ATTRACTION AND MEAN CONVERGENCE THEOREMS FOR TWO COMMUTATIVE NONLINEAR MAPPINGS IN BANACH SPACES

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Dedicated to Professor Ravi Agarwal on the occasion of his 70th birthday

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ABSTRACT. In this paper, using the class of 2-generalized nonspreading mappings which was defined by [29] in a Banach space and covers 2-generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem in a Banach space. Then we prove a mean convergence theorem of Baillon’s type [2] without convexity for commutative 2-generalized nonspreading mappings in a Banach space.

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1. INTRODUCTION

Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. Let $T$ be a mapping of $C$ into $H$. Then we denote by $F(T)$ the set of fixed points of $T$ and by $A(T)$ the set of attractive points [27] of $T$, i.e.,

(i) $F(T) = \{ z \in C : Tz = z \}$;

(ii) $A(T) = \{ z \in H : \|Tx - z\| \leq \|x - z\|, \forall x \in C \}$.

We know from [27] that $A(T)$ is closed and convex. This property is important for proving mean convergence theorems. Such a concept of attractive points was defined in a Banach space; see [20]. A mapping $T : C \rightarrow H$ is said to be nonexpansive [4] if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Baillon [2] proved the first mean convergence theorem in a Hilbert space.
Theorem 1.1 ([2]). Let $C$ be a bounded, closed and convex subset of $H$ and let $T : C \to C$ be nonexpansive. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

This theorem for nonexpansive mappings has been extended to Banach spaces by many authors; see, for example, [3, 5]. On the other hand, in 2010, Kocourek, Takahashi and Yao [13] defined a broad class of nonlinear mappings in a Hilbert space: Let $H$ be a Hilbert space and let $C$ be a nonempty subset of $H$. A mapping $T : C \to H$ is called \textit{generalized hybrid} [13] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping $T$ is called $(\alpha, \beta)$-\textit{generalized hybrid}. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a $(1,0)$-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$ 

It is \textit{nonsparing} [17, 18] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$ 

It is also \textit{hybrid} [25] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$ 

In general, nonsparing and hybrid mappings are not continuous; see [10]. The mean convergence theorem by Baillon [2] for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [13]. Furthermore, Takahashi and Takeuchi [27] proved the following mean convergence theorem without convexity in a Hilbert space.

Theorem 1.2 ([27]). Let $H$ be a Hilbert space and let $C$ be a nonempty subset of $H$. Let $T$ be a generalized hybrid mapping from $C$ into itself. Assume that $\{T^n z\}$ for some $z \in C$ is bounded and define

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

for all $x \in C$ and $n \in \mathbb{N}$. Then $\{S_n x\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n \to \infty} P_{A(T)} T^n x$ and $P_{A(T)}$ is the metric projection of $H$ onto $A(T)$. 
Maruyama, Takahashi and Yao [21] also defined a more broad class of nonlinear mappings called 2-generalized hybrid which covers generalized hybrid mappings in a Hilbert space. Let $C$ be a nonempty subset of $H$ and let $T$ be a mapping of $C$ into $H$. A mapping $T : C \to H$ is 2-generalized hybrid [21] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$
\alpha_1 \|T^2 x - Ty\|^2 + \alpha_2 \|T x - Ty\|^2 + (1 - \alpha_1 - \alpha_2) \|x - Ty\|^2 \\
\leq \beta_1 \|T^2 x - y\|^2 + \beta_2 \|T x - y\|^2 + (1 - \beta_1 - \beta_2) \|x - y\|^2
$$

for all $x, y \in C$.

Recently, Hojo, Takahashi and Takahashi [6] proved attractive and mean convergence theorems without convexity for commutative 2-generalized hybrid mappings in a Hilbert space. These results generalize Takahashi and Takeuchi’s theorem (Theorem 1.2) and Kohsaka’s theorem [15] which is a mean convergence theorem for commutative $\lambda$-hybrid mappings in a Hilbert space.

In this paper, using the class of 2-generalized nonspreading mappings which was defined by [29] in a Banach space and covers 2-generalized hybrid mappings in a Hilbert space, we prove an attractive point theorem in a Banach space. This theorem generalizes Hojo, Takahashi and Takahashi’s attractive point theorem [6] in a Hilbert space. Then we prove a mean convergence theorem of Baillon’s type [2] without convexity for commutative 2-generalized nonspreading mappings in a Banach space. This result is a general mean convergence theorem which extends Baillon’s theorem (Theorem 1.1) to a Banach space.

2. PRELIMINARIES

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ be the topological dual space of $E$. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus $\delta$ of convexity of $E$ is defined by

$$
\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}
$$

for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let $C$ be a nonempty subset of a Banach space $E$. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of $T$. If $C$ is a nonempty, closed and convex subset of a strictly convex Banach space $E$ and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [11].
Let \( E \) be a Banach space. The duality mapping \( J \) from \( E \) into \( 2^{E^*} \) is defined by
\[
J x = \{ x^* \in E^* : \langle x, x^* \rangle = \| x \|^2 = \| x^* \|^2 \}
\]
for every \( x \in E \). Let \( U = \{ x \in E : \| x \| = 1 \} \). The norm of \( E \) is said to be Gâteaux differentiable if for each \( x, y \in U \), the limit
\[
\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}
\]
exists. In this case, \( E \) is called smooth. We know that \( E \) is smooth if and only if \( J \) is a single-valued mapping of \( E \) into \( E^* \). We also know that \( E \) is reflexive if and only if \( J \) is surjective, and \( E \) is strictly convex if and only if \( J \) is one-to-one. Therefore, if \( E \) is a smooth, strictly convex and reflexive Banach space, then \( J \) is a single-valued bijection. The norm of \( E \) is said to be uniformly Gâteaux differentiable if for each \( y \in U \), the limit (2.1) is attained uniformly for \( x \in U \). It is also said to be Fréchet differentiable if for each \( x \in U \), the limit (2.1) is attained uniformly for \( y \in U \). A Banach space \( E \) is called uniformly smooth if the limit (2.1) is attained uniformly for \( x, y \in U \). It is known that if the norm of \( E \) is uniformly Gâteaux differentiable, then \( J \) is uniformly norm to weak* continuous on each bounded subset of \( E \), and if the norm of \( E \) is Fréchet differentiable, then \( J \) is norm to norm continuous. If \( E \) is uniformly smooth, \( J \) is uniformly norm to norm continuous on each bounded subset of \( E \). For more details, see [23, 24].

**Lemma 2.1** ([23, 24]). Let \( E \) be a smooth Banach space and let \( J \) be the duality mapping on \( E \). Then \( \langle x - y, Jx - Jy \rangle \geq 0 \) for all \( x, y \in E \). Furthermore, if \( E \) is strictly convex and \( \langle x - y, Jx - Jy \rangle = 0 \), then \( x = y \).

Let \( E \) be a smooth Banach space. The function \( \phi : E \times E \to (-\infty, \infty) \) is defined by
\[
\phi(x, y) = \| x \|^2 - 2\langle x, Jy \rangle + \| y \|^2
\]
for \( x, y \in E \), where \( J \) is the duality mapping of \( E \); see [1] and [12]. We have from the definition of \( \phi \) that
\[
\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle
\]
for all \( x, y, z \in E \). From \( (\| x \| - \| y \|)^2 \leq \phi(x, y) \) for all \( x, y \in E \), we can see that \( \phi(x, y) \geq 0 \). Furthermore, we can obtain the following equality:
\[
2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w)
\]
for \( x, y, z, w \in E \). If \( E \) is additionally assumed to be strictly convex, then from Lemma 2.1 we have
\[
\phi(x, y) = 0 \iff x = y.
\]

The following lemma is in Xu [33].
Lemma 2.2 ([33]). Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and
\[
\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)
\]
for all $x, y \in B_r$ and $\lambda$ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Using Lemma 2.2, we have the following lemma by Kamimura and Takahashi [12].

Lemma 2.3 ([12]). Let $E$ be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and
\[
g(\|x - y\|) \leq \phi(x, y)
\]
for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let $E$ be a smooth Banach space. Let $C$ be a nonempty subset of $E$ and let $T$ be a mapping of $C$ into $E$. We denote by $A(T)$ the set of attractive points of $T$, i.e., $A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x), \forall x \in C\}$; see [20].

Lemma 2.4 ([20]). Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping from $C$ into $E$. Then $A(T)$ is a closed and convex subset of $E$.

Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Then a mapping $T : C \to E$ is called generalized nonexpansive [8] if $F(T) \neq \emptyset$ and
\[
\phi(Tx, y) \leq \phi(x, y)
\]
for all $x \in C$ and $y \in F(T)$; see also [32]. Let $D$ be a nonempty subset of a Banach space $E$. A mapping $R : E \to D$ is said to be sunny if
\[
R(Rx + t(x - Rx)) = Rx
\]
for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset $D$ of a smooth Banach space $E$ is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of $E$ if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) $R$ from $E$ onto $D$; see [8] for more details. The following results are in Ibaraki and Takahashi [8].

Lemma 2.5 ([8]). Let $C$ be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space $E$. Then the sunny generalized nonexpansive retraction from $E$ onto $C$ is uniquely determined.
Lemma 2.6 ([8]). Let $C$ be a nonempty closed subset of a smooth and strictly convex Banach space $E$ such that there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following hold:

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [16] proved the following results:

Lemma 2.7 ([16]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed subset of $E$. Then the following are equivalent:

(a) $C$ is a sunny generalized nonexpansive retract of $E$;
(b) $C$ is a generalized nonexpansive retract of $E$;
(c) $JC$ is closed and convex.

Lemma 2.8 ([16]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed sunny generalized nonexpansive retract of $E$. Let $R$ be the sunny generalized nonexpansive retraction from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following are equivalent:

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Ibaraki and Takahashi [9] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.9 ([9]). Let $E$ be a reflexive, strictly convex and smooth Banach space and let $T$ be a generalized nonexpansive mapping from $E$ into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.

The following theorem is proved by using Lemmas 2.7 and 2.9.

Lemma 2.10 ([9]). Let $E$ be a reflexive, strictly convex and smooth Banach space and let $T$ be a generalized nonexpansive mapping from $E$ into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of $E$.

Using Lemma 2.7, we also have the following result.

Lemma 2.11 ([26]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $\{C_i : i \in I\}$ be a family of sunny generalized nonexpansive retracts of $E$ such that $\cap_{i \in I} C_i$ is nonempty. Then $\cap_{i \in I} C_i$ is a sunny generalized nonexpansive retract of $E$.

Let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then, we denote by $\mu(f)$ the value of $\mu$ at
\[ f = (x_1, x_2, x_3, \ldots) \in l^\infty. \] Sometimes, we denote by \( \mu_n(x_n) \) the value \( \mu(f) \). A linear functional \( \mu \) on \( l^\infty \) is called a mean if \( \mu(e) = ||\mu|| = 1 \), where \( e = (1, 1, 1, \ldots) \). A mean \( \mu \) is called a Banach limit on \( l^\infty \) if \( \mu_n(x_{n+1}) = \mu_n(x_n) \). We know that there exists a Banach limit on \( l^\infty \). If \( \mu \) is a Banach limit on \( l^\infty \), then for \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \),

\[
\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.
\]

In particular, if \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \) and \( x_n \to a \in \mathbb{R} \), then we have \( \mu(f) = \mu_n(x_n) = a \). For the proof of existence of a Banach limit and its other elementary properties, see [23].

3. FIXED POINT THEOREMS

Let \( E \) be a smooth Banach space and let \( C \) be a nonempty subset of \( E \). Then a mapping \( T : C \to E \) is called 2-generalized nonscattering [29] if there exist \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R} \) such that

\[
\begin{align*}
\alpha_1 \phi(T^2x, Ty) + \alpha_2 \phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2) \phi(x, Ty) \\
+ \gamma_1 \{ \phi(Ty, T^2x) - \phi(Ty, x) \} + \gamma_2 \{ \phi(Ty, Tx) - \phi(Ty, x) \} \\
\leq \beta_1 \phi(T^2x, y) + \beta_2 \phi(Tx, y) + (1 - \beta_1 - \beta_2) \phi(x, y) \\
+ \delta_1 \{ \phi(y, T^2x) - \phi(y, x) \} + \delta_2 \{ \phi(y, Tx) - \phi(y, x) \}
\end{align*}
\]

for all \( x, y \in C \); see also [30]. Such a mapping is called \((\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)\)-generalized nonscattering. We know that a \((0, \alpha_2, 0, \beta_2, 0, \gamma_2, 0, \delta_2)\)-generalized nonscattering mapping is generalized nonscattering in the sense of [14]. We also know that a \((0, 1, 0, 1, 0, 1, 0, 0)\)-generalized nonscattering mapping is nonscattering in the sense of [18].

Now we prove an attractive point theorem for commutative 2-generalized nonscattering mappings in a Banach space. Before proving it, we prove the following result.

**Lemma 3.1.** Let \( E \) be a smooth, strictly convex and reflexive Banach space with the duality mapping \( J \) and let \( C \) be a nonempty subset of \( E \). Let \( S \) and \( T \) be mappings of \( C \) into itself. Let \( \{x_n\} \) be a bounded sequence of \( E \) and let \( \mu \) be a mean on \( l^\infty \). Suppose that

\[
\mu_n \phi(x_n, Sy) \leq \mu_n \phi(x_n, y) \text{ and } \mu_n \phi(x_n, Ty) \leq \mu_n \phi(x_n, y)
\]

for all \( y \in C \). Then \( A(S) \cap A(T) \) is nonempty. Additionally, if \( C \) is closed and convex and \( \{x_n\} \subset C \), then \( F(S) \cap F(T) \) is nonempty.

**Proof.** Using a mean \( \mu \) and a bounded sequence \( \{x_n\} \), we define a function \( g : E^* \to \mathbb{R} \) as follows:

\[
g(x^*) = \mu_n(x_n, x^*)
\]
for all \( x^* \in E^* \). Since \( \mu \) is linear, \( g \) is also linear. Furthermore, we have
\[
|g(x^*)| = |\mu_n(x_n, x^*)| \\
\leq \|\mu\| \sup_{n \in \mathbb{N}} |\langle x_n, x^* \rangle| \\
\leq \|\mu\| \sup_{n \in \mathbb{N}} \|x_n\|\|x^*\| \\
= \sup_{n \in \mathbb{N}} \|x_n\|\|x^*\|
\]
for all \( x^* \in E^* \). Then \( g \) is a linear and bounded real-valued function on \( E^* \). Since \( E \) is reflexive, there exists a unique element \( z \) of \( E \) such that
\[
g(x^*) = \mu_n(x_n, x^*) = \langle z, x^* \rangle
\]
for all \( x^* \in E^* \). From (2.3) we have that for \( y \in C \) and \( n \in \mathbb{N} \),
\[
\phi(x, y) = \phi(x_n, Sy) + \phi(Sy, y) + 2\langle x_n - Sy, JSy - Jy \rangle.
\]
So, we have that for \( y \in C \),
\[
\mu_n\phi(x_n, y) = \mu_n\phi(x_n, Sy) + \mu_n\phi(Sy, y) + 2\mu_n\langle x_n - Sy, JSy - Jy \rangle \\
= \mu_n\phi(x_n, Sy) + \phi(Sy, y) + 2\langle z - Sy, JSy - Jy \rangle.
\]
Since, by assumption, \( \mu_n\phi(x_n, Sy) \leq \mu_n\phi(x_n, y) \) for all \( y \in C \), we have
\[
\mu_n\phi(x_n, y) \leq \mu_n\phi(x_n, y) + \phi(Sy, y) + 2\langle z - Sy, JSy - Jy \rangle.
\]
This implies that
\[
0 \leq \phi(Sy, y) + 2\langle z - Sy, JSy - Jy \rangle.
\]
Using (2.4), we have that
\[
0 \leq \phi(Sy, y) + \phi(z, y) + \phi(Sy, Sy) - \phi(z, Sy) - \phi(Sy, y)
\]
and hence \( \phi(z, Sy) \leq \phi(z, y) \). This implies that \( z \) is an element of \( A(S) \). Similarly, we have that \( \phi(z, Ty) \leq \phi(z, y) \) and hence \( z \in A(T) \). Therefore we have \( z \in A(S) \cap A(T) \). Additionally, if \( C \) is closed and convex and \( \{x_n\} \subset C \), we have that \( z \in \text{co}\{x_n : n \in \mathbb{N}\} \subset C \). In fact, if \( z \notin C \), then there exists \( y^* \in E^* \) by the separation theorem [23] such that \( \langle z, y^* \rangle < \inf_{y \in C} \langle y, y^* \rangle \). So, from \( \{x_n\} \subset C \) we have
\[
\langle z, y^* \rangle < \inf_{y \in C} \langle y, y^* \rangle \leq \inf_{n \in \mathbb{N}} \langle x_n, y^* \rangle \leq \mu_n \langle x_n, y^* \rangle = \langle z, y^* \rangle.
\]
This is a contradiction. Then we have \( z \in C \). Since \( z \in A(S) \cap A(T) \) and \( z \in C \), we have that
\[
\phi(z, Sz) \leq \phi(z, z) = 0 \quad \text{and} \quad \phi(z, Tz) \leq \phi(z, z) = 0
\]
and hence \( \phi(z, Sz) = 0 \) and \( \phi(z, Tz) = 0 \). Since \( E \) is strictly convex, we have \( z \in F(S) \cap F(T) \). This completes the proof. \( \square \)
Using Lemma 3.1, we prove an attractive point theorem for commutative 2-generalized nonspreading mappings in a Banach space.

**Theorem 3.2.** Let $C$ be a nonempty subset of a smooth, strictly convex and reflexive Banach space $E$ and let $S$ and $T$ be commutative 2-generalized nonspreading mappings of $C$ into itself. Suppose that there exists an element $z \in C$ such that $\{S^{kT^l}z : k,l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if $C$ is closed and convex, then $F(S) \cap F(T)$ is nonempty.

**Proof.** Since $S$ is a 2-generalized nonspreading mapping of $C$ into itself, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that for all $x, y \in C$,

\[
\alpha_1 \phi(S^2x, Sy) + \alpha_2 \phi(Sx, Sy) + (1 - \alpha_1 - \alpha_2) \phi(x, Sy)
\]

\[
+ \gamma_1 \{\phi(Sy, S^2x) - \phi(Sy, x)\} + \gamma_2 \{\phi(Sy, Sx) - \phi(Sy, x)\}
\]

\[
\leq \beta_1 \phi(S^2x, y) + \beta_2 \phi(Sx, y) + (1 - \beta_1 - \beta_2) \phi(x, y)
\]

\[
+ \delta_1 \{\phi(y, S^2x) - \phi(y, x)\} + \delta_2 \{\phi(y, Sx) - \phi(y, x)\}.
\]

By assumption, we can take $z \in C$ such that $\{S^{kT^l}z : k,l \in \mathbb{N} \cup \{0\}\}$ is bounded. Replacing $x$ by $S^{kT^l}x$ in (3.2), we have that for any $y \in C$ and $k, l \in \mathbb{N} \cup \{0\}$,

\[
\alpha_1 \phi(S^{k+2T^l}z, Sy) + \alpha_2 \phi(S^{k+1T^l}z, Sy) + (1 - \alpha_1 - \alpha_2) \phi(S^{kT^l}z, Sy)
\]

\[
+ \gamma_1 \{\phi(Sy, S^{k+2T^l}z) - \phi(Sy, S^{kT^l}z)\} + \gamma_2 \{\phi(Sy, S^{k+1T^l}z) - \phi(Sy, S^{kT^l}z)\}
\]

\[
\leq \beta_1 \phi(S^{k+2T^l}z, y) + \beta_2 \phi(S^{k+1T^l}z, y) + (1 - \beta_1 - \beta_2) \phi(S^{kT^l}z, y)
\]

\[
+ \delta_1 \{\phi(y, S^{k+2T^l}z) - \phi(y, S^{kT^l}z)\} + \delta_2 \{\phi(y, S^{k+1T^l}z) - \phi(y, S^{kT^l}z)\}
\]

\[
= \beta_1 \{\phi(S^{k+2T^l}z, Sy) + \phi(Sy, y) + 2(S^{k+2T^l}z - Sy, JSy - Jy)\}
\]

\[
+ \beta_2 \{\phi(S^{k+1T^l}z, Sy) + \phi(Sy, y) + 2(S^{k+1T^l}z - Sy, JSy - Jy)\}
\]

\[
+ (1 - \beta_1 - \beta_2) \{\phi(S^{kT^l}z, Sy) + \phi(Sy, y) + 2(S^{kT^l}z - Sy, JSy - Jy)\}
\]

\[
+ \delta_1 \{\phi(y, S^{k+2T^l}z) - \phi(y, S^{kT^l}z)\} + \delta_2 \{\phi(y, S^{k+1T^l}z) - \phi(y, S^{kT^l}z)\}.
\]

This implies that

\[
0 \leq (\beta_1 - \alpha_1) \{\phi(S^{k+2T^l}z, Sy) - \phi(S^{kT^l}z, Sy)\}
\]

\[
+ (\beta_2 - \alpha_2) \{\phi(S^{k+1T^l}z, Sy) - \phi(S^{kT^l}z, Sy)\} + \phi(Sy, y)
\]

\[
+ 2(S^{kT^l}z - Sy + \beta_1(S^{k+2T^l}z - S^{kT^l}z) + \beta_2(S^{k+1T^l}z - S^{kT^l}z), JSy - Jy)
\]

\[
- \gamma_1 \{\phi(Sy, S^{k+2T^l}z) - \phi(Sy, S^{kT^l}z)\} - \gamma_2 \{\phi(Sy, S^{k+1T^l}z) - \phi(Sy, S^{kT^l}z)\}
\]

\[
+ \delta_1 \{\phi(y, S^{k+2T^l}z) - \phi(y, S^{kT^l}z)\} + \delta_2 \{\phi(y, S^{k+1T^l}z) - \phi(y, S^{kT^l}z)\}.
\]

Summing up these inequalities with respect to $k = 0, 1, \ldots, n$, we have

\[
0 \leq (\beta_1 - \alpha_1) \{\phi(S^{n+2T^l}z, Sy) + \phi(S^{n+1T^l}z, Sy)
\]

\[
- \phi(ST^l z, Sy) - \phi(T^l z, Sy)\}.
Furthermore, summing up these inequalities with respect to \( l = 0, 1, \ldots, n \), we have

\[
0 \leq (\beta_1 - \alpha_1) \sum_{l=0}^{n} \left\{ \phi(S^{n+2}T^l z, Sy) + \phi(S^{n+1}T^l z, Sy) \right\}
- \phi(ST^l z, Sy) - \phi(T^l z, Sy)

+ (\beta_2 - \alpha_2) \sum_{l=0}^{n} \left\{ \phi(S^{n+1}T^l z, Sy) - \phi(T^l z, Sy) \right\}
+ (n + 1)^2 \phi(Sy, y)

+ 2 \left\{ \sum_{l=0}^{n} \sum_{k=0}^{n} S^k T^l z + \beta_1 \sum_{l=0}^{n} (S^{n+2}T^l z + S^{n+1}T^l z - ST^l z - T^l z)
+ \beta_2 \sum_{l=0}^{n} (S^{n+1}T^l z - T^l z) - (n + 1)^2 Sy, JSy - Jy \right\}

- \gamma_1 \sum_{l=0}^{n} \left\{ \phi(Sy, S^{n+2}T^l z) + \phi(Sy, S^{n+1}T^l z) - \phi(Sy, ST^l z) - \phi(Sy, T^l z) \right\}

- \gamma_2 \sum_{l=0}^{n} \left\{ \phi(Sy, S^{n+1}T^l z) - \phi(Sy, T^l z) \right\}

+ \delta_1 \sum_{l=0}^{n} \left\{ \phi(y, S^{n+2}T^l z) + \phi(y, S^{n+1}T^l z) - \phi(y, ST^l z) - \phi(y, T^l z) \right\}

+ \delta_2 \sum_{l=0}^{n} \left\{ \phi(y, S^{n+1}T^l z) - \phi(y, T^l z) \right\}.
\]

Dividing by \((n + 1)^2\), we have

\[
0 \leq (\beta_1 - \alpha_1) \frac{1}{(n + 1)^2} \sum_{l=0}^{n} \left\{ \phi(S^{n+2}T^l z, Sy) + \phi(S^{n+1}T^l z, Sy) \right\}
- \phi(ST^l z, Sy) - \phi(T^l z, Sy)

+ (\beta_2 - \alpha_2) \frac{1}{(n + 1)^2} \sum_{l=0}^{n} \left\{ \phi(S^{n+1}T^l z, Sy) - \phi(T^l z, Sy) \right\}
+ (n + 1)^2 \phi(Sy, y).
\]
\begin{align*}
+ 2 \left( S_n z + \beta_1 \frac{1}{(n + 1)^2} \sum_{l=0}^{n} (S^{n+2} T^l z + S^{n+1} T^l z - S T^l z - T^l z) \\
+ \beta_2 \frac{1}{(n + 1)^2} \sum_{l=0}^{n} (S^{n+1} T^l z - T^l z) - S y, J S y - J y \right) \\
- \gamma_1 \frac{1}{(n + 1)^2} \sum_{l=0}^{n} \{ \phi(S y, S^{n+2} T^l z) + \phi(S y, S^{n+1} T^l z) \\
- \phi(S y, S T^l z) - \phi(S y, T^l z) \} \\
- \gamma_2 \frac{1}{(n + 1)^2} \sum_{l=0}^{n} \{ \phi(S y, S^{n+1} T^l z) - \phi(S y, T^l z) \} \\
+ \delta_1 \frac{1}{(n + 1)^2} \sum_{l=0}^{n} \{ \phi(y, S^{n+2} T^l z) + \phi(y, S^{n+1} T^l z) - \phi(y, S T^l z) - \phi(y, T^l z) \} \\
+ \delta_2 \frac{1}{(n + 1)^2} \sum_{l=0}^{n} \{ \phi(y, S^{n+1} T^l z) - \phi(y, T^l z) \},
\end{align*}

where $S_n z = \frac{1}{(n + 1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l z$. Since \{S^k T^l z\} is bounded by assumption, \{S_n z\} is bounded. Taking a Banach limit $\mu$ to both sides of this inequality, we have that

$$0 \leq \phi(S y, y) + 2 \mu_n \langle S_n z - S y, J S y - J y \rangle$$

and hence

$$0 \leq \phi(S y, y) + \mu_n \phi(S_n z, y) + \phi(S y, S y) - \mu_n \phi(S_n z, S y) - \phi(S y, y).$$

Thus, we have

$$\mu_n \phi(S_n z, S y) \leq \mu_n \phi(S_n z, y).$$

Similarly, replacing $S$ and $T$ by $T$ and $S$, respectively, we have

$$\mu_n \phi(S_n z, T y) \leq \mu_n \phi(S_n z, y).$$

Using Lemma 3.1, we have that $A(S) \cap A(T)$ is nonempty. Additionally, if $C$ is closed and convex, then $F(S) \cap F(T)$ is nonempty. \hfill \Box

Since commutative 2-generalized hybrid mappings in a Hilbert space are commutative 2-generalized nonspreading mappings in a Banach space, as a direct sequence of Theorem 3.2, we have the following theorem proved by Hojo, Takahashi and Takahashi [6] in a Hilbert space.

**Theorem 3.3** ([6]). Let $H$ be a Hilbert space, let $C$ be a nonempty subset of $H$ and let $S$ and $T$ be commutative 2-generalized hybrid mappings of $C$ into itself. Suppose that there exists an element $z \in C$ such that \{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\} is bounded. Then $A(S) \cap A(T)$ is nonempty. Additionally, if $C$ is closed and convex, then $F(S) \cap F(T)$ is nonempty.
4. NONLINEAR ERGODIC THEOREMS

Let $E$ be a smooth Banach space, let $C$ be a nonempty subset of $E$ and let $J$ be the duality mapping from $E$ into $E^*$. Observe that if $T : C \to E$ is a 2-generalized nonspreading mapping and $F(T) \neq \emptyset$, then

$$\phi(u, Ty) \leq \phi(u, y)$$

for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (3.1), we obtain that

$$\alpha_1 \phi(u, Ty) + \alpha_2 \phi(u, Ty) + (1 - \alpha_1 - \alpha_2) \phi(u, Ty)$$
$$+ \gamma_1 \{\phi(Ty, u) - \phi(Ty, u)\} + \gamma_2 \{\phi(Ty, u) - \phi(Ty, u)\}$$
$$\leq \beta_1 \phi(u, y) + \beta_2 \phi(u, y) + (1 - \beta_1 - \beta_2) \phi(u, y)$$
$$+ \delta_1 \{\phi(y, u) - \phi(y, u)\} + \delta_2 \{\phi(y, u) - \phi(y, u)\}.$$

So, we have that

(4.1) \hspace{1cm} \phi(u, Ty) \leq \phi(u, y)

for all $u \in F(T)$ and $y \in C$. Similarly, putting $y = u \in F(T)$ in (3.1), we obtain that for $x \in C,$

$$\alpha_1 \phi(T^2 x, u) + \alpha_2 \phi(Tx, u) + (1 - \alpha_1 - \alpha_2) \phi(x, u)$$
$$+ \gamma_1 \{\phi(u, T^2 x) - \phi(u, x)\} + \gamma_2 \{\phi(u, Tx) - \phi(u, x)\}$$
$$\leq \beta_1 \phi(T^2 x, u) + \beta_2 \phi(Tx, u) + (1 - \beta_1 - \beta_2) \phi(x, u)$$
$$+ \delta_1 \{\phi(u, T^2 x) - \phi(u, x)\} + \delta_2 \{\phi(u, Tx) - \phi(u, x)\}$$

and hence

$$(\alpha_1 - \beta_1)\{\phi(T^2 x, u) - \phi(x, u)\} + (\alpha_2 - \beta_2)\{\phi(Tx, u) - \phi(x, u)\}$$
$$+ (\gamma_1 - \delta_1)\{\phi(u, T^2 x) - \phi(u, x)\} + (\gamma_2 - \delta_2)\{\phi(u, Tx) - \phi(u, x)\} \leq 0.$$

If $\alpha_1 - \beta_1 = 0$, $\gamma_1 \leq \delta_1$, $\gamma_2 \leq \delta_2$ and $\alpha_2 > \beta_2$, then we have from (4.1) that

$$(\alpha_2 - \beta_2)\{\phi(Tx, u) - \phi(x, u)\} \leq (\delta_2 - \gamma_2)\{\phi(u, T^2 x) - \phi(u, x)\}$$
$$+ (\delta_2 - \gamma_2)\{\phi(u, Tx) - \phi(u, x)\} \leq 0.$$

So, we have that

(4.2) \hspace{1cm} \phi(Tx, u) \leq \phi(x, u)

for all $x \in C$ and $u \in F(T)$. This implies that $T$ is generalized nonexpansive in the sense of [8].

Now using the technique developed by [22] and [28], we can prove a mean convergence theorem without convexity for commutative 2-generalized nonspreading mappings in a Banach space. For proving this result, we need the following lemmas.
Lemma 4.1. Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $S$ and $T$ be commutative $2$-generalized nonspringing mappings of $C$ into itself. If $\{S^kT^l : k, l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$ is bounded and

$$S_n x = \frac{1}{(1+n)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$, then every weak cluster point of $\{S_n x\}$ is a point of $A(S) \cap A(T)$. Additionally, if $C$ is closed and convex, then every weak cluster point of $\{S_n x\}$ is a point of $F(S) \cap F(T)$.

Proof. Since $S : C \to C$ is $2$-generalized nonspringing, we have that for all $x, y \in C$, (3.2) holds. Since there exists $z \in C$ such that $\{S^kT^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded, $\{S^kT^l x : k, l \in \mathbb{N} \cup \{0\}\}$ for all $x \in C$ is bounded. Then as in the proof of Theorem 3.2, we have that for any $y \in C$

$$0 \leq (\beta_1 - \alpha_1) \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(S^{n+2}T^l x, S y) + \phi(S^{n+1}T^l x, S y) - \phi(S^l x, S y)\}$$

$$+ (\beta_2 - \alpha_2) \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(S^{n+1}T^l x, S y) - \phi(T^l x, S y)\} + \phi(S y, y)$$

$$+ 2 \left( \sum_{l=0}^{n} (S^{n+2}T^l x + S^{n+1}T^l x - ST^l x - T^l x) \right)$$

$$+ \frac{\beta_2}{(n+1)^2} \sum_{l=0}^{n} (S^{n}T^l x - T^l x - Sy, JSy - Jy)$$

$$- \gamma_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(S y, S^{n+2}T^l x) + \phi(S y, S^{n+1}T^l x) - \phi(S y, S^l x)\}$$

$$- \phi(S y, ST^l x) - \phi(S y, T^l x)\}$$

$$- \frac{\gamma_2}{(n+1)^2} \sum_{l=0}^{n} \{\phi(S y, S^{n+1}T^l x) - \phi(S y, T^l x)\}$$

$$+ \frac{\delta_1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(y, S^{n+2}T^l x) + \phi(y, S^{n+1}T^l x) - \phi(y, ST^l x) - \phi(y, T^l x)\}$$

$$+ \frac{\delta_2}{(n+1)^2} \sum_{l=0}^{n} \{\phi(y, S^{n+1}T^l x) - \phi(y, T^l x)\}.$$

Since $\{S^kT^l x\}$ is bounded, $\{S_n x\}$ is bounded. Thus we have a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ such that $\{S_{n_i} x\}$ converges weakly to a point $u \in E$. Letting $n_i \to \infty$, we obtain

$$0 \leq \phi(S y, y) + 2(u - Sy, JSy - Jy).$$
Using (2.4), we have that
\[ 0 \leq \phi(Sy, y) + \phi(u, y) + \phi(Sy, Sy) - \phi(u, Sy) - \phi(Sy, y) \]
and hence
\[ \phi(u, Sy) \leq \phi(u, y). \]
This implies that \( u \) is an element of \( A(S) \). Similarly, we have that
\[ \phi(u, Ty) \leq \phi(u, y). \]
and hence \( u \in A(T) \). Therefore we have \( u \in A(S) \cap A(T) \). Additionally, if \( C \) is closed and convex, we have that \( \{S_n z \} \subset C \) and then
\[ u \in \overline{\sigma} \{S_n x : n \in \mathbb{N} \} \subset C. \]
Since \( u \in A(S) \cap A(T) \) and \( u \in C \), we have that
\[ \phi(u, Su) \leq \phi(u, u) = 0 \quad \text{and} \quad \phi(u, Tu) \leq \phi(u, u) = 0 \]
and hence
\[ \phi(u, Su) = 0 \quad \text{and} \quad \phi(u, Tu) = 0. \]
Since \( E \) is strictly convex, we have \( u \in F(S) \cap F(T) \). This completes the proof. \( \square \)

Let \( E \) be a smooth Banach space. Let \( C \) be a nonempty subset of \( E \) and let \( T \) be a mapping of \( C \) into \( E \). We denote by \( B(T) \) the set of skew-attractive points of \( T \), i.e., \( B(T) = \{ z \in E : \phi(Tx, z) \leq \phi(x, z), \forall x \in C \} \). The following result was proved by Lin and Takahashi [20].

**Lemma 4.2** ([20]). Let \( E \) be a smooth Banach space and let \( C \) be a nonempty subset of \( E \). Let \( T \) be a mapping from \( C \) into \( E \). Then \( B(T) \) is closed.

Let \( E \) be a smooth, strictly convex and reflexive Banach space and let \( C \) be a nonempty subset of \( E \). Let \( T \) be a mapping of \( C \) into \( E \). Define a mapping \( T^* \) as follows:
\[ T^* x^* = JT J^{-1} x^*, \quad \forall x^* \in JC, \]
where \( J \) is the duality mapping on \( E \) and \( J^{-1} \) is the duality mapping on \( E^* \). A mapping \( T^* \) is called the duality mapping of \( T \); see also [31] and [7]. It is easy to show that if \( T \) is a mapping of \( C \) into itself, then \( T^* \) is a mapping of \( JC \) into itself. In fact, for \( x^* \in JC \), we have \( J^{-1} x^* \in C \) and hence \( TJ^{-1} x^* \in C \). So, we have
\[ T^* x^* = JT J^{-1} x^* \in JC. \]
Then, \( T^* \) is a mapping of \( JC \) into itself. Using Lemma 2.4, we have the following result.
Lemma 4.3 ([20]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty subset of $E$. Let $T$ be a mapping of $C$ into $E$ and let $T^*$ be the duality mapping of $T$. Then, the following hold:

1. $JB(T) = A(T^*)$;
2. $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Let $D = \{(k, l) : k, l \in \mathbb{N} \cup \{0\}\}$. Then $D$ is a directed set by the binary relation:

$(k, l) \leq (i, j)$ if $k \leq i$ and $l \leq j$.

Theorem 4.4. Let $E$ be a uniformly convex Banach space with a Fréchet differentiable norm and let $C$ be a nonempty subset of $E$. Let $S, T : C \to C$ be commutative 2-generalized nonspreading mappings such that $\{S^kT^lz : k, l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$ is bounded, $A(S) = B(S)$ and $A(T) = B(T)$. Let $R$ be the sunny generalized nonexpansive retraction of $E$ onto $B(S) \cap B(T)$. Then, for any $x \in C$,

$$S_nx = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^kT^lx$$

converges weakly to an element $q$ of $A(S) \cap A(T)$, where $q = \lim_{(k,l) \in D} RS^kT^lx$.

Proof. We have from Theorem 3.2 that $A(S) \cap A(T) = B(S) \cap B(T)$ is nonempty. We know from Lemmas 2.11, 4.2 and 4.3 that $B(S) \cap B(T)$ is closed, and

$$J(B(S) \cap B(T)) = JB(S) \cap JB(T)$$

is closed and convex. So, from Lemma 2.5 and Lemma 2.7 there exists the sunny generalized nonexpansive retraction $R$ of $E$ onto $B(S) \cap B(T)$. From Lemma 2.8, this retraction $R$ is characterized by

$$Rx = \arg\min_{u \in B(S) \cap B(T)} \phi(x, u).$$

We also know from Lemma 2.6 that

$$0 \leq \langle v - Rv, JRv - Ju \rangle, \quad \forall u \in B(S) \cap B(T), \ v \in C.$$ 

Adding up $\phi(Rv, u)$ to both sides of this inequality, we have

$$\phi(Rv, u) \leq \phi(Rv, u) + 2 \langle v - Rv, JRv - Ju \rangle
\quad (4.3)
= \phi(Rv, u) + \phi(v, u) + \phi(Rv, Rv) - \phi(v, Rv) - \phi(Rv, u)
= \phi(v, u) - \phi(v, Rv).$$

Since $\phi(Sz, u) \leq \phi(z, u)$ and $\phi(Tz, u) \leq \phi(z, u)$ for any $u \in B(S) \cap B(T)$ and $z \in C$, it follows that for any $(k, l), (i, j) \in D$ with $(k, l) \leq (i, j)$,

$$\phi(S^iT^jx, RS^iT^jx) \leq \phi(S^iT^jx, RS^kT^lx)$$
Hence the net \( \phi(S^kT^lx, RS^kT^lx) \) is nonincreasing. Putting \( u = RS^kT^lx \) and \( v = S^jx \) with \((k,l) \leq (i,j)\) in (4.3), we have from Lemma 2.3 that

\[
\begin{align*}
\phi(S^kT^lx, RS^kT^lx) & \leq \phi(S^kT^lx, RS^kT^lx) \\
& \leq \phi(S^jx, RS^kT^lx) - \phi(S^jx, RS^kT^lx) \\
& \leq \phi(S^kT^lx, RS^kT^lx) - \phi(S^jx, RS^kT^lx),
\end{align*}
\]

where \( g \) is a strictly increasing, continuous and convex real-valued function with \( g(0) = 0 \). From the properties of \( g \), \( \{RS^kT^lx\} \) is a Cauchy net; see [19]. Therefore \( \{RS^kT^lx\} \) converges strongly to a point \( q \in B(S) \cap B(T) \). Next, consider a fixed \( x \in C \) and an arbitrary subsequence \( \{S_nx\} \) of \( \{S_nx\} \) which converges weakly to a point \( v \). From the proof of Lemma 4.1, we know that \( v \in A(S) \cap A(T) = B(S) \cap B(T) \).

Rewriting the characterization of the retraction \( R \), we have that for any \( u \in B(S) \cap B(T) \),

\[
0 \leq \left( S^kT^lx - RS^kT^lx, JRS^kT^lx - Ju \right)
\]

and hence

\[
\begin{align*}
\langle S^kT^lx - RS^kT^lx, Ju - Jq \rangle & \leq \langle S^kT^lx - RS^kT^lx, JRS^kT^lx - Jq \rangle \\
& \leq \|S^kT^lx - RS^kT^lx\| \cdot \|JRS^kT^lx - Jq\| \\
& \leq K \|JRS^kT^lx - Jq\|,
\end{align*}
\]

where \( K \) is an upper bound for \( \|S^kT^lx - RS^kT^lx\| \). Summing up these inequalities for \( k = 0, 1, \ldots, n \) and \( l = 0, 1, \ldots, n \) and dividing by \((n + 1)^2\), we arrive to

\[
\begin{align*}
\left( S_nx - \frac{1}{(n + 1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} RS^kT^lx, Ju - Jq \right) & \leq K \frac{1}{(n + 1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \|JRS^kT^lx - Jq\|,
\end{align*}
\]

where \( S_nx = \frac{1}{(n + 1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^kT^lx \). Letting \( n_i \to \infty \) and remembering that \( J \) is continuous, we get

\[
\langle v - q, Ju - Jq \rangle \leq 0.
\]

This holds for any \( u \in B(S) \cap B(T) \). Therefore \( Rv = q \). But because \( v \in B(S) \cap B(T) \), we have \( v = q \). Thus the sequence \( \{S_nx\} \) converges weakly to the point \( q \in A(S) \cap A(T) \).

Using Theorem 4.4, we obtain the following theorems.

**Theorem 4.5.** Let \( E \) be a uniformly convex Banach space with a Fréchet differentiable norm. Let \( S, T : E \to E \) be commutative \((\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2)\) and \((\alpha'_1, \alpha'_2, \beta'_1, \beta'_2, \gamma'_1, \gamma'_2, \delta'_1, \delta'_2)\)-generalized nonspraying mappings such that \( \alpha_1 - \beta_1 = 0 \), \( \gamma_1 \leq \delta_1 \), \( \gamma_2 \leq \delta_2 \), \( \alpha_2 > \beta_2 \) and \( \alpha'_1 - \beta'_1 = 0 \), \( \gamma'_1 \leq \delta'_1 \), \( \gamma'_2 \leq \delta'_2 \), \( \alpha'_2 > \beta'_2 \), respectively.
Assume that \( \{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\} \) for some \( z \in C \) is bounded. Let \( R \) be the sunny generalized nonexpansive retraction of \( E \) onto \( F(S) \cap F(T) \). Then, for any \( x \in E \),

\[
S_n x = \frac{1}{(n + 1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x
\]

converges weakly to an element \( q \) of \( F(S) \cap F(T) \), where \( q = \lim_{(k,l) \in D} R S^k T^l x \).

**Proof.** Since \( \{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\} \) for some \( z \in C \) is bounded, we have that \( A(S) \cap A(T) = F(S) \cap F(T) \) is nonempty. We also know that \( \alpha_2 > \beta_2 \) together with \( \alpha_1 - \beta_1 = 0, \gamma_1 \leq \delta_1 \) and \( \gamma_2 \leq \delta_2 \) implies that

\[
\phi(Sx, u) \leq \phi(x, u)
\]

for all \( x \in E \) and \( u \in F(S) \). Similarly, \( \alpha'_2 > \beta'_2 \) together with \( \alpha'_1 - \beta'_1 = 0, \gamma'_1 \leq \delta'_1 \) and \( \gamma'_2 \leq \delta'_2 \) implies that

\[
\phi(Tx, v) \leq \phi(x, v)
\]

for all \( x \in E \) and \( v \in F(T) \). Thus, we have that \( F(S) = B(S) \) and \( F(T) = B(T) \). Therefore, we have the desired result from Theorem 4.4.

**Theorem 4.6** ([6]). Let \( H \) be a Hilbert space and let \( C \) be a nonempty subset of \( H \). Let \( S \) and \( T \) be commutative 2-generalized hybrid mappings of \( C \) into itself such that \( \{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\} \) for some \( z \in C \) is bounded. Let \( P \) be the metric projection of \( H \) onto \( A(S) \cap A(T) \). Then, for any \( x \in C \),

\[
S_n x = \frac{1}{(n + 1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x
\]

converges weakly to an element \( q \) of \( A(S) \cap A(T) \), where \( q = \lim_{(k,l) \in D} P S^k T^l x \). In particular, if \( C \) is closed and convex, \( \{S_n x\} \) converges weakly to an element \( q \) of \( F(S) \cap F(T) \).

**Proof.** We have from Theorem 3.2 that \( A(S) \cap A(T) \) is nonempty. We also have that \( A(S) = B(S) \) and \( A(T) = B(T) \). Since \( A(S) \cap A(T) \) is a nonempty, closed and convex subset of \( H \), there exists the metric projection of \( H \) onto \( A(S) \cap A(T) \). In a Hilbert space, the metric projection of \( H \) onto \( A(S) \cap A(T) \) is equivalent to the sunny generalized nonexpansive retraction of \( H \) onto \( A(S) \cap A(T) \). On the other hand, commutative 2-generalized hybrid mappings \( S, T : C \to C \) are commutative 2-generalized nonspreading mappings. So, we have the desired result from Theorem 4.6. Furthermore, if \( C \) is closed and convex, we have that \( q \in F(S) \cap F(T) \) and then \( \{S_n x\} \) converges weakly to \( q \in F(S) \cap F(T) \).

**Remark** We do not know whether a mean convergence theorem of Baillon’s type for nonspreading mappings in a Banach space holds or not.
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