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# TOPOLOGICAL STRUCTURE OF THE COINCIDENCE SET FOR ABSTRACT CLASSES OF MAPS

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**ABSTRACT.** The existence of a coincidence point is discussed in an abstract setting. In addition we consider the case when the coincidence set contains a continuum intersecting a given set.

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## 1. INTRODUCTION

In this paper we present general results for homotopies $H$ for which the maps $H_t$ may be defined in different domains. Conditions are put not only to guarantee the existence of a coincidence point but also to guarantee that the coincidence set contains a continuum (i.e. a compact connected set) which intersects a given set. Our theory is based on the notion of $\Phi$-essentiality (see [1, 10] and the references therein). The results in this paper were motivated in part by results in [1, 3, 4, 8, 9, 11].

## 2. PRELIMINARIES

We recall some results from the literature. Let $X$ be a completely regular topological space and $V$ an open subset of $X$.

We consider classes $\mathbf{A}$ and $\mathbf{B}$ of maps.

**Definition 2.1.** We say $F \in MA(\overline{V}, X)$ (respectively $F \in B(\overline{V}, X)$) if $F : \overline{V} \to 2^X$ and $F \in \mathbf{A}(\overline{V}, X)$ (respectively $F \in \mathbf{B}(\overline{V}, X)$); here $2^X$ denotes the family of nonempty subsets of $X$ and $\overline{V}$ denotes the closure of $V$ in $X$.

<u>Fix</u> a $\Psi \in B(\overline{V}, X)$.

**Definition 2.2.** We say $F \in MA_{\partial V}(\overline{V}, X)$ if $F \in MA(\overline{V}, X)$ with $F(x) \cap \Psi(x) = \emptyset$ for $x \in \partial V$; here $\partial V$ denotes the boundary of $V$ in $X$.

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**Definition 2.3.** Let $F \in MA_{\partial V}(\overline{V}, X)$. We say $F : \overline{V} \to 2^X$ is $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$ if for every map $J \in MA_{\partial V}(\overline{V}, X)$ with $J|_{\partial V} = F|_{\partial V}$ there exists $x \in V$ with $J(x) \cap \Psi(x) \neq \emptyset$.

The following result was established in [5, 10].

**Theorem 2.4.** *Let $X$ be a completely regular (respectively normal) topological space, $V$ an open subset of $X$ and let $F \in MA_{\partial V}(\overline{V}, X)$ be $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$. Suppose there exists a map $H : \overline{V} \times [0,1] \to 2^X$ with $H(\cdot, \eta(\cdot)) \in MA(\overline{V}, X)$ for any continuous function $\eta : \overline{V} \to [0,1]$ with $\eta(\partial V) = 0$, $\Psi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial V$ and $t \in (0,1]$, $H_0 = F$ and $\{x \in \overline{V} : \Psi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively closed). Then there exists $x \in V$ with $\Psi(x) \cap H_1(x) \neq \emptyset$; here $H_t(x) = H(x,t)$.*

Next we present a homotopy result for $\Psi$-essential maps. To achieve this we need to change Definition 2.3 (see Definition 2.6 below).

**Definition 2.5.** Let $X$ be a completely regular (respectively normal) topological space, and $V$ an open subset of $X$. Let $F, G \in MA_{\partial V}(\overline{V}, X)$. We say $F \cong G$ in $MA_{\partial V}(\overline{V}, X)$ if there exists a map $H : \overline{V} \times [0,1] \to 2^X$ with $H(\cdot, \eta(\cdot)) \in MA(\overline{V}, X)$ for any continuous function $\eta : \overline{V} \to [0,1]$ with $\eta(\partial V) = 0$, $H_t(x) \cap \Psi(x) = \emptyset$ for any $x \in \partial V$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\{x \in \overline{V} : \Psi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact (respectively closed); here $H_t(x) = H(x,t)$.

The following conditions will be assumed in our next result:

$$\cong \text{ is an equivalence relation in } MA_{\partial V}(\overline{V}, X), \tag{2.1}$$

and for any map $\Lambda \in MA_{\partial V}(\overline{V}, X)$ we have

$$\begin{cases} \text{if there exists a map } J \in MA_{\partial V}(\overline{V}, X) \text{ with } J \cong \Lambda \\ \text{in } MA_{\partial V}(\overline{V}, X) \text{and } J(x) \cap \Psi(x) = \emptyset \text{ for all } x \in \overline{V} \\ \text{and if } H : \overline{V} \times [0,1] \text{ is a map with } H(\cdot, \eta(\cdot)) \in MA(\overline{V}, X) \\ \text{for any continuous function } \eta : \overline{V} \to [0,1] \text{ with } \eta(\partial V) = 0, \\ H_t(x) \cap \Psi(x) = \emptyset \text{ for any } x \in \partial V \text{ and } t \in [0,1], H_1 = \Lambda, H_0 = J \\ \text{and } \{x \in \overline{V} : \Psi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{is compact (respectively closed) and if } \mu : \overline{V} \to [0,1] \text{ is a} \\ \text{continuous map with } \mu(\partial V) = 0, \text{ then} \\ \{x \in \overline{V} : \emptyset \neq \Psi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0,1]\} \text{ is closed.} \end{cases} \tag{2.2}$$

**Definition 2.6.** Let $F \in MA_{\partial V}(\overline{V}, X)$. We say $F : \overline{V} \to 2^X$ is $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$ if for every map $J \in MA_{\partial V}(\overline{V}, X)$ with $J|_{\partial V} = F|_{\partial V}$ and $J \cong F$ in $MA_{\partial V}(\overline{V}, X)$ there exists $x \in V$ with $J(x) \cap \Psi(x) \neq \emptyset$.

The following result was established in [10].

**Theorem 2.7.** *Let $X$ be a completely regular (respectively normal) topological space, $V$ an open subset of $X$ and assume* (2.1) *and* (2.2) *hold. Suppose $F$ and $G$ are two maps in $MA_{\partial V}(\overline{V}, X)$ with $F \cong G$ in $MA_{\partial V}(\overline{V}, X)$. Then $F$ is $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$ if and only if $G$ is $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$.*

**Remark 2.8.** Another homotopy result without conditions (2.1) and (2.2) can be found in [6]: Let $X$ be a completely regular (respectively normal) topological space, $V$ an open subset of $X$, $F \in MA_{\partial V}(\overline{V}, X)$ and let $G \in MA_{\partial V}(\overline{V}, X)$ be $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$ (Definition 2.3). For any map $R \in MA_{\partial V}(\overline{V}, X)$ with $R|_{\partial V} = F|_{\partial V}$ assume there exists a map $H^R : \overline{V} \times [0,1] \to 2^X$ with $H^R(\cdot, \eta(\cdot)) \in MA(\overline{V}, X)$ for any continuous function $\eta : \overline{V} \to [0,1]$ with $\eta(\partial V) = 0$, $\Psi(x) \cap H^R_t(x) = \emptyset$ for any $x \in \partial V$ and $t \in (0,1)$ and $\{x \in \overline{V} : \Psi(x) \cap H^R(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (respectively closed) and $H^R_0 = G$, $H^R_1 = R$; here $H^R_t(x) = H^R(x,t)$. Then $F$ is $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$. There is an analogue result also in [6] if $G \in MA_{\partial V}(\overline{V}, X)$ is $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$ (Definition 2.3) is changed to $G \in MA_{\partial V}(\overline{V}, X)$ is $\Psi$-essential in $MA_{\partial V}(\overline{V}, X)$ (Definition 2.6).

**Remark 2.9.** The ideas presented in this paper could be applied to other natural situations. Let $X$ be a Hausdorff topological vector space, $Y$ a topological vector space, and $V$ an open subset of $X$. Also let $L : dom\, L \subseteq X \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of $X$. Finally $T : X \to Y$ will be a linear single valued map with $L + T : dom\, L \to Y$ a bijection; for convenience we say $T \in H_L(X, Y)$. We say $F \in MA(\overline{V}, Y; L, T)$ (respectively $F \in B(\overline{V}, Y; L, T)$) if $F : \overline{V} \to 2^Y$ and $(L+T)^{-1}(F+T) \in MA(\overline{V}, X)$ (respectively $(L+T)^{-1}(F+T) \in B(\overline{V}, X)$). Fix a $\Phi \in B(\overline{V}, Y; L, T)$. We say $F \in MA_{\partial V}(\overline{V}, Y; L, T)$ if $F \in MA(\overline{V}, Y; L, T)$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Psi+T)(x) = \emptyset$ for $x \in \partial V$. Now $F \in MA_{\partial V}(\overline{V}, Y; L, T)$ is $(L,T)\Psi$-essential in $MA_{\partial V}(\overline{V}, Y; L, T)$ if for every map $J \in MA_{\partial V}(\overline{V}, Y; L, T)$ with $J|_{\partial V} = F|_{\partial V}$ there exists $x \in V$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Psi+T)(x) \neq \emptyset$ (this is the analogue of Definition 2.3). There are analogues of Theorem 2.4, Theorem 2.7 and Remark 2.8 in this situation; see [5, 6, 10] where the results and proofs are presented. For example the analogue of Theorem 2.4 in this situation is: Let $X$ be a topological vector space (so automatically completely regular), $Y$ a topological vector space, $V$ an open subset of $X$, $L : dom\, L \subseteq X \to Y$ a linear single valued map and $T \in H_L(X, Y)$. Let $F \in MA_{\partial V}(\overline{V}, Y; L, T)$ be $(L,T)\Psi$-essential in $MA_{\partial V}(\overline{V}, Y; L, T)$. Suppose there exists a map $H : \overline{V} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in MA(\overline{V}, X)$ for any continuous function $\eta : \overline{V} \to [0,1]$ with $\eta(\partial V) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Psi+T)(x) = \emptyset$ for any $x \in \partial V$ and $t \in (0,1]$, $H_0 = F$ (here $H_t(x) = H(x,t)$) and $D = \{x \in \overline{V} : (L+T)^{-1}(\Psi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact. Then there exists $x \in V$ with $(L+T)^{-1}(H_1+T)(x) \cap (L+T)^{-1}(\Psi+T)(x) \neq \emptyset$. If $X$ is a normal topological vector space then the assumption that $D$ is compact, can be replaced by $D$ is closed. It is easy to state and prove analogues of the results in Section 3 in this situation; we leave the details to the reader.

**Remark 2.10.** It is of interest also to note other general classes of maps in the literature. Consider classes $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ of maps. We say $F \in D(\overline{V}, X)$ (respectively $F \in B(\overline{V}, X)$) if $F : \overline{V} \to 2^X$ and $F \in \mathbf{D}(\overline{V}, X)$ (respectively $F \in \mathbf{B}(\overline{V}, X)$). We say $F \in A(\overline{V}, X)$ if $F : \overline{V} \to 2^X$ and $F \in \mathbf{A}(\overline{V}, X)$ and there exists a selection $\theta \in D(\overline{V}, X)$ of $F$. There are analogues of Theorem 2.4 and Remark 2.8 for these maps; see for example [7].

Recall a compact connected set is called a continuum. Whyburn's lemma from topology can be stated as follows.

**Theorem 2.11.** *Let $A$ and $B$ be disjoint closed subsets of a compact Hausdorff topological space $K$ such that no connected component of $K$ intersects both $A$ and $B$. Then there exists a partition $K = K_1 \cup K_2$ where $K_1$ and $K_2$ are disjoint compact sets containing $A$ and $B$ respectively.*

An easy consequence of Theorem 2.11 is the following (see [3]).

**Theorem 2.12.** *Let $X$ be a metric space and $K$ a compact subset of $X$. Assume that $A$ and $B$ are two disjoint closed subsets of $K$ such that no connected component of $K$ intersects both. Then there exists an open bounded set $U$ such that*

$$A \subset U, \quad \overline{U} \cap B = \emptyset \quad \text{and } \partial U \cap K = \emptyset.$$

## 3. MAIN RESULTS

In many applications results are needed for homotopies $H$ for which the maps $H_t$ may be defined on different domains. The idea is to reduce the study of this family to that of a new family (of course depending on the old one) defined on the same domain. For notational purposes let $Z$ be a topological space and $\Omega$ a subset of $Z \times [0,1]$. We write $\Omega_\lambda = \{x \in Z : (x, \lambda) \in \Omega\}$ to denote the section of $\Omega$ at $\lambda$.

In our next results we assume $E$ is a completely regular topological space and $U$ an open subset of $E \times [0,1]$ (note $E \times [0,1]$ is a completely regular topological space). We begin by presenting some results which guarantee the existence of a coincidence point.

**Theorem 3.1.** *Let $E$ be a completely regular topological space and $U$ an open subset of $E \times [0,1]$. Suppose $N \in MA(\overline{U}, E)$ and fix $\Phi : \overline{U} \to 2^E$ with $\Phi^\star \in B(\overline{U}, E \times [0,1])$; here $\Phi^\star(x,\lambda) = (\Phi(x,\lambda), \lambda)$ for $(x,\lambda) \in \overline{U}$. Let $H : \overline{U} \times [0,1] \to 2^{E \times [0,1]}$ be given* *by* $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ *for* $(x,\lambda) \in \overline{U}$ *and* $\mu \in [0,1]$ *and assume* $H(\cdot,\cdot,\eta(\cdot)) \in MA(\overline{U}, E \times [0,1])$ *for any continuous function* $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$. *Also suppose the following conditions are satisfied:*

$$\text{(3.1)} \qquad \begin{cases} D = \big\{(x,\lambda) \in \overline{U} : \Phi^\star(x,\lambda) \cap H(x,\lambda,\mu) \neq \emptyset \text{ for some } \mu \in [0,1]\big\} \\ \text{is compact} \end{cases}$$

$$\text{(3.2)} \qquad \begin{cases} H_0 \text{ is } \Phi^\star\text{-essential in } MA_{\partial U}(\overline{U}, E \times [0,1]) \text{ (Definition 2.3);} \\ \text{here } H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \text{ for } (x,\lambda) \in \overline{U} \end{cases}$$

*and*

$$\text{(3.3)} \qquad \Phi(x,\lambda) \cap N(x,\lambda) = \emptyset \quad \text{for } (x,\lambda) \in \partial U.$$

*Then there exists* $x \in U_1 = \{y \in E : (y,1) \in U\}$ *with* $\Phi(x,1) \cap N(x,1) \neq \emptyset$.

*Proof.* Suppose there exists $(x_0,\lambda_0) \in \partial U$ and $\mu_0 \in [0,1]$ with $\Phi^\star(x_0,\lambda_0) \cap H(x_0,\lambda_0,\mu_0) \neq \emptyset$ i.e. $(\Phi(x_0,\lambda_0),\lambda_0) \cap (N(x_0,\lambda_0),\mu_0) \neq \emptyset$. Then $\mu_0 = \lambda_0$ and $\Phi(x_0,\lambda_0) \cap N(x_0,\lambda_0) \neq \emptyset$, which contradicts (3.3). Thus

$$\Phi^\star(x,\lambda) \cap H(x,\lambda,\mu) = \emptyset \quad \text{for } (x,\lambda) \in \partial U \text{ and } \mu \in [0,1].$$

Now Theorem 2.4 (with $X = E \times [0,1]$, $V = U$ and $\Psi = \Phi^\star$) guarantees that there exists $(x,\lambda) \in U$ with $\Phi^\star(x,\lambda) \cap H(x,\lambda,1) \neq \emptyset$ i.e. $(\Phi(x,\lambda),\lambda) \cap (N(x,\lambda),1) \neq \emptyset$ i.e. $\Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset$ and $\lambda = 1$ i.e. $x \in U_1$ and $\Phi(x,1) \cap N(x,1) \neq \emptyset$. □

**Remark 3.2.** If $E \times [0,1]$ is a normal topological space then (3.1) can be changed to: $D$ is closed.

**Theorem 3.3.** *Let* $E$ *be a completely regular topological space and* $U$ *an open subset of* $E \times [0,1]$. *Suppose* $N \in MA(\overline{U}, E)$ *and fix* $\Phi : \overline{U} \to 2^E$ *with* $\Phi^\star \in B(\overline{U}, E \times [0,1])$*; here* $\Phi^\star(x,\lambda) = (\Phi(x,\lambda),\lambda)$ *for* $(x,\lambda) \in \overline{U}$. *Also suppose* (2.1) *and* (2.2) *hold with* $X = E \times [0,1]$, $V = U$ *and* $\Psi = \Phi^\star$. *Let* $H : \overline{U} \times [0,1] \to 2^{E \times [0,1]}$ *be given by* $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ *for* $(x,\lambda) \in \overline{U}$ *and* $\mu \in [0,1]$ *and assume* $H(\cdot,\cdot,\eta(\cdot)) \in MA(\overline{U}, E \times [0,1])$ *for any continuous function* $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$. *In addition assume* (3.1), (3.2) *(with Definition* 2.6*) and* (3.3) *hold. Then* $H_1$ *is* $\Phi$*-essential in* $MA_{\partial U}(\overline{U}, E \times [0,1])$ *(in particular there exists* $x \in U_1$ *with* $\Phi(x,1) \cap N(x,1) \neq \emptyset$*); here* $H_1(x,\lambda) = H(x,\lambda,1) = (N(x,1),1)$ *for* $(x,\lambda) \in \overline{U}$.

*Proof.* As in Theorem 3.1 note

$$\Phi^\star(x,\lambda) \cap H(x,\lambda,\mu) = \emptyset \text{ for } (x,\lambda) \in \partial U \text{ and } \mu \in [0,1].$$

Also the conditions in the statement of Theorem 3.3 guarantees that $H_0 \cong H_1$ in $MA_{\partial U}(\overline{U}, E \times [0,1])$. Theorem 2.7 guarantees that $H_1$ is $\Phi$-essential in $MA_{\partial U}(\overline{U}, E \times [0,1])$. □

**Remark 3.4.** If $E \times [0,1]$ is a normal topological space then (3.1) can be changed to: $D$ is closed.

**Remark 3.5.** We now consider the situation in Remark 2.9. Let $E$ be a Hausdorff topological vector space, $Y$ a topological vector space, and $U$ an open subset of $E \times [0,1]$. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map. Now let $\mathbf{L} : dom\, \mathbf{L} = dom\, L \times [0,1] \to Y \times [0,1]$ be given by $\mathbf{L}(y,\lambda) = (Ly,\lambda)$. Let $T : E \to Y$ be a linear single valued map with $L+T : dom\, L \to Y$ a bijection and let $\mathbf{T} : E \times [0,1] \to Y \times [0,1]$ be given by $\mathbf{T}(y,\lambda) = (Ty, 0)$. Notice $(\mathbf{L}+\mathbf{T})^{-1}(y,\lambda) = ((L+T)^{-1}y,\lambda)$ for $(y,\lambda) \in Y \times [0,1]$. There are analogues of Theorem 3.1 and Theorem 3.3 in this situation. For example the analogue of Theorem 3.1 is: Suppose $N \in MA(\overline{U}, Y; L, T)$ and fix $\Phi : \overline{U} \to 2^Y$ with $\Phi^\star \in B(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T})$; here $\Phi^\star(x,\lambda) = (\Phi(x,\lambda),\lambda)$ for $(x,\lambda) \in \overline{U}$. Let $H : \overline{U} \times [0,1] \to 2^{Y \times [0,1]}$ be a map given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$ and assume $(\mathbf{L}+\mathbf{T})^{-1}(H(\cdot,\cdot,\eta(\cdot)) + \mathbf{T}) \in MA(\overline{U}, E \times [0,1])$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$. Also suppose the following conditions are satisfied:

$$(3.4) \qquad \begin{cases} D = \{(x,\lambda) \in \overline{U} : (\mathbf{L}+\mathbf{T})^{-1}(\Phi^\star + \mathbf{T})(x,\lambda) \cap \\ (\mathbf{L}+\mathbf{T})^{-1}(H_\mu + \mathbf{T})(x,\lambda) \neq \emptyset \text{ for some } \mu \in [0,1]\} \\ \text{is compact} \end{cases}$$

$$(3.5) \qquad \begin{cases} H_0 \text{ is } (\mathbf{L},\mathbf{T})\Phi^\star\text{-essential in } MA_{\partial U}(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T});\ \text{here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \text{ for } (x,\lambda) \in \overline{U} \end{cases}$$

and

$$(3.6) \qquad \begin{cases} (L+T)^{-1}(\Phi+T)(x,\lambda) \cap (L+T)^{-1}(N+T)(x,\lambda) = \emptyset \\ \text{for } (x,\lambda) \in \partial U;\ \text{here } (N+T)(x,\lambda) = N(x,\lambda) + T(x). \end{cases}$$

Then there exists $x \in U_1 = \{y \in E : (y,1) \in U\}$ with $(L+T)^{-1}(\Phi+T)(x,1) \cap (L+T)^{-1}(N+T)(x,1) \neq \emptyset$. If $E \times [0,1]$ is a normal topological vector space then $D$ compact above can be changed to $D$ closed.

Next we discuss the topological structure of the coincidence set.

**Theorem 3.6.** *Let $E$ be a completely regular topological space and $U$ an open subset of $E \times [0,1]$. Suppose $N \in MA(\overline{U}, E)$ and fix $\Phi : \overline{U} \to 2^E$ with $\Phi^\star \in B(\overline{U}, E \times [0,1])$; here $\Phi^\star(x,\lambda) = (\Phi(x,\lambda),\lambda)$ for $(x,\lambda) \in \overline{U}$. Let $H : \overline{U} \times [0,1] \to 2^{E \times [0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$ and assume* (3.2) *(with Definition* 2.3*) and* (3.3) *hold. For any continuous map $\mu : \overline{U} \to [0,1]$ assume $\Lambda \in MA(\overline{U}, E \times [0,1])$ where*

$$\Lambda(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) \textit{ for } (x,\lambda) \in \overline{U}.$$

*Also suppose*

$$(3.7)\qquad \begin{cases} \Omega = \{(x,\lambda) \in \overline{U} : \Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset\} \\ \textit{is compact and } \Omega_1 \neq \emptyset; \end{cases}$$

*here* $\Omega_t = \{x \in E : (x,t) \in \Omega\}$ *for* $t \in [0,1]$. *Then* $\Omega$ *contains a continuum intersecting* $\Omega_0 \times \{0\}$ *and* $\Omega_1 \times \{1\}$.

*Proof.* Note $A = \Omega_0 \times \{0\} \subseteq \Omega$ and $B = \Omega_1 \times \{1\} \subseteq \Omega$ are closed and compact. If there is no continuum intersecting $A$ and $B$ then from Theorem 2.11, $\Omega$ can be represented as $\Omega = \Omega^\star \cup \Omega^{\star\star}$ where $\Omega^\star$ and $\Omega^{\star\star}$ are disjoint compact sets with $A \subseteq \Omega^\star$ and $B \subseteq \Omega^{\star\star}$. Notice $\Omega^\star$ and $\Omega^{\star\star} \cup \partial U$ are closed and disjoint (note $\Omega^\star \cap \partial U = \emptyset$ since if there exists a $(x,\lambda) \in \partial U$ and $(x,\lambda) \in \Omega^\star$ then (note $(x,\lambda) \in \Omega^\star \subseteq \Omega$) $\Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset$ which contradicts (3.3)). Now there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\Omega^{\star\star} \cup \partial U) = 0$ and $\mu(\Omega^\star) = 1$. Let

$$\Lambda(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) \quad \text{for } (x,\lambda) \in \overline{U}.$$

From the statement of Theorem 3.6 note $\Lambda \in MA(\overline{U}, E \times [0,1])$ and in fact $\Lambda \in MA_{\partial U}(\overline{U}, E \times [0,1])$ since if there exists a $(x,\lambda) \in \partial U$ with $\Phi^\star(x,\lambda) \cap \Lambda(x,\lambda) \neq \emptyset$ then $(\Phi(x,\lambda),\lambda) \cap (N(x,\lambda),\mu(x,\lambda)) \neq \emptyset$ i.e. $(\Phi(x,\lambda),\lambda) \cap (N(x,\lambda),0) \neq \emptyset$ i.e. $\Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset$ with $\lambda = 0$, and this contradicts (3.3). Note $H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0)$ so

$$\Lambda|_{\partial U} = H_0|_{\partial U}$$

since if $(x,\lambda) \in \partial U$ then $\Lambda(x,\lambda) = (N(x,\lambda),\mu(x,\lambda)) = (N(x,\lambda),0)$ because $\mu(\Omega^{\star\star} \cup \partial U) = 0$. Now (3.2) guarantees that there exists a $(x,\lambda) \in U$ with $\Phi^\star(x,\lambda) \cap \Lambda(x,\lambda) \neq \emptyset$ i.e. $(\Phi(x,\lambda),\lambda) \cap (N(x,\lambda),\mu(x,\lambda)) \neq \emptyset$ i.e. $\Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset$ and $\lambda = \mu(x,\lambda)$. Note $(x,\lambda) \in \Omega$ since $(x,\lambda) \in U$ and $\Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset$. Now either $(x,\lambda) \in \Omega^\star$ or $(x,\lambda) \in \Omega^{\star\star}$. Suppose $(x,\lambda) \in \Omega^\star$. Then $\mu(x,\lambda) = 1$ so $\lambda = \mu(x,\lambda) = 1$ and $\Phi(x,1) \cap N(x,1) \neq \emptyset$ i.e. $(x,1) \in B \subseteq \Omega^{\star\star}$ which contradicts $(x,1) = (x,\lambda) \in \Omega^\star$. Next suppose $(x,\lambda) \in \Omega^{\star\star}$. Then $\mu(x,\lambda) = 0$ so $\lambda = \mu(x,\lambda) = 0$ and $\Phi(x,0) \cap N(x,0) \neq \emptyset$ i.e. $(x,0) \in A \subseteq \Omega^\star$ which contradicts $(x,0) = (x,\lambda) \in \Omega^{\star\star}$. □

**Remark 3.7.** We now consider the situation in Remark 2.9 (and Remark 3.5) and the corresponding result is: Suppose $N \in A(\overline{U}, Y; L, T)$ and fix $\Phi : \overline{U} \to 2^Y$ with $\Phi^\star \in B(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T})$; here $\Phi^\star(x,\lambda) = (\Phi(x,\lambda),\lambda)$ for $(x,\lambda) \in \overline{U}$. Let $H : \overline{U} \times [0,1] \to 2^{Y \times [0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$ and assume (3.5) and (3.6) hold. For any continuous map $\mu : \overline{U} \to [0,1]$ assume $\Lambda \in MA(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T})$ where

$$\Lambda(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) \quad \text{for } (x,\lambda) \in \overline{U}.$$

Also suppose

$$(3.8)\qquad \begin{cases} \Omega = \{(x,\lambda) \in \overline{U} : (L+T)^{-1}(\Phi+T)(x,\lambda) \cap \\ (L+T)^{-1}(N+T)(x,\lambda) \neq \emptyset\} \text{ is compact} \\ \text{and } \Omega_1 \neq \emptyset. \end{cases}$$

Then $\Omega$ contains a continuum intersecting $\Omega_0 \times \{0\}$ and $\Omega_1 \times \{1\}$.

In our next result (3.3) is not assumed.

**Theorem 3.8.** *Let $E$ be a metric space and $U$ an open subset of $E \times [0,1]$. Suppose $N \in MA(\overline{U}, E)$ and fix $\Phi : \overline{U} \to 2^E$ with $\Phi^\star \in B(\overline{U}, E \times [0,1])$; here $\Phi^\star(x,\lambda) = (\Phi(x,\lambda),\lambda)$ for $(x,\lambda) \in \overline{U}$. Assume*

$$(3.9)\qquad \Phi(x,0) \cap N(x,0) = \emptyset \quad \text{for } (x,0) \in \partial U.$$

*Let $H : \overline{U} \times [0,1] \to 2^{E\times[0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$ and assume* (3.2) *(with Definition* 2.3*) and* (3.7) *hold. For any continuous map $\mu : \overline{U} \to [0,1]$ assume $\Lambda \in MA(\overline{U}, E \times [0,1])$ where*

$$(3.10)\qquad \Lambda(x,\lambda) = (N(x,\lambda),\mu(x,\lambda)) \quad \text{for } (x,\lambda) \in \overline{U}.$$

*In addition for open bounded subsets $W$ of $U$ with $\Omega_0 \times \{0\} \subseteq W \subseteq U$ (so $\Phi(x,0) \cap N(x,0) = \emptyset$ for $(x,0) \in U \backslash W$), $\partial W \cap \Omega = \emptyset$ and $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$ assume $N \in MA(\overline{W}, E)$ and the following conditions holds:*

$$(3.11)\qquad H_0 \text{ is } \Phi^\star\text{-essential in } MA_{\partial W}(\overline{W}, E \times [0,1])$$

$$(3.12)\qquad \begin{cases} \text{for any continuous map } \mu : \overline{W} \to [0,1] \text{ assume} \\ \Lambda \in MA(\overline{W}, E \times [0,1]) \text{where } \Lambda(x,\lambda) = (N(x,\lambda),\mu(x,\lambda)) \\ \text{for } (x,\lambda) \in \overline{W} \end{cases}$$

*and*

$$(3.13)\qquad \Sigma \text{ is closed and } \Sigma_1 \neq \emptyset;$$

*here $\Sigma = \{(x,\lambda) \in \overline{W} : \Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset\}$ and $\Sigma_t = \{x \in E : (x,t) \in \Sigma\}$ for $t \in [0,1]$. Then $\Omega$ contains a continuum intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$.*

*Proof.* There are two cases to consider, namely $\Omega \cap \partial U = \emptyset$ or $\Omega \cap \partial U \neq \emptyset$. If $\Omega \cap \partial U = \emptyset$ then (3.3) holds so the result follows from Theorem 3.6. Now suppose $\Omega \cap \partial U \neq \emptyset$. Let $A = \Omega_0 \times \{0\}$, $B = \Omega_1 \times \{1\}$ and $C = \Omega \cap \partial U (\neq \emptyset)$. Notice $C \subseteq \Omega$ is closed and (3.9) guarantees that $C \cap A = \emptyset$. Now from Theorem 2.11 either

(1). there exists a continuum of $\Omega$ which intersects $A$ and $C$ (and we are finished), or

(2). $\Omega = \Omega^\star \cup \Omega^{\star\star}$ where $\Omega^\star$ and $\Omega^{\star\star}$ are disjoint compact sets with $A \subseteq \Omega^\star$ and $C \subseteq \Omega^{\star\star}$.

Suppose (2) occurs. From Theorem 2.12 there exists an open bounded set $V$ with

$$\Omega^\star \subseteq V, \quad \overline{V} \cap \Omega^{\star\star} = \emptyset \quad \text{and} \quad \partial V \cap \Omega = \emptyset. \tag{3.14}$$

Let $W = U \cap V$. We now show

$$A \subseteq W \subseteq U, \quad \partial W \cap \Omega = \emptyset \quad \text{and} \quad \overline{W} \cap (\partial U \cap \Omega) = \emptyset. \tag{3.15}$$

Note $A \subseteq W$ since $A \subseteq \Omega^\star \subseteq V$ and $A \subseteq U$ from (3.9). Next notice that

$$\begin{aligned}\partial W &= (\overline{U \cap V})\backslash(U \cap V) \subseteq (\overline{U} \cap \overline{V})\backslash(U \cap V)\\ &= ((\overline{U}\backslash U) \cap \overline{V}) \cup ((\overline{V}\backslash V) \cap \overline{U})\\ &= (\partial U \cap \overline{V}) \cup (\partial V \cap \overline{U}) \subseteq (\partial U \cap \overline{V}) \cup \partial V,\end{aligned}$$

and note $\partial V \cap \Omega = \emptyset$ (see (3.14)) and $(\partial U \cap \overline{V}) \cap \Omega = \emptyset$ (from (3.14) we have $\overline{V} \cap \Omega^{\star\star} = \emptyset$ and note $C = \Omega \cap \partial U \subseteq \Omega^{\star\star}$ so we have $\overline{V} \cap \Omega \cap \partial U = \emptyset$) and so $\partial W \cap \Omega = \emptyset$. Finally notice $\overline{W} \cap \Omega^{\star\star} = \emptyset$ since $\overline{W} \subseteq \overline{U} \cap \overline{V} \subseteq \overline{V}$ and $\overline{V} \cap \Omega^{\star\star} = \emptyset$ from (3.9), so $\overline{W} \cap \Omega^{\star\star} = \emptyset$ and $C \subseteq \Omega^{\star\star}$ implies $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$. Thus (3.15) holds.

Let

$$\Sigma = \left\{(x, \lambda) \in \overline{W} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\right\}.$$

Note $\partial W \cap \Sigma = \emptyset$ (see (3.15) and note $\Sigma \subseteq \Omega$). Now Theorem 3.6 (note $\Sigma$ is compact) implies that $\Sigma$ contains a continuum intersecting $\Sigma_0 \times \{0\}$ ($\subseteq \Omega_0 \times \{0\}$) and $\Sigma_1 \times \{1\}$ ($\subseteq \Omega_1 \times \{1\}$) and our result follows. □

**Remark 3.9.** We now consider the situation in Remark 2.9 (and Remarks 3.5 and 3.7) and the corresponding result is: Let $E$ be a metric space and $U$ an open subset of $E \times [0, 1]$. Suppose $N \in MA(\overline{U}, Y; L, T)$ and fix $\Phi : \overline{U} \to 2^Y$ with $\Phi^\star \in B(\overline{U}, Y \times [0, 1]; \mathbf{L}, \mathbf{T})$; here $\Phi^\star(x, \lambda) = (\Phi(x, \lambda), \lambda)$ for $(x, \lambda) \in \overline{U}$. Assume

$$(L + T)^{-1}(\Phi + T)(x, 0) \cap (L + T)^{-1}(N + T)(x, 0) = \emptyset \quad \text{for } (x, 0) \in \partial U. \tag{3.16}$$

Let $H : \overline{U} \times [0, 1] \to 2^{Y \times [0,1]}$ be given by $H(x, \lambda, \mu) = (N(x, \lambda), \mu)$ for $(x, \lambda) \in \overline{U}$ and $\mu \in [0, 1]$ and assume (3.5) and (3.8) hold. For any continuous map $\mu : \overline{U} \to [0, 1]$ assume $\Lambda \in MA(\overline{U}, Y \times [0, 1]; \mathbf{L}, \mathbf{T})$ where

$$\Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \quad \text{for } (x, \lambda) \in \overline{U}. \tag{3.17}$$

In addition for open bounded subsets $W$ of $U$ with $\Omega_0 \times \{0\} \subseteq W \subseteq U$, $\partial W \cap \Omega = \emptyset$, and $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$ assume $N \in MA(\overline{W}, Y; L, T)$ and the following conditions hold:

$$H_0 \text{ is } (\mathbf{L}, \mathbf{T})\Phi^\star\text{-essential in } MA_{\partial W}(\overline{W}, Y \times [0, 1]; \mathbf{L}, \mathbf{T}) \tag{3.18}$$

$$\begin{cases} \text{for any continuous map } \mu : \overline{W} \to [0, 1] \text{ assume} \\ \Lambda \in MA(\overline{W}, Y \times [0, 1]; \mathbf{L}, \mathbf{T}) \text{ where} \\ \Lambda(x, \lambda) = (N(x, \lambda), \mu(x, \lambda)) \text{ for } (x, \lambda) \in \overline{W} \end{cases} \tag{3.19}$$

and

$$\Sigma \text{ is closed and } \Sigma_1 \neq \emptyset; \tag{3.20}$$

here $\Sigma = \big\{(x,\lambda) \in \overline{W} : (L+T)^{-1}(\Phi+T)(x,\lambda) \cap (L+T)^{-1}(N+T)(x,\lambda) \neq \emptyset\big\}$. Then $\Omega$ contains a continuum intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$.

In our next result $\{(x,\lambda) \in \overline{U} : \Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset\}$ is compact is not assumed. For convenience we assume $E$ is a normed space (the proof when $E$ is a metric space is similar).

**Theorem 3.10.** *Let $E$ be a normed space and $U$ an open subset of $E \times [0,1]$. Suppose $N \in MA(\overline{U}, E)$ and fix $\Phi : \overline{U} \to 2^E$ with $\Phi^\star \in B(\overline{U}, E \times [0,1])$; here $\Phi^\star(x,\lambda) = (\Phi(x,\lambda),\lambda)$ for $(x,\lambda) \in \overline{U}$. Assume* (3.9) *and the following condition holds:*

$$\Omega_0 \textit{ is nonempty and compact;} \tag{3.21}$$

*here $\Omega_0 = \{x \in E : (x,0) \in \Omega\}$ where $\Omega = \{(x,\lambda) \in \overline{U} : \Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset\}$. Let $H : \overline{U} \times [0,1] \to 2^{E\times[0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$. In addition for open bounded subsets $W$ of $U$ with $\Omega_0 \times \{0\} \subseteq W \subseteq U$ (so $\Phi(x,0) \cap N(x,0) = \emptyset$ for $(x,0) \in U \backslash W$) assume $N \in MA(\overline{W}, E)$ and the following conditions hold:*

$$H_0 \textit{ is } \Phi^\star\textit{-essential in } MA_{\partial W}(\overline{W}, E \times [0,1]) \tag{3.22}$$

$$\begin{cases} \textit{for any continuous map } \mu : \overline{W} \to [0,1] \textit{ assume} \\ \Lambda \in MA(\overline{W}, E \times [0,1]) \textit{ where } \Lambda(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) \\ \textit{for } (x,\lambda) \in \overline{W} \end{cases} \tag{3.23}$$

*and*

$$\begin{cases} \Sigma = \{(x,\lambda) \in \overline{W} : \Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset\} \\ \textit{is compact and } \Sigma_1 \neq \emptyset. \end{cases} \tag{3.24}$$

*Then $\Omega$ contains a connected component intersecting $\Omega_0 \times \{0\}$ and which either intersects $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$ or is unbounded.*

*Proof.* Since $\Omega_0$ is compact there exists $n_0 \in \mathbf{N}$ with $\Omega_0 \subseteq B(0,n_0)$. For $n \geq n_0$ let

$$U^n = U \cap (B(0,n) \times [0,1]) \quad \text{and} \quad \Omega^n = \{(x,\lambda) \in \overline{U^n} : \Phi(x,\lambda) \cap N(x,\lambda) \neq \emptyset\}.$$

Note (3.9) implies $\Omega_0 \times \{0\} \subseteq U$ and as a result $\Omega_0 \times \{0\} \subseteq U^n$. Also note if there exists $(x,0) \in U \backslash U^n$ with $\Phi(x,0) \cap N(x,0) \neq \emptyset$ then $(x,0) \in \Omega_0 \times \{0\} \subseteq U^n$, a contradiction. Thus

$$\Phi(x,0) \cap N(x,0) = \emptyset \quad \text{for } (x,0) \in U \backslash U^n.$$

For each $n \geq n_0$, Theorem 3.8 (with $U^n$ replacing $U$ and note (3.7) holds with $U^n$ replacing $U$ (see (3.24) with $W = U^n$)) guarantees that there exists $(x_n, 0) \in$

$\Omega_0 \times \{0\}$ and a connected component $\mathbf{C}_n$ of $\Omega^n$ containing $(x_n, 0)$ and intersecting $(\partial U^n \cap \Omega^n) \cup (\Omega_1^n \times \{1\})$ (here $\Omega_1^n = \{x \in E : (x, 1) \in \Omega^n\}$). Since $\Omega_0$ is compact the sequence $(x_n)$ has an accumulation point $x_0 \in \Omega_0$. Assume that there is NO connected component of $\Omega$ intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$. Let $\mathbf{C}_0$ be the connected component containing $x_0$ (and not intersecting $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$).

Our result follows if we show $\mathbf{C}_0$ is unbounded. Assume $\mathbf{C}_0$ is bounded. Note $\mathbf{C}_0 \subseteq \overline{U}$ and $\mathbf{C}_0 \cap \partial U = \emptyset$ (since $\mathbf{C}_0$ does not intersect $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$) so $\mathbf{C}_0 \subseteq U$, and note $\mathbf{C}_0$ , $\Omega_0 \times \{0\}$ are closed and bounded and as a result we can choose an open bounded set $V$ with

$$\mathbf{C}_0 \cup (\Omega_0 \times \{0\}) \subseteq V \subseteq U.$$

Suppose $\partial V \cap \Omega = \emptyset$. Note if there exists $(x, 0) \in U \backslash V$ with $\Phi(x, 0) \cap N(x, 0) \neq \emptyset$ then $(x, 0) \in \Omega_0 \times \{0\} \subseteq V$, a contradiction. Thus

$$\Phi(x, 0) \cap N(x, 0) = \emptyset \quad \text{ for } (x, 0) \in U \backslash V.$$

Now Theorem 3.8 with $V$ replacing $U$ (note $\tilde{\Omega}_0 \times \{0\} \subseteq V \subseteq U$ and $\partial V \cap \tilde{\Omega} = \emptyset$ since $\tilde{\Omega} \subseteq \Omega$) implies that $\tilde{\Omega} = \{(x, \lambda) \in \overline{V} : \Phi(x, \lambda) \cap N(x, \lambda) \neq \emptyset\}$ has a connected component intersecting $\tilde{\Omega}_0 \times \{0\}$ ($\subseteq \Omega_0 \times \{0\}$) and $\tilde{\Omega}_1 \times \{1\}$ ($\subseteq \Omega_1 \times \{1\}$), which contradicts the assumption that there is no connected component of $\Omega$ intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$; here $\tilde{\Omega}_t = \{x \in E : (x, t) \in \tilde{\Omega}\}$ for $t \in [0, 1]$. Thus

$$\partial V \cap \Omega \neq \emptyset.$$

Note $(x_0, 0) \in \Omega_0 \times \{0\} \subseteq V$ so $(x_0, 0)$ and $\partial V \cap \Omega$ are closed disjoint subsets of the compact set $\tilde{\Omega}$ and the connected component of $\tilde{\Omega}$ containing $(x_0, 0)$ does not intersect $\partial V \cap \Omega$ (since $\mathbf{C}_0 \subseteq V$). Now from Theorem 2.12 there exists an open bounded neighborhood $V_0$ of $(x_0, 0)$ with

$$(x_0, 0) \in V_0, \quad \overline{V_0} \cap (\Omega \cap \partial V) = \emptyset \quad \text{and} \quad \partial V_0 \cap \tilde{\Omega} = \emptyset. \tag{3.25}$$

Let $W = V \cap V_0$. Now $W \subseteq V$ with

$$(x_0, 0) \in W \quad \text{and} \quad \partial W \cap \Omega = \emptyset \tag{3.26}$$

since $\partial W \subseteq (\partial V \cap \overline{V_0}) \cup (\partial V_0 \cap \overline{V})$ and note $(\partial V \cap \overline{V_0}) \cap \Omega = \overline{V_0} \cap (\partial V \cap \Omega) = \emptyset$ from (3.25) and $(\partial V_0 \cap \overline{V}) \cap \Omega = \partial V_0 \cap (\overline{V} \cap \Omega) = \partial V_0 \cap \tilde{\Omega} = \emptyset$ from (3.25).

Now $V$ is bounded and $W$ is an open neighborhood of $(x_0, 0)$ so there exists a $n_1 \geq n_0$ with

$$(x_{n_1}, 0) \in W \quad \text{and} \quad V \subseteq B(0, n_1) \times [0, 1].$$

Note $(x_{n_1}, 0) \in W \cap \mathbf{C}_{n_1}$ so $W \cap \mathbf{C}_{n_1} \neq \emptyset$. Also note that $\mathbf{C}_{n_1}$ meets $(E \times [0, 1]) \backslash W$ since $\mathbf{C}_{n_1}$ intersects $(\partial U^{n_1} \cap \Omega^{n_1}) \cup (\Omega_1^{n_1} \times \{1\})$ (and does not intersect $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$). Now $\mathbf{C}_{n_1}$ is connected so $\mathbf{C}_{n_1} \cap \partial W \neq \emptyset$. This is a contradiction since $\mathbf{C}_{n_1} \cap \partial W \subseteq \Omega^{n_1} \cap \partial W \subseteq \Omega \cap \partial W = \emptyset$ from (3.26). □

**Remark 3.11.** We now consider the situation in Remark 2.9 (and Remarks 3.5, 3.7 and 3.9) and the corresponding result is: Let $E$ be a normed space and $U$ an open subset of $E \times [0,1]$. Suppose $N \in MA(\overline{U}, Y; L, T)$ and fix $\Phi : \overline{U} \to 2^Y$ with $\Phi^\star \in B(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T})$; here $\Phi^\star(x,\lambda) = (\Phi(x,\lambda), \lambda)$ for $(x,\lambda) \in \overline{U}$. Assume (3.16) and the following conditions holds:

$$\Omega_0 \text{ is nonempty and compact;} \tag{3.27}$$

here $\Omega_0 = \{x \in E : (x,0) \in \Omega\}$ where $\Omega = \{(x,\lambda) \in \overline{U} : (L+T)^{-1}(\Phi+T)(x,\lambda) \cap (L+T)^{-1}(N+T)(x,\lambda) \neq \emptyset\}$. Let $H : \overline{U} \times [0,1] \to 2^{Y \times [0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$. In addition for open bounded subsets $W$ of $U$ with $\Omega_0 \times \{0\} \subseteq W \subseteq U$ assume $N \in MA(\overline{W}, Y; L, T)$ and the following conditions hold:

$$H_0 \text{ is } (\mathbf{L},\mathbf{T})\Phi^\star\text{-essential in } MA_{\partial W}(\overline{W}, Y \times [0,1]; \mathbf{L}, \mathbf{T}) \tag{3.28}$$

$$\begin{cases} \text{for any continuous map } \mu : \overline{W} \to [0,1] \text{ assume} \\ \Lambda \in MA(\overline{W}, Y \times [0,1]; \mathbf{L}, \mathbf{T}) \text{ where} \\ \Lambda(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) \text{ for } (x,\lambda) \in \overline{W} \end{cases} \tag{3.29}$$

and

$$\begin{cases} \Sigma = \{(x,\lambda) \in \overline{W} : (L+T)^{-1}(\Phi+T)(x,\lambda) \cap \\ (L+T)^{-1}(N+T)(x,\lambda) \neq \emptyset\} \text{is compact} \\ \text{and } \Sigma_1 \neq \emptyset. \end{cases} \tag{3.30}$$

Then $\Omega$ contains a connected component intersecting $\Omega_0 \times \{0\}$ and which either intersects $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$ or is unbounded.

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