Dynamic Systems and Applications 26 (2017) 395-410

FIXED POINT APPROXIMATION OF GENERALIZED NONEXPANSIVE MULTI-VALUED MAPPINGS IN BANACH SPACES VIA NEW ITERATIVE ALGORITHMS

ANUPAM SHARMA[a*] AND MOHAMMAD IMDAD[b]

[a]Department of Mathematics and Statistics, Indian Institute of Technology
Kanpur, Kanpur 208 016, India
anupam@iitk.ac.in
[b]Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
mhimdad@gmail.com

ABSTRACT. In this paper, we introduce a new iterative algorithm to approximate the fixed points of generalized nonexpansive multi-valued mappings in Banach spaces and utilize the same to establish weak as well as strong convergence theorems. Our results generalize and improve several previously known results of the existing literature.

AMS (MOS) Subject Classification. 47H05, 47H09, 47H10.

1. INTRODUCTION

The fixed point theory of multi-valued nonexpansive mappings is relatively more involved and cumbersome than the corresponding theory of single-valued nonexpansive mappings. Fixed point theory for multi-valued mappings has many fruitful applications in diverse fields, e.g. game theory, mathematical economics and several others. Therefore, it is natural to extend the known fixed point results for single-valued mappings to multi-valued mappings. However, some classical fixed point theorems for single-valued nonexpansive mappings have already been extended to multi-valued mappings. The earliest results in this direction were respectively established by Markin [10] in Hilbert spaces while by Browder [4] for spaces admitting weakly continuous duality mapping. Dozo [5] generalized these results in a Banach space satisfying Opial's condition. Though nonexpansive mappings are most extensively studied class of mappings in metric fixed point theory, yet there also exists considerable literature on the classes of mappings enlarging the class of nonexpansive mappings.

Throughout the paper, E stands for a real Banach space with the norm $\|\cdot\|$ and K a nonempty subset of E. Let $\mathbb{N}$ denotes the set of all positive integers. Let $CB(K)$, $C(K)$ and $P(K)$ denote the families of nonempty closed and bounded

Received December 12, 2016

subsets, nonempty compact subsets and nonempty proximinal bounded subsets of K respectively. Recall that the set K is said to be proximinal if for any $x \in E$, there exists an element $y \in K$ such that $d(x, y) = \text{dist}(x, K)$, where $\text{dist}(x, K) = \inf\{\|x - y\|; y \in K\}$.

Let H be the Hausdorff metric on $CB(E)$ defined by

$$H(A, B) = \max\left\{\sup_{x\in A} \text{dist}(x, B), \sup_{y\in B} \text{dist}(y, A)\right\} \text{ for all } A, B \in CB(E).$$

A multi-valued mapping $T : K \to CB(E)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\| \text{ for all } x, y \in K.$$

A point $z \in K$ is called a fixed point of T if $z \in Tz$. As usual, $F(T)$ stands for the set of fixed points of a multi-valued mapping T. A multi-valued mapping $T : K \to CB(K)$ is said to be quasi-nonexpansive ([20]) if $F(T) \neq \emptyset$ and

$$H(Tx, Tz) \leq \|x - z\| \text{ for all } x \in K \text{ and } z \in F(T).$$

As mentioned earlier, the study of fixed points for multi-valued nonexpansive mappings using the Hausdorff metric was initiated by Markin [10] while the existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces can be found in Lim [9].

In 2008, Suzuki [23] defined a generalization of nonexpansive mapping and called it a mapping satisfying condition (C). Further, García-Falset et al. [7] proposed two new generalizations of condition (C) and term them as condition (E) and condition (C_λ) and studied the existence of fixed points for these classes of mappings whose set-valued versions were studied in [1, 2, 8] whose relevant details can be described as follows:

Definition 1.1 ([8])**.** Let $T : K \to CB(E)$ be a multi-valued mapping. Then T is said to satisfy condition (C_λ) if for some $\lambda \in (0, 1)$ and for each $x, y \in K$

$$\lambda \text{ dist}(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\|.$$

For $\lambda = \frac{1}{2}$, we recapture the class of mappings satisfying condition (C). It is easy to see that for $0 < \lambda_1 < \lambda_2 < 1$, condition (C_{λ_1}) implies condition (C_{λ_2}).

Lemma 1.2 ([8])**.** *Let $T : K \to CB(E)$ be a multi-valued mapping.*

(i) *If T is nonexpansive, then T satisfies condition (C_λ).*
(ii) *If T satisfies condition (C_λ) and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

Lemma 1.3 ([6])**.** *Let K be a nonempty subset of a Banach space E and $T : K \to P(E)$ a multi-valued map satisfying condition (C). Then*

$$H(Tx, Ty) \leq 2 \text{ dist}(x, Tx) + \|x - y\| \textit{ for all } x, y \in K.$$

Very recently, Abkar and Eslamian [2] used a modified Suzuki condition for multi-valued mappings which runs as follows:

Definition 1.4 ([2])**.** A multi-valued mapping $T : K \to CB(E)$ is said to satisfy condition (E_μ) if for some $\mu \geq 1$, for all $x, y \in K$

$$\text{dist}(x, Ty) \leq \mu\ \text{dist}(x, Tx) + \|x - y\|.$$

We say that T satisfies condition (E) on K whenever T satisfies condition (E_μ) for some $\mu \geq 1$.

Lemma 1.5 ([2])**.** *Let $T : K \to CB(E)$ be a multi-valued nonexpansive mapping. Then T satisfies condition (E_1).*

In the sequel we need the following definitions:

Definition 1.6 ([11])**.** A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E with $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \ \forall\ y \in E,\ y \neq x.$$

Examples of Banach spaces satisfying Opial's condition are Hilbert spaces and all l_p spaces $(1 < p < \infty)$. On the other hand, $L_p[0, 2]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

Definition 1.7 ([21])**.** A multi-valued mapping $T : K \to CB(K)$ is said to satisfy condition (I) if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(r) > 0$ for all $r \in (0, \infty)$ such that $\text{dist}(x, Tx) \geq h(\text{dist}(x, F(T))$ for all $x \in K$.

Definition 1.8. Multi-valued mappings $T, S : K \to CB(K)$ are said to satisfy condition (I') if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$ such that either $\text{dist}(x, Tx) \geq g(\text{dist}(x, F))$ or $\text{dist}(x, Sx) \geq g(\text{dist}(x, F))$ for all $x \in K$ where $F = F(T) \cap F(S)$.

Remark 1.9. With $S = T$ in Definition 1.8, condition (I') reduces to condition (I).

Definition 1.10 ([5], [7])**.** A multi-valued mapping $T : K \to P(E)$ is said to be demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in K weakly convergent to an element x and $y_n \in Tx_n$ strongly convergent to y, we have $y \in Tx$.

Definition 1.11 ([3])**.** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences of real numbers that converge to a and b, respectively. Assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

If $l = 0$, then we say that $\{a_n\}_{n\in\mathbb{N}}$ converges to a faster than $\{b_n\}_{n\in\mathbb{N}}$ to b.

An important property for the class of uniformly convex Banach spaces is contained in following lemma due to Schu [15].

Lemma 1.12 ([15]). *Let E be a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in \mathbb{N}$. If $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\lim_{n\to\infty} \sup \|x_n\| \le r$, $\lim_{n\to\infty} \sup \|y_n\| \le r$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \ge 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The purpose of this paper is to approximate the fixed points of generalized nonexpansive multi-valued mappings in Banach spaces via new iterative algorithms and establish weak and strong convergence theorems of these iterative algorithms under suitable conditions. For further details one can be referred to [16]–[19].

2. PRELIMINARIES

Different iterative schemes have been utilized to approximate the fixed points of multi-valued nonexpansive mappings. Sastry and Babu [14] studied the Mann and Ishikawa iterative schemes for multi-valued mappings and proved that these schemes for a multi-valued map T with a fixed point z converges to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from z. To describe some relevant iterative processes, let K be a nonempty convex subset of E and $T : K \to P(K)$ a multi-valued mapping with $z \in Tz$. Then, the sequence of Mann iterates is defined by with $u_1 \in K$,

$$u_{n+1} = (1-a_n)u_n + a_n t_n, \quad n \in \mathbb{N}, \tag{2.1}$$

where $t_n \in Tu_n$ is such that $\|t_n - z\| = \operatorname{dist}(z, Tu_n)$ and $\{a_n\}$ is a sequence of numbers in $(0,1)$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum a_n = \infty$.

The sequence of Ishikawa iterates is defined by $v_1 \in K$,

$$\begin{cases} v_{n+1} = (1-a_n)v_n + a_n u_n, \\ q_n = (1-b_n)v_n + b_n t'_n, \qquad n \in \mathbb{N}, \end{cases}$$

where $u_n \in Tq_n$, $t'_n \in Tv_n$ are such that $\|u_n - z\| = \operatorname{dist}(z, Tq_n)$ and $\|t'_n - z\| = \operatorname{dist}(z, Tv_n)$ and $\{a_n\}, \{b_n\}$ are real sequences of numbers in $(0,1)$ satisfying $\lim_{n\to\infty} b_n = 0$ and $\sum a_n b_n = \infty$.

Panyanak [12] extended the result of Sastry and Babu [14] by modifying the iteration schemes of Sastry and Babu [14] in the setting of uniformly convex Banach spaces but the domain of T remains compact while Song and Wang [21] employed the condition $Tz = \{z\}$ to prove their results.

Recently, Sahu [13] introduced an iterative scheme, which has been studied extensively in connection with fixed points of single-valued nonexpansive mappings as follows: Let K be a nonempty convex subset of E and $f : K \to K$ a single-valued mapping. Then, for arbitrary $w_1 \in K$, the iterative process is defined by

$$\begin{cases} w_{n+1} = f s_n \\ s_n = (1 - a_n) w_n + a_n f w_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{a_n\} \in (0, 1)$.

In the following, we extend the above iterative scheme to the case of multi-valued nonexpansive mappings on convex subset of E modifying the above ones. Let K be a nonempty convex subset of E and $T : K \to P(K)$ a multi-valued mapping with $z \in Tz$. Then, the sequence of iterates is defined by

$$\begin{cases} w_1 \in K, \\ w_{n+1} = v_n \\ s_n = (1 - a_n) w_n + a_n z_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.2}$$

where $v_n \in Ts_n$, $z_n \in Tw_n$ are such that $\|v_n - z\| = \operatorname{dist}(z, Ts_n)$ and $\|z_n - z\| = \operatorname{dist}(z, Tw_n)$ and $\{a_n\}$ is a sequence of numbers in $(0, 1)$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum a_n < \infty$.

Motivated and inspired by the work of Sahu [13], we introduced a new iterative scheme in the context of multi-valued mappings as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = u_n \\ y_n = (1 - a_n) v_n + a_n w_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.3}$$

where $u_n \in Ty_n$, $v_n \in Tx_n$ and $w_n \in Sx_n$ are such that $\|v_n - z\| = \operatorname{dist}(z, Tx_n)$, $\|u_n - z\| = \operatorname{dist}(z, Ty_n)$ and $\|w_n - z\| = \operatorname{dist}(z, Sx_n)$ and $\{a_n\}$ is a sequence of numbers in $(0, 1)$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum a_n < \infty$.

3. CONVERGENCE THEOREMS VIA ALGORITHM (2.2)

In this section we prove some weak and strong convergence theorems by approximating the fixed points of a multi-valued quasi-nonexpansive mapping and generalized nonexpansive multi-valued mappings by using iterative scheme (2.2). In the sequel, $F(T)$ denotes the set of fixed point of mapping T.

Theorem 3.1. *Let E be a uniformly convex Banach space satisfying Opial's condition and K a nonempty closed and convex subset of E. Let $T : K \to P(K)$ be a multi-valued quasi-nonexpansive mapping and $\{w_n\}$ a sequence as defined by* (2.2). *If $F(T) \neq \emptyset$ and $(I - T)$ is demiclosed at zero, then $\{w_n\}$ converges weakly to a fixed point of T.*

Proof. Let $z \in F(T)$. Hence from (2.2) we have

$$\|w_{n+1} - z\| = \|v_n - z\| = \text{dist}(Ts_n, z) \leq H(Ts_n, Tz) \leq \|s_n - z\|, \tag{3.1}$$

and

$$\begin{aligned}
\|s_n - z\| &= \|(1 - a_n)w_n + a_n z_n - z\| \\
&\leq (1 - a_n)\|w_n - z\| + a_n\|z_n - z\| \\
&= (1 - a_n)\|w_n - z\| + a_n \ \text{dist}(Tw_n, z) \\
&\leq (1 - a_n)\|w_n - z\| + a_n \ H(Tw_n, Tz) \\
&\leq \|w_n - z\|. \qquad (3.2)
\end{aligned}$$

Hence from (3.1) and (3.2), we have

$$\|w_{n+1} - z\| \leq \|w_n - z\|. \tag{3.3}$$

Therefore, $\lim_{n\to\infty} \|w_n - z\|$ exists for each $z \in F(T)$. Let $\lim_{n\to\infty} \|w_n - z\| = a$ for some $a \geq 0$. Then if $a = 0$, we are done. Suppose that $a > 0$. Next, we show that $\lim_{n\to\infty} \text{dist}(Tw_n, w_n) = 0$. Taking lim sup on both sides of (3.2), we have

$$\lim_{n\to\infty} \sup \|s_n - z\| \leq a. \tag{3.4}$$

As,

$$\lim_{n\to\infty} \sup \|z_n - z\| \leq \lim_{n\to\infty} \sup H(Tw_n, Tz) \leq \lim_{n\to\infty} \sup \|w_n - z\| = a. \tag{3.5}$$

Moreover, $\lim_{n\to\infty} \|w_{n+1} - z\| = a$ means that

$$\begin{aligned}
a = \lim_{n\to\infty} \inf \|w_{n+1} - z\| = \lim_{n\to\infty} \inf \|v_n - z\| &\leq \lim_{n\to\infty} \inf \text{dist}(Ts_n, z) \\
&\leq \lim_{n\to\infty} \inf H(Ts_n, Tz) \\
&\leq \lim_{n\to\infty} \inf \|s_n - z\|. \qquad (3.6)
\end{aligned}$$

From (3.4) and (3.6), we have

$$\lim_{n\to\infty} \|s_n - z\| = a.$$

As,

$$\begin{aligned}
a &= \lim_{n\to\infty} \|s_n - z\| = \lim_{n\to\infty} \|(1 - a_n)w_n + a_n z_n - z\| \\
&= \lim_{n\to\infty} \|(1 - a_n)(w_n - z) + a_n(z_n - z)\|. \qquad (3.7)
\end{aligned}$$

Therefore from (3.5), (3.7), and Lemma 1.12, we have

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{3.8}$$

As $\text{dist}(Tw_n, w_n) \leq \|z_n - w_n\|$, we have, $\lim_{n\to\infty} \text{dist}(Tw_n, w_n) = 0$. Now, we prove that $\{w_n\}$ has a unique weak subsequential limit in $F(T)$. Let $\{w_{n_k}\}$ and $\{w_{n_j}\}$ be the subsequences of $\{w_n\}$ while z_1 and z_2 be the weak limits of $\{w_{n_k}\}$ and $\{w_{n_j}\}$

respectively. Since $(I-T)$ is demiclosed at zero, therefore using the fact $z_n \in Tw_n$ and equation (3.8), we obtain that $z_1 \in F(T)$. Similarly we can show that $z_2 \in F(T)$. Now, we show the uniqueness of weak limit. Let us suppose that $z_1 \neq z_2$. Since $w_{n_k} \rightharpoonup z_1$ and $z_1 \neq z_2$, by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|w_n - z_1\| = \liminf_{k\to\infty} \|w_{n_k} - z_1\| < \liminf_{k\to\infty} \|w_{n_k} - z_2\| &= \liminf_{j\to\infty} \|w_{n_j} - z_2\| \\ &< \liminf_{j\to\infty} \|w_{n_j} - z_1\| \\ &= \lim_{n\to\infty} \|w_n - z_1\|,\end{aligned}$$

which is a contradiction. Hence $\{w_n\}$ converges weakly to a fixed point of T. □

Now, we prove some strong convergence theorems involving generalized nonexpansive multi-valued mapping.

Theorem 3.2. *Let E be a Banach space and K a nonempty closed convex subset of E. Let $T : K \to P(K)$ be a multi-valued quasi-nonexpansive mapping and satisfies condition (E). Let $\{w_n\}$ be a sequence as defined by (2.2). If $F(T) \neq \emptyset$, then $\{w_n\}$ converges strongly to a fixed point of T if and only if* $\lim_{n\to\infty} \inf \operatorname{dist}(w_n, F(T)) = 0$.

Proof. The necessary part is evident. For the reverse part, let us suppose that

$$\liminf_{n\to\infty} \operatorname{dist}(w_n, F(T)) = 0.$$

Then by (3.3), we have

$$\|w_{n+1} - z\| \leq \|w_n - z\| \Rightarrow \operatorname{dist}(w_{n+1}, F(T)) \leq \operatorname{dist}(w_n, F(T)),$$

which implies that $\lim_{n\to\infty} \operatorname{dist}(w_n, F(T))$ exists. Therefore by hypothesis, we have $\lim_{n\to\infty} \operatorname{dist}(w_n, F(T)) = 0$. Now, we show that $\{w_n\}$ is a Cauchy sequence in K. Let $\epsilon > 0$. As $\lim_{n\to\infty} \operatorname{dist}(w_n, F(T)) = 0$, there exists a positive integer m such that for all $n \geq m$, we have $\operatorname{dist}(w_n, F(T)) < \frac{\epsilon}{4}$. In particular,

$$\inf\{\|w_m - z\| : z \in F(T)\} < \frac{\epsilon}{4}.$$

Therefore there exists $l \in F(T)$ such that $\|w_m - l\| < \frac{\epsilon}{2}$. Now for $n, p \geq m$, we have

$$\|w_{n+p} - w_n\| \leq \|w_{n+p} - l\| + \|w_n - l\| \leq 2\|w_m - l\| < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Hence $\{w_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E. Therefore it converges in K. Let $\lim_{n\to\infty} w_n = w$. Then by using condition (E), we have

$$\begin{aligned}\operatorname{dist}(w, Tw) &\leq \|w_n - w\| + \operatorname{dist}(w_n, Tw) \\ &\leq \|w_n - w\| + \mu \operatorname{dist}(w_n, Tw_n) + \|w_n - w\| \\ &\leq 2\|w_n - w\| + \mu \|z_n - w_n\| \to 0 \text{ as } n \to \infty \qquad \text{(by using (3.8))}.\end{aligned}$$

Thus $\operatorname{dist}(w, Tw) = 0$, which in turn implies that $w \in F(T)$. □

Theorem 3.3. *Theorem 3.2 also holds if condition (E) is replaced by condition (C).*

Proof. From Theorem 3.2, we conclude that the sequence $\{w_n\}$ converges to $w \in K$. Hence, by using condition (C) and Lemma 1.3, we have

$$\begin{aligned}\operatorname{dist}(w, Tw) &\leq \|w_n - w\| + \operatorname{dist}(w_n, Tw_n) + H(Tw_n, Tw)\\ &\leq \|w_n - w\| + \operatorname{dist}(w_n, Tw_n) + 2\operatorname{dist}(w_n, Tw_n) + \|w_n - w\|\\ &= 2\|w_n - w\| + 3\operatorname{dist}(w_n, Tw_n)\\ &\leq 2\|w_n - w\| + 3\|z_n - w_n\| \to 0 \text{ as } n \to \infty \qquad \text{(by using (3.8))}.\end{aligned}$$

Therefore $\operatorname{dist}(w, Tw) = 0$, which in turn implies that $w \in F(T)$. □

We now apply Theorem 3.2 to obtain our next result in a uniformly convex Banach space wherein T satisfies condition (I).

Theorem 3.4. *Let E be a uniformly convex Banach space and $K, T, F(T)$ and $\{w_n\}$ be as in Theorem 3.2. If T satisfies condition (I) and $F(T) \neq \emptyset$, then $\{w_n\}$ converges strongly to a fixed point of T.*

Proof. By Theorem 3.1, $\lim_{n\to\infty} \|w_n - z\|$ exists for all $z \in F(T)$. Let $\lim_{n\to\infty} \|w_n - z\| = a$, for some $a \geq 0$. If $a = 0$, then there is nothing to prove. Suppose $a > 0$. Then again from Theorem 3.1, $\|w_{n+1} - z\| \leq \|w_n - z\|$, which implies that, $\inf_{z\in F(T)} \|w_{n+1} - z\| \leq \inf_{z\in F(T)} \|w_n - z\|$, so that $\operatorname{dist}(w_{n+1}, F(T)) \leq \operatorname{dist}(w_n, F(T))$ and $\lim_{n\to\infty} \operatorname{dist}(w_n, F(T))$ exists. On using condition (I) and Theorem 3.1, we have,

$$\lim_{n\to\infty} h(\operatorname{dist}(w_n, F(T))) \leq \lim_{n\to\infty} \operatorname{dist}(w_n, Tw_n) = 0,$$

that is, $\lim_{n\to\infty} h(\operatorname{dist}(w_n, F(T))) = 0$. Since h is a nondecreasing function and $h(0) = 0$, it follows that $\lim_{n\to\infty} \operatorname{dist}(w_n, F(T)) = 0$. Now applying Theorem 3.2, we obtain the result. □

Theorem 3.5. *Theorem 3.4 also holds if condition (E) is replaced by condition (C).*

Proof. The proof of this theorem is same as that of Theorem 3.4. □

4. CONVERGENCE THEOREMS VIA ALGORITHM (2.3)

We start this section to approximate the common fixed points of generalized nonexpansive multi-valued mappings to prove some weak and strong convergence theorems by using iterative algorithm (2.3). In the sequel, $F = F(T) \cap F(S)$ denotes the set of common fixed points of mappings T and S.

Lemma 4.1. *Let K be a nonempty closed and convex subset of a uniformly convex Banach space E and $T, S : K \to P(K)$ two multi-valued quasi-nonexpansive mappings. Let $\{x_n\}$ be a sequence as defined by (2.3). If $F \neq \emptyset$ and*

$$\mathrm{dist}(x, Tx) \leq H(Tx, Sx) \quad \forall\, x \in K,$$

then $\lim_{n\to\infty} \mathrm{dist}(Tx_n, x_n) = 0 = \lim_{n\to\infty} \mathrm{dist}(Sx_n, x_n)$.

Proof. Let $z \in F$. Then from (2.3) we have

$$\|x_{n+1} - z\| = \|u_n - z\| = \mathrm{dist}(Ty_n, z) \leq H(Ty_n, Tz) \leq \|y_n - z\|, \tag{3.9}$$

and

$$\begin{aligned} \|y_n - z\| &= \|(1-a_n)v_n + a_n w_n - z\| \\ &\leq (1-a_n)\|v_n - z\| + a_n\|w_n - z\| \\ &= (1-a_n)\mathrm{dist}(Tx_n, z) + a_n\ \mathrm{dist}(Sx_n, z) \\ &\leq (1-a_n)H(Tx_n, Tz) + a_n\ H(Sx_n, Sz) \\ &\leq (1-a_n)\|x_n - z\| + a_n\|x_n - z\| \\ &= \|x_n - z\|. \qquad (3.10) \end{aligned}$$

Hence from (3.9) and (3.10), we have

$$\|x_{n+1} - z\| \leq \|x_n - z\|. \tag{3.11}$$

Therefore, $\lim_{n\to\infty} \|x_n - z\|$ exists for each $z \in F(T)$. Assume that $\lim_{n\to\infty} \|x_n - z\| = b$ for some $b \geq 0$. Then if $b = 0$, we are done. Suppose that $b > 0$. Next, we show that $\lim_{n\to\infty} \mathrm{dist}(Tx_n, x_n) = 0$. Taking limit as $n \to \infty$ on both sides of (3.10), we have

$$\lim_{n\to\infty} \|y_n - z\| \leq \lim_{n\to\infty} \|x_n - z\| = b. \tag{3.12}$$

Moreover, as $\lim_{n\to\infty} \|x_{n+1} - z\| = b$, from (3.9) we have

$$b = \lim_{n\to\infty} \|x_{n+1} - z\| \leq \lim_{n\to\infty} \|y_n - z\|. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\lim_{n\to\infty} \|y_n - z\| = b.$$

As,

$$\limsup_{n\to\infty} \|v_n - z\| \leq \limsup_{n\to\infty} H(Tx_n, Tz) \leq \limsup_{n\to\infty} \|x_n - z\| = b. \tag{3.14}$$

In the same way, we get

$$\limsup_{n\to\infty} \|w_n - z\| \leq b. \tag{3.15}$$

As,

$$b = \lim_{n\to\infty} \|y_n - z\| = \lim_{n\to\infty} \|(1-a_n)(v_n - z) + a_n(w_n - z)\|,$$

hence by using (3.14), (3.15) and by applying Lemma 1.12, we get,

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{3.16}$$

As $H(Tx_n, Sx_n) \leq \|v_n - w_n\|$, we have $\lim_{n\to\infty} H(Tx_n, Sx_n) = 0$. Now, we have

$$\operatorname{dist}(Tx_n, x_n) \leq H(Tx_n, Sx_n).$$

Taking limit as $n \to \infty$ on both sides, we get $\lim_{n\to\infty} \operatorname{dist}(Tx_n, x_n) = 0$. Again, we have for each $n \in \mathbb{N}$,

$$\operatorname{dist}(Sx_n, x_n) \leq H(Sx_n, Tx_n) + \operatorname{dist}(Tx_n, x_n),$$

which on taking limit as $n \to \infty$ follows, $\lim_{n\to\infty} \operatorname{dist}(Sx_n, x_n) = 0$. □

Remark 4.2. It is well known that every nonexpansive mapping satisfies conditions (C) and (E). Hence for the sake of simplicity, we present the following example of nonexpansive mappings with the common non empty fixed point set and satisfies the inequality

$$\operatorname{dist}(x, Tx) \leq H(Tx, Sx) \quad \forall\, x \in K.$$

Example 4.3. Let $E = \mathbb{R}$ and $K = [1, \infty)$. Let us define the mappings T and S by $T, S : K \to CB(K)$ by

$$Tx = \left[0, \frac{1+x}{2}\right], \quad Sx = \left[0, \frac{5-2x}{3}\right] \quad \forall\, x \in K.$$

Then obviously, S and T are nonexpansive mappings with the common fixed points 0 and 1 as follows: for $x, y \in K$,

$$H(Tx, Ty) = \max\left\{\left|\frac{1+x}{2} - \frac{1+y}{2}\right|, 0\right\} = \frac{1}{2}|x-y| \leq |x-y|.$$

In a similar way,

$$H(Sx, Sy) = \max\left\{\left|\frac{5-2x}{3} - \frac{5-2y}{3}\right|, 0\right\} = \frac{2}{3}|x-y| \leq |x-y|.$$

Now, for any $x, y \in K$,

$$\operatorname{dist}(x, Tx) = \operatorname{dist}\left(x, \left[0, \frac{1+x}{2}\right]\right) = \left|x - \frac{1+x}{2}\right| = \frac{1}{2}|x-1|,$$

and

$$H(Tx, Sx) = \max\left\{\left|\frac{1+x}{2} - \frac{5-2x}{3}\right|, 0\right\} = \left|\frac{1+x}{2} - \frac{5-2x}{3}\right| = \frac{7}{6}|x-1|,$$

that is, for all $x \in K$,

$$\operatorname{dist}(x, Tx) \leq H(Tx, Sx).$$

Theorem 4.4. *Let E be a uniformly convex Banach space satisfying Opial's condition and K, T, S and $\{x_n\}$ be same as in Lemma 4.1. If $F \neq \emptyset$, $(I - T)$ and $(I - S)$ are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of T and S.*

Proof. Let $z \in F$. Then as proved in Lemma 4.1, $\lim_{n\to\infty} \|x_n - z\|$ exists. Since E is a uniformly convex Banach space. Thus there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $z_1 \in K$. From Lemma 4.1, we have $\lim_{n\to\infty} \operatorname{dist}(Tx_{n_i}, x_{n_i}) = 0$ and $\lim_{n\to\infty} \operatorname{dist}(Sx_{n_i}, x_{n_i}) = 0$. Since $(I - T)$ and $(I - S)$ are demiclosed at zero, therefore $Sz_1 = z_1$. Similarly $Tz_1 = z_1$. Finally, we prove that $\{x_n\}$ converges weakly to z_1. Let on contrary that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $z_2 \in K$ and $z_1 \neq z_2$. Again in the same way, we can prove that $z_2 \in F$. Again from Lemma 4.1, $\lim_{n\to\infty} \|x_n - z_1\|$ and $\lim_{n\to\infty} \|x_n - z_2\|$ exist. Let $z_1 \neq z_2$. Then by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z_1\| = \lim_{i\to\infty} \inf \|x_{n_i} - z_1\| < \lim_{i\to\infty} \inf \|x_{n_i} - z_2\| &= \lim_{j\to\infty} \inf \|x_{n_j} - z_2\| \\ &< \lim_{j\to\infty} \inf \|x_{n_j} - z_1\| \\ &= \lim_{n\to\infty} \|x_n - z_1\|,\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a common fixed point of T and S. □

Theorem 4.5. *Let E be a Banach space and K a nonempty closed and convex subset of E. Let $T, S : K \to P(K)$ be two multi-valued quasi-nonexpansive mappings satisfying condition (E). Let $\{x_n\}$ be a sequence as defined by* (2.3)*. If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T and S if and only if $\lim_{n\to\infty} \inf \operatorname{dist}(x_n, F) = 0$.*

Proof. The first part is obvious. Let us suppose that $\lim_{n\to\infty} \inf \operatorname{dist}(x_n, F) = 0$. Then from (3.11), we have

$$\|x_{n+1} - z\| \le \|x_n - z\| \Rightarrow \operatorname{dist}(x_{n+1}, F) \le \operatorname{dist}(x_n, F),$$

which implies that $\lim_{n\to\infty} \operatorname{dist}(x_n, F)$ exists. Then on the similar lines of proof of Theorem 3.2, we can say that $\{x_n\}$ converges in K. Let $\lim_{n\to\infty} x_n = x$. Then by using condition (E) and Lemma 4.1, we have

$$\begin{aligned}\operatorname{dist}(x, Tx) &\le \|x_n - x\| + \operatorname{dist}(x_n, Tx) \\ &\le \|x_n - x\| + \mu \operatorname{dist}(x_n, Tx_n) + \|x_n - x\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

that is, $\operatorname{dist}(x, Tx) = 0$ that is, $x \in F(T)$. Similarly, by using condition (E), we have

$$\operatorname{dist}(x, Sx) \le 2\|x_n - x\| + \mu \operatorname{dist}(x_n, Sx_n).$$

On taking limit $n \to \infty$, we have $\operatorname{dist}(x, Sx) = 0$, as from Lemma 4.1, $\lim_{n\to\infty} \operatorname{dist}(x_n, Sx_n) = 0$. Therefore we have $x \in F(S)$, which implies that $x \in F(T) \cap F(S) = F$. □

Theorem 4.6. *Theorem* 4.5 *also holds if condition (E) is replaced by condition (C).*

Proof. From Theorem 4.5, we conclude that the sequence $\{x_n\}$ converges to $x \in K$. Hence, by using condition (C) and Lemma 1.3, we have

$$\begin{aligned}\operatorname{dist}(x, Tx) &\leq \|x_n - x\| + \operatorname{dist}(x_n, Tx_n) + H(Tx_n, Tx)\\ &\leq \|x_n - x\| + \operatorname{dist}(x_n, Tx_n) + 2\operatorname{dist}(x_n, Tx_n) + \|x_n - x\|\\ &= 2\|x_n - x\| + 3\operatorname{dist}(x_n, Tx_n) \to 0 \text{ as } n \to \infty \quad \text{(by Lemma 4.1).}\end{aligned}$$

Therefore $\operatorname{dist}(x, Tx) = 0$, that is, $x \in F(T)$. Similarly, we have $\operatorname{dist}(x, Sx) = 0$, that is, $x \in F(S)$, which implies that $x \in F(T) \cap F(S) = F$. □

Now by using Theorem 4.5, we obtain a strong convergence theorem of the iterative scheme (2.2) under condition (I').

Theorem 4.7. *Let E be a uniformly convex Banach space and K, T, S, F and $\{x_n\}$ be as in Theorem 4.5. If mappings T and S satisfy condition (I') and $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T and S.*

Proof. By Lemma 4.1, $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F$. Let $\lim_{n\to\infty} \|x_n - z\| = b$, for some $b \geq 0$. If $b = 0$, then there is nothing to prove. Suppose that $b > 0$. Then again from Lemma 4.1, $\|x_{n+1} - z\| \leq \|x_n - z\|$, which implies that, $\inf_{z\in F} \|x_{n+1} - z\| \leq \inf_{z\in F} \|x_n - z\|$, so that $\operatorname{dist}(x_{n+1}, F) \leq \operatorname{dist}(x_n, F)$ and $\lim_{n\to\infty} \operatorname{dist}(x_n, F)$ exists. On using condition (I') and Lemma 4.1, we have,

$$\begin{aligned}\lim_{n\to\infty} g(\operatorname{dist}(x_n, F(T))) &\leq \lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0,\\ \lim_{n\to\infty} g(\operatorname{dist}(x_n, F(S))) &\leq \lim_{n\to\infty} \operatorname{dist}(x_n, Sx_n) = 0,\end{aligned}$$

that is,

$$\lim_{n\to\infty} g(\operatorname{dist}(x_n, F(T) \cap F(S))) = 0, \text{ or } \lim_{n\to\infty} g(\operatorname{dist}(x_n, F)) = 0.$$

Since g is a nondecreasing function and $g(0) = 0$, it follows that $\lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0$. Now applying Theorem 4.5, we obtain the result. □

Remark 4.8. The rate of convergence of iterative algorithm (2.3) is faster than iterative algorithms (2.1) and (2.2) for contraction mappings as shown by the given proposition.

Proposition 4.9. *Let K be a nonempty closed and convex subset of a Banach space E. Let $T, S : K \to P(K)$ be multi-valued contraction mappings with Lipschitz constants k_1 and k_2 respectively where k_1, $k_2 < k < 1$, and a unique fixed point z. Define sequences $\{u_n\}$, $\{w_n\}$ and $\{x_n\}$ in K by (2.1), (2.2) and (2.3) respectively. Then we have the following:*

1. $\|u_{n+1} - z\| \leq [1 - a_n(1 - k)]^n \|u_1 - z\|$ *for all* $n \in \mathbb{N}$,
2. $\|w_{n+1} - z\| \leq k^n [1 - a_n(1 - k)]^n \|w_1 - z\|$ *for all* $n \in \mathbb{N}$,
3. $\|x_{n+1} - z\| \leq k^{2n} \|x_1 - z\|$ *for all* $n \in \mathbb{N}$.

Proof. Suppose that z is a common fixed point of mappings T and S. Then from iterative algorithm (2.1), we have

$$\begin{aligned}\|u_{n+1}-z\| &\le \|(1-a_n)(u_n-z)+a_n(t_n-z)\|\\ &\le (1-a_n)\|u_n-z\|+a_n\text{dist}(Tu_n,z)\\ &\le (1-a_n)\|u_n-z\|+a_nH(Tu_n,Tz)\\ &\le (1-a_n)\|u_n-z\|+a_nk_1\|u_n-z\|\\ &\le (1-a_n)\|u_n-z\|+a_nk\|u_n-z\|\\ &= [1-(1-k)a_n]\|u_n-z\|\\ &\quad\vdots\\ &\le [1-(1-k)a_n]^n\|u_1-z\|.\end{aligned}$$

Let $A_n=[1-(1-k)a_n]^n\|u_1-z\|$.

Now, from iterative algorithm (2.2), we have

$$\begin{aligned}\|w_{n+1}-z\|=\|v_n-z\| &= \text{dist}(Ts_n,z)\\ &\le H(Ts_n,Tz)\\ &\le k_1\|s_n-z\|\\ &\le k\|(1-a_n)(w_n-z)+a_n(z_n-z)\|\\ &\le k[(1-a_n)\|(w_n-z)\|+a_n\text{dist}(Tw_n,z)]\\ &\le k[(1-a_n)\|(w_n-z)\|+a_nH(Tw_n,Tz)]\\ &\le k[(1-a_n)\|(w_n-z)\|+a_nk\|w_n-z\|]\\ &= k[1-(1-k)a_n]\|w_n-z\|]\\ &\quad\vdots\\ &\le k^n[1-(1-k)a_n]^n\|w_1-z\|.\end{aligned}$$

Assume that $B_n=k^n[1-(1-k)a_n]^n\|w_1-z\|$.

By iterative algorithm (2.3), we have

$$\begin{aligned}\|x_{n+1}-z\|=\|u_n-z\| &= \text{dist}(Ty_n,z)\\ &\le H(Ty_n,Tz)\\ &\le k_1\|y_n-z\|\\ &= k_1\|(1-a_n)(v_n-z)+a_n(w_n-z)\|\\ &\le k_1[(1-a_n)\text{dist}(Tx_n,z)+a_n\text{dist}(Sx_n,z)]\\ &\le k_1[(1-a_n)H(Tx_n,Tz)+a_nH(Sx_n,Sz)]\\ &\le k_1[(1-a_n)k_1\|x_n-z\|+a_nk_2\|x_n-z\|]\end{aligned}$$

$$= k^2\|x_n - z\| \qquad (\text{as } k_1,\, k_2 < k)$$
$$\vdots$$
$$\leq k^{2n}\|x_1 - z\|.$$

Assume that $C_n = k^{2n}\|x_1 - z\|$.

Now,

$$\lim_{n\to\infty}\frac{C_n}{A_n} = \lim_{n\to\infty}\frac{k^{2n}\|x_1 - z\|}{[1-(1-k)a_n]^n\|u_1 - z\|} = \lim_{n\to\infty}\frac{k^{2n}}{[1-(1-k)a_n]^n} \times \lim_{n\to\infty}\frac{\|x_1 - z\|}{\|u_1 - z\|}.$$

Since $k < 1$,$\lim_{n\to\infty} k^{2n} = 0$ and as $a_n < 1$ with $\lim_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty}\frac{C_n}{A_n} = 0$. Thus $\{x_n\}$ converges faster than $\{u_n\}$ to z. Similarly,

$$\lim_{n\to\infty}\frac{C_n}{B_n} = \lim_{n\to\infty}\frac{k^{2n}\|x_1 - z\|}{k^n[1-(1-k)a_n]^n\|w_1 - z\|} = \lim_{n\to\infty}\frac{k^{n}}{[1-(1-k)a_n]^n} \times \lim_{n\to\infty}\frac{\|x_1 - z\|}{\|w_1 - z\|},$$

so that $\lim_{n\to\infty}\frac{C_n}{B_n} = 0$. Therefore $\{x_n\}$ converges faster than $\{w_n\}$ to z. □

REFERENCES

[1] A. Abkar and M. Eslamian, *Fixed Point Theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces*, Fixed Point Theory Appl., vol. 2010. ID 457935, 10 pages.

[2] A. Abkar and M. Eslamian, *A fixed point theorem for generalized nonexpansive multivalued mappings*, Fixed Point Theory, 12 (2011), 241–246.

[3] V. Berinde, *Picard iteration converges faster than Mann iteration for a class of quasi contractive operators*, Fixed point theory and applications, 2 (2004), 97–105.

[4] F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces, in : Nonlinear Functional Analysis* (Proc. Sympos. Pure Math., vol. 18, Part 2, Chicago, III., 1968), American Mathematical Society, Rhode Island, (1976), 1–308.

[5] E. L. Dozo, *Multivalued nonexpansive mappings and Opial's condition*, Proc. Amer. Math. Soc. 38 (1973) 286–292.

[6] M. Eslamian and A. Abkar, *Onestep iterative process for a finite family of multivalued mappings*, Math. Comput. Modell. 54 (2011) 105–111.

[7] J. García-Falset, E. Llorens-Fuster and T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Math. Anal. Appl. 375 (2011), 185–195.

[8] A. Kaewcharoen and B. Panyanak, *Fixed point theorems for some generalized multivalued nonexpansive mappings*, Nonlinear Anal. 74 (2011), 5578–5584.

[9] T. C. Lim, *A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach spaces*, Bull. Amer. Math. Soc. 80, 1123–1126 (1974).

[10] J. T. Markin, *Continuous dependence of fixed point sets*, Proc. Amer. Math. Soc. 38, 545–547 (1973).

[11] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73, 591–597 (1967).

[12] B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comp. Math. Appl. 54, 872–877 (2007).

[13] D. R. Sahu, *Applications of the S-iteration process to constrained minimization problems and split feasibility problems*, Fixed Point Theory, Vol. 12, No. 1 (2011), 187–204.

[14] K. P. R. Sastry and G. V. R. Babu, *Convergence of Ishikawa iterates for a multivalued mapping with a fixed point*, Czechoslovak Math J. 55, 817–826 (2005).

[15] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. 43, 153–159 (1991).

[16] A. Sharma and M. Imdad, *Approximating fixed points of generalized nonexpansive mappings by faster iteration schemes*, Advances in Fixed Point Theory, 4 (4), 605–623, (2014).

[17] A. Sharma, D. Bahuguna and M. Imdad, *Approximating fixed points of generalized nonexpansive mappings in CAT(k) spaces via modified S-iteration process*, Preprint.

[18] A. Sharma and D. Bahuguna, *On the convergence of a new iterative algorithm of three infinite families of generalized nonexpansive multi-valued mappings*, Preprint.

[19] A. Sharma and M. Imdad, *On an iterative process for generalized nonexpansive multi-valued mappings in Banach spaces*, Vietnam J. Math., 44, 777–787, (2016).

[20] C. Shiau, K. K. Tan, and C. S. Wong, *Quasi-nonexpansive multi-valued maps and selections*, Fund. Math. 87 (1975), 109–119.

[21] Y. Song and H. Wang, *Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces"* [Comp. Math. Appl. 54, 872–877 (2007)], Comp. Math. Appl. 55, 2999–3002 (2008).

[22] Y. Song and Y. J. Cho, *Some notes on Ishikawa iteration for multivalued mappings*, Bull. Korean Math. Soc. 48(3):575–584 (2011). doi:10.4134/BKMS.2011.48.3.575.

[23] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. 340 (2008), 1088–1095.