there exists a \((x, \lambda) \in \partial U\) with \(\Phi^*(x, \lambda) \cap \Lambda(x, \lambda) \neq \emptyset\) then \((\Phi(x, \lambda), \lambda) \cap (N(x, \lambda), \mu(x, \lambda)) \neq \emptyset\) i.e. \((\Phi(x, \lambda), \lambda) \cap (N(x, \lambda), 0) \neq \emptyset\) i.e. \(\Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\) with \(\lambda = 0\), and this contradicts (3.3). Note \(H_0(x, \lambda) = H(x, \lambda, 0) = (N(x, \lambda), 0)\) so

\[\Lambda|_{\partial U} = H_0|_{\partial U}\]

since if \((x, \lambda) \in \partial U\) then \(\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) = (N(x, \lambda), 0)\) because \(\mu(\Omega^{**} \cup \partial U) = 0\). Now (3.2) guarantees that there exists a \((x, \lambda) \in U\) with \(\Phi^*(x, \lambda) \cap \Lambda(x, \lambda) \neq \emptyset\) i.e. \((\Phi(x, \lambda), \lambda) \cap (N(x, \lambda), \mu(x, \lambda)) \neq \emptyset\) i.e. \(\Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\) and \(\lambda = \mu(x, \lambda)\). Note \((x, \lambda) \in \Omega\) since \((x, \lambda) \in U\) and \(\Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\). Now either \((x, \lambda) \in \Omega^*\) or \((x, \lambda) \in \Omega^{**}\). Suppose \((x, \lambda) \in \Omega^*\). Then \(\mu(x, \lambda) = 1\) so \(\lambda = \mu(x, \lambda) = 1\) and \(\Phi(x, 1) \cap N(x, 1) \neq \emptyset\) i.e. \((x, 1) \in B \subseteq \Omega^{**}\) which contradicts \((x, 1) = (x, \lambda) \in \Omega^*\). Next suppose \((x, \lambda) \in \Omega^{**}\). Then \(\mu(x, \lambda) = 0\) so \(\lambda = \mu(x, \lambda) = 0\) and \(\Phi(x, 0) \cap N(x, 0) \neq \emptyset\) i.e. \((x, 0) \in A \subseteq \Omega^*\) which contradicts \((x, 0) = (x, \lambda) \in \Omega^{**}\). \(\square\)

**Remark 3.7.** We now consider the situation in Remark 2.9 (and Remark 3.5) and the corresponding result is: Suppose \(N \in A(\overline{U}, Y; L, T)\) and fix \(\Phi : \overline{U} \to 2^Y\) with \(\Phi^* \in B(\overline{U}, Y \times [0,1]; L, T)\); here \(\Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda)\) for \((x, \lambda) \in \overline{U}\). Let \(H : \overline{U} \times [0,1] \to 2^{Y \times [0,1]}\) be given by \(H(x, \lambda, \mu) = (N(x, \lambda), \mu)\) for \((x, \lambda) \in \overline{U}\) and \(\mu \in [0,1]\) and assume (3.5) and (3.6) hold. For any continuous map \(\mu : \overline{U} \to [0,1]\) assume \(\Lambda \in MA(\overline{U}, Y \times [0,1]; L, T)\) where

\[\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \quad \text{for} \quad (x, \lambda) \in \overline{U}.\]
Also suppose
\[
\begin{align*}
\Omega &= \{(x, \lambda) \in U : (L + T)^{-1}(\Phi + T)(x, \lambda) \cap (L + T)^{-1}(N + T)(x, \lambda) \neq \emptyset \} \text{ is compact} \\
\text{and } \Omega_1 &\neq \emptyset.
\end{align*}
\]

Then \( \Omega \) contains a continuum intersecting \( \Omega_0 \times \{0\} \) and \( \Omega_1 \times \{1\} \).

In our next result (3.3) is not assumed.

**Theorem 3.8.** Let \( E \) be a metric space and \( U \) an open subset of \( E \times [0, 1] \). Suppose \( N \in MA(U, E) \) and fix \( \Phi : U \to 2^E \) with \( \Phi^* \in B(U, E \times [0, 1]) \); here \( \Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda) \) for \( (x, \lambda) \in \overline{U} \). Assume
\[
\Phi(x, 0) \cap N(x, 0) = \emptyset \quad \text{for} \ (x, 0) \in \partial U.
\]

Let \( H : \overline{U} \times [0, 1] \to 2^{E \times [0, 1]} \) be given by \( H(x, \lambda, \mu) = (N(x, \lambda), \mu) \) for \( (x, \lambda) \in \overline{U} \) and \( \mu \in [0, 1] \) and assume (3.2) (with Definition 2.3) and (3.7) hold. For any continuous map \( \mu : \overline{U} \to [0, 1] \) assume \( \Lambda \in MA(U, E \times [0, 1]) \) where
\[
\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \quad \text{for} \ (x, \lambda) \in \overline{U}.
\]

In addition for open bounded subsets \( W \) of \( E \) with \( \Omega_0 \times \{0\} \subseteq W \subseteq U \) (so \( \Phi(x, 0) \cap N(x, 0) = \emptyset \) for \( (x, 0) \in U \setminus W \), \( \partial W \cap \Omega = \emptyset \) and \( \overline{W} \cap (\partial U \cap \Omega) = \emptyset \) assume \( N \in MA(W, E) \) and the following conditions holds:
\[
H_0 \text{ is } \Phi^*\text{-essential in } MA_{\partial W}(\overline{W}, E \times [0, 1])
\]
\[
\begin{align*}
\Lambda \in MA(\overline{W}, E \times [0, 1]) \text{ where } \Lambda(x, \lambda) &= (N(x, \lambda), \mu(x, \lambda)) \\
\text{for } (x, \lambda) \in \overline{W}
\end{align*}
\]

and
\[
\Sigma \text{ is closed and } \Sigma_1 \neq \emptyset;
\]

here \( \Sigma = \{(x, \lambda) \in \overline{W} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset \} \) and \( \Sigma_t = \{x \in E : (x, t) \in \Sigma \} \) for \( t \in [0, 1] \). Then \( \Omega \) contains a continuum intersecting \( \Omega_0 \times \{0\} \) and \( (\partial U \cap \Omega) \cup (\Omega_1 \times \{1\}) \).

**Proof.** There are two cases to consider, namely \( \Omega \cap \partial U = \emptyset \) or \( \Omega \cap \partial U \neq \emptyset \). If \( \Omega \cap \partial U = \emptyset \) then (3.3) holds so the result follows from Theorem 3.6. Now suppose \( \Omega \cap \partial U \neq \emptyset \). Let \( A = \Omega_0 \times \{0\} \), \( B = \Omega_1 \times \{1\} \) and \( C = \Omega \cap \partial U (\neq \emptyset) \). Notice \( C \subseteq \Omega \) is closed and (3.9) guarantees that \( C \cap A = \emptyset \). Now from Theorem 2.11 either

(1). there exists a continuum of \( \Omega \) which intersects \( A \) and \( C \) (and we are finished), or

(2). \( \Omega = \Omega^* \cup \Omega^{**} \) where \( \Omega^* \) and \( \Omega^{**} \) are disjoint compact sets with \( A \subseteq \Omega^* \) and \( C \subseteq \Omega^{**} \).
Suppose (2) occurs. From Theorem 2.12 there exists an open bounded set $V$ with
\begin{equation}
\Omega^* \subseteq V, \quad \overline{V} \cap \Omega^{**} = \emptyset \quad \text{and} \quad \partial V \cap \Omega = \emptyset.
\end{equation}
Let $W = U \cap V$. We now show
\begin{equation}
A \subseteq W \subseteq U, \quad \partial W \cap \Omega = \emptyset \quad \text{and} \quad \overline{W} \cap (\partial U \cap \Omega) = \emptyset.
\end{equation}
Note $A \subseteq W$ since $A \subseteq \Omega^* \subseteq V$ and $A \subseteq U$ from (3.9). Next notice that
\begin{equation}
\partial W = (\overline{U \cap V}) \setminus (U \cap V) \subseteq (U \setminus V) \setminus (U \cap V)
= ((U \setminus U) \cap \overline{V}) \cup ((V \setminus V) \cap \overline{U})
= (\partial U \cap \overline{V}) \cup (\partial V \cap \overline{U}) \subseteq (\partial U \cap V) \cup \partial V,
\end{equation}
and note $\partial V \cap \Omega = \emptyset$ (see (3.14)) and $(\partial U \cap \overline{V}) \cap \Omega = \emptyset$ (from (3.14) we have $\overline{V} \cap \Omega^{**} = \emptyset$ and note $C = \Omega \cap \partial U \subseteq \Omega^{**}$ so we have $\overline{V} \cap \Omega \cap \partial U = \emptyset$) and so $\partial W \cap \Omega = \emptyset$. Finally notice $\overline{W} \cap \Omega^{**} = \emptyset$ since $\overline{W} \subseteq \overline{U \cap V} \subseteq \overline{V}$ and $\overline{V} \cap \Omega^{**} = \emptyset$ from (3.9), so $\overline{W} \cap \Omega^{**} = \emptyset$ and $C \subseteq \Omega^{**}$ implies $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$. Thus (3.15) holds.

Let
\begin{equation}
\Sigma = \{(x, \lambda) \in \overline{W} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\}.
\end{equation}
Note $\partial W \cap \Sigma = \emptyset$ (see (3.15) and note $\Sigma \subseteq \Omega$). Now Theorem 3.6 (note $\Sigma$ is compact) implies that $\Sigma$ contains a continuum intersecting $\Sigma_0 \times \{0\} \subseteq \Omega_0 \times \{0\}$ and $\Sigma_1 \times \{1\}$ ($\subseteq \Omega_1 \times \{1\}$) and our result follows. \hfill \square

**Remark 3.9.** We now consider the situation in Remark 2.9 (and Remarks 3.5 and 3.7) and the corresponding result is: Let $E$ be a metric space and $U$ an open subset of $E \times [0, 1]$. Suppose $N \in MA(\overline{U}, Y; L, T)$ and fix $\Phi : \overline{U} \to 2^Y$ with $\Phi^* \in B(\overline{U}, Y \times [0, 1]; L, T)$; here $\Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda)$ for $(x, \lambda) \in \overline{U}$. Assume
\begin{equation}
(L + T)^{-1}(\Phi + T)(x, 0) \cap (L + T)^{-1}(N + T)(x, 0) = \emptyset \quad \text{for} \quad (x, 0) \in \partial U.
\end{equation}
Let $H : \overline{U} \times [0, 1] \to 2^{Y \times [0, 1]}$ be given by $H(x, \lambda, \mu) = (N(x, \lambda), \mu)$ for $(x, \lambda) \in \overline{U}$ and $\mu \in [0, 1]$ and assume (3.5) and (3.8) hold. For any continuous map $\mu : \overline{U} \to [0, 1]$ assume $\Lambda \in MA(\overline{U}, Y \times [0, 1]; L, T)$ where
\begin{equation}
\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \quad \text{for} \quad (x, \lambda) \in \overline{U}.
\end{equation}
In addition for open bounded subsets $W$ of $U$ with $\Omega_0 \times \{0\} \subseteq W \subseteq U$, $\partial W \cap \Omega = \emptyset$, and $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$ assume $N \in MA(\overline{W}, Y; L, T)$ and the following conditions hold:
\begin{equation}
H_0 \text{ is } (L, T)\Phi^*-essential in } MA_{\partial W}(\overline{W}, Y \times [0, 1]; L, T)
\end{equation}
\begin{equation}
\begin{cases}
\text{for any continuous map } \mu : \overline{W} \to [0, 1] \text{ assume} \\
\quad \Lambda \in MA(\overline{W}, Y \times [0, 1]; L, T) \text{ where} \\
\quad \Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \text{ for } (x, \lambda) \in \overline{W}
\end{cases}
\end{equation}
and

\(\Sigma \) is closed and \(\Sigma_1 \neq \emptyset\);

here \(\Sigma = \{ (x, \lambda) \in \overline{W} : (L + T)^{-1}(\Phi + T)(x, \lambda) \cap (L + T)^{-1}(N + T)(x, \lambda) \neq \emptyset \}\). Then \(\Omega\) contains a continuum intersecting \(\Omega_0 \times \{ 0 \}\) and \((\partial U \cap \Omega) \cup (\Omega_1 \times \{ 1 \})\).

In our next result \(\{(x, \lambda) \in \overline{U} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\}\) is compact is not assumed. For convenience we assume \(E\) is a normed space (the proof when \(E\) is a metric space is similar).

**Theorem 3.10.** Let \(E\) be a normed space and \(U\) an open subset of \(E \times [0, 1]\). Suppose \(N \in MA(U, E)\) and fix \(\Phi : \overline{U} \to 2^E\) with \(\Phi^* \in B(\overline{U}, E \times [0, 1])\); here \(\Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda)\) for \((x, \lambda) \in \overline{U}\). Assume (3.9) and the following condition holds:

\[(3.20) \quad \Omega_0 \text{ is nonempty and compact;}
\]

here \(\Omega_0 = \{ x \in E : (x, 0) \in \Omega \}\) where \(\Omega = \{ (x, \lambda) \in \overline{U} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset \}\). Let \(H : \overline{U} \times [0, 1] \to 2^{E \times [0, 1]}\) be given by \(H(x, \lambda, \mu) = (N(x, \lambda), \mu)\) for \((x, \lambda) \in \overline{U}\) and \(\mu \in [0, 1]\). In addition for open bounded subsets \(W\) of \(U\) with \(\Omega_0 \times \{ 0 \} \subseteq W \subseteq U\) (so \(\Phi(x, 0) \cap N(x, 0) = \emptyset\) for \((x, 0) \in U \setminus W\)) assume \(N \in MA(W, E)\) and the following conditions hold:

\[(3.21) \quad H_0 \text{ is } \Phi^*\text{-essential in } MA_\partial W(\overline{W}, E \times [0, 1])
\]

\[(3.22) \quad \text{for any continuous map } \mu : \overline{W} \to [0, 1] \text{ assume}
\]

\[(3.23) \quad \lambda \in MA(\overline{W}, E \times [0, 1]) \text{ where } \Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda))
\]

and

\[(3.24) \quad \Sigma = \{ (x, \lambda) \in \overline{W} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset \}
\]

is compact and \(\Sigma_1 \neq \emptyset\).

Then \(\Omega\) contains a connected component intersecting \(\Omega_0 \times \{ 0 \}\) and which either intersects \((\partial U \cap \Omega) \cup (\Omega_1 \times \{ 1 \})\) or is unbounded.

**Proof.** Since \(\Omega_0\) is compact there exists \(n_0 \in \mathbb{N}\) with \(\Omega_0 \subseteq B(0, n_0)\). For \(n \geq n_0\) let

\[U^n = U \cap (B(0, n) \times [0, 1])\quad \text{and} \quad \Omega^n = \{ (x, \lambda) \in \overline{U^n} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset \}\]

Note (3.9) implies \(\Omega_0 \times \{ 0 \} \subseteq U\) and as a result \(\Omega_0 \times \{ 0 \} \subseteq U^n\). Also note if there exists \((x, 0) \in U \setminus U^n\) with \(\Phi(x, 0) \cap N(x, 0) \neq \emptyset\) then \((x, 0) \in \Omega_0 \times \{ 0 \} \subseteq U^n\), a contradiction. Thus

\[\Phi(x, 0) \cap N(x, 0) = \emptyset \quad \text{for } (x, 0) \in U \setminus U^n.
\]

For each \(n \geq n_0\), Theorem 3.8 (with \(U^n\) replacing \(U\) and note (3.7) holds with \(U^n\) replacing \(U\) (see (3.24) with \(W = U^n\))) guarantees that there exists \((x_n, 0) \in\)
\( \Omega_0 \times \{0\} \text{ and a connected component } C_n \text{ of } \Omega^n \text{ containing } (x_n, 0) \text{ and intersecting } (\partial U^n \cap \Omega^n) \cup (\Omega^n_i \times \{1\}) \text{ (here } \Omega^n_i = \{x \in E : (x, 1) \in \Omega^n\}). \) Since \( \Omega_0 \) is compact the sequence \( (x_n) \) has an accumulation point \( x_0 \in \Omega_0 \). Assume that there is NO connected component of \( \Omega \) intersecting \( \Omega_0 \times \{0\} \) and \( (\partial U \cap \Omega) \cup (\Omega_1 \times \{1\}) \). Let \( C_0 \) be the connected component containing \( x_0 \) (and not intersecting \( (\partial U \cap \Omega) \cup (\Omega_1 \times \{1\}) \)).

Our result follows if we show \( C_0 \) is unbounded. Assume \( C_0 \) is bounded. Note \( C_0 \subseteq \Omega \) and \( C_0 \cap \partial U = \emptyset \) (since \( C_0 \) does not intersect \( (\partial U \cap \Omega) \cup (\Omega_1 \times \{1\}) \)) so \( C_0 \subseteq U \); and note \( C_0, \Omega_0 \times \{0\} \) are closed and bounded and as a result we can choose an open bounded set \( V \) with

\[
C_0 \cup (\Omega_0 \times \{0\}) \subseteq V \subseteq U.
\]

Suppose \( \partial V \cap \Omega = \emptyset \). Note if there exists \( (x, 0) \in U \setminus V \) with \( \Phi(x, 0) \cap N(x, 0) \neq \emptyset \) then \( (x, 0) \in \Omega_0 \times \{0\} \subseteq V \), a contradiction. Thus

\[
\Phi(x, 0) \cap N(x, 0) = \emptyset \quad \text{for } (x, 0) \in U \setminus V.
\]

Now Theorem 3.8 with \( V \) replacing \( U \) (note \( \tilde{\Omega}_0 \times \{0\} \subseteq V \subseteq U \) and \( \partial V \cap \tilde{\Omega} = \emptyset \) since \( \tilde{\Omega} \subseteq \Omega \)) implies that \( \tilde{\Omega} = \{(x, \lambda) \in \nabla : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\} \) has a connected component intersecting \( \Omega_0 \times \{0\} \) \( (\subseteq \Omega_0 \times \{0\}) \) and \( \Omega_1 \times \{1\} \) \( (\subseteq \Omega_1 \times \{1\}) \), which contradicts the assumption that there is no connected component of \( \Omega \) intersecting \( \Omega_0 \times \{0\} \) and \( (\partial U \cap \Omega) \cup (\Omega_1 \times \{1\}) \); here \( \tilde{\Omega}_t = \{x \in E : (x, t) \in \tilde{\Omega}\} \) for \( t \in [0, 1] \). Thus

\[
\partial V \cap \Omega \neq \emptyset.
\]

Note \( (x_0, 0) \in \Omega_0 \times \{0\} \subseteq V \) so \( (x_0, 0) \) and \( \partial V \cap \Omega \) are closed disjoint subsets of the compact set \( \tilde{\Omega} \) and the connected component of \( \tilde{\Omega} \) containing \( (x_0, 0) \) does not intersect \( \partial V \cap \Omega \) (since \( C_0 \subseteq V \)). Now from Theorem 2.12 there exists an open bounded neighborhood \( V_0 \) of \( (x_0, 0) \) with

\[
(x_0, 0) \in V_0, \quad \nabla_0 \cap (\Omega \cap \partial V) = \emptyset \quad \text{and} \quad \partial V_0 \cap \tilde{\Omega} = \emptyset.
\]

Let \( W = V \cap V_0 \). Now \( W \subseteq V \) with

\[
(x_0, 0) \in W \quad \text{and} \quad \partial W \cap \Omega = \emptyset
\]

since \( \partial W \subseteq (\partial V \cap \nabla) \cup (\partial V_0 \cap \nabla) \) and note \( (\partial V \cap \nabla) \cap \Omega = \nabla_0 \cap (\partial V \cap \Omega) = \emptyset \) from (3.25) and \( (\partial V_0 \cap \nabla) \cap \Omega = \partial V_0 \cap (\nabla \cap \Omega) = \partial V_0 \cap \tilde{\Omega} = \emptyset \) from (3.25).

Now \( V \) is bounded and \( W \) is an open neighborhood of \( (x_0, 0) \) so there exists a \( n_1 \geq n_0 \) with

\[
(x_{n_1}, 0) \in W \quad \text{and} \quad V \subseteq B(0, n_1) \times [0, 1].
\]

Note \( (x_{n_1}, 0) \in W \cap C_{n_1} \) so \( W \cap C_{n_1} \neq \emptyset \). Also note that \( C_{n_1} \) meets \( (E \times [0, 1]) \setminus W \) since \( C_{n_1} \) intersects \( (\partial U^{n_1} \cap \Omega^{n_1}) \cup (\Omega^{n_1}_i \times \{1\}) \) (and does not intersect \( (\partial U \cap \Omega) \cup (\Omega_1 \times \{1\}) \)).

Now \( C_{n_1} \) is connected so \( C_{n_1} \cap \partial W \neq \emptyset \). This is a contradiction since \( C_{n_1} \cap \partial W \subseteq \Omega^{n_1} \cap \partial W \subseteq \Omega \cap \partial W = \emptyset \) from (3.26).

\[\square\]
Remark 3.11. We now consider the situation in Remark 2.9 (and Remarks 3.5, 3.7 and 3.9) and the corresponding result is: Let $E$ be a normed space and $U$ an open subset of $E \times [0, 1]$. Suppose $N \in MA(U, Y; L, T)$ and fix $\Phi : U \to 2^Y$ with $\Phi^* \in B(U, Y \times [0, 1]; L, T)$; here $\Phi^*(x, \lambda) = (\Phi(x, \lambda), \lambda)$ for $(x, \lambda) \in U$. Assume (3.16) and the following conditions holds:

\begin{equation}
\Omega_0 \text{ is nonempty and compact;}
\end{equation}

here $\Omega_0 = \{ x \in E : (x, 0) \in \Omega \}$ where $\Omega = \{ (x, \lambda) \in \overline{U} : (L + T)^{-1}(\Phi + T)(x, \lambda) \cap (L + T)^{-1}(N + T)(x, \lambda) \neq \emptyset \}$. Let $H : \overline{U} \times [0, 1] \to 2^{Y \times [0, 1]}$ be given by $H(x, \lambda, \mu) = (N(x, \lambda), \mu)$ for $(x, \lambda) \in \overline{U}$ and $\mu \in [0, 1]$. In addition for open bounded subsets $W$ of $U$ with $\Omega_0 \times \{ 0 \} \subseteq W \subseteq U$ assume $N \in MA(W, Y; L, T)$ and the following conditions hold:

\begin{equation}
H_0 \text{ is } (L, T)\Phi^*\text{-essential in } MA_{\partial W}(W, Y \times [0, 1]; L, T)
\end{equation}

\begin{equation}
\begin{cases}
\Lambda \in MA(W, Y \times [0, 1]; L, T) \\
\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \text{ for } (x, \lambda) \in W
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
\Sigma = \{ (x, \lambda) \in \overline{W} : (L + T)^{-1}(\Phi + T)(x, \lambda) \cap (L + T)^{-1}(N + T)(x, \lambda) \neq \emptyset \} \text{is compact} \\
\text{and } \Sigma_1 \neq \emptyset.
\end{cases}
\end{equation}

Then $\Omega$ contains a connected component intersecting $\Omega_0 \times \{ 0 \}$ and which either intersects $(\partial U \cap \Omega) \cup (\Omega_1 \times \{ 1 \})$ or is unbounded.

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