Dynamic Systems and Applications 26 (2017) 383-394

LIMIT OF INVERSE SYSTEMS AND COINCIDENCE PRINCIPLES IN FRÉCHET SPACE

DONAL O'REGAN

School of Mathematics, Statistics and Applied Mathematics,
National University of Ireland, Galway, Ireland

ABSTRACT. Using the notion of Φ-essential or Φ-epi maps we present a variety of coincidence principles for multimaps defined on subsets of Fréchet spaces.

AMS (MOS) Subject Classification. 54H25, 47H10.

1. INTRODUCTION

Applicable coincidence principles for set valued maps defined on subsets of Fréchet spaces are presented in this paper. The idea is to use recent coincidence principles in the literature [1, 3, 6, 7, 8] for maps defined on Banach spaces and view our Fréchet space E as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (here $N = \{1, 2, \dots\}$); see [1, 2, 5] and the references therein. We use maps F_n and Φ_n defined on subsets of E_n whose coincidence points satisfy some closure property which guarantee that our original operators F and Φ have a coincidence point. We now recall some coincidence results [3, 6, 7] established in the literature.

Let E be a normal topological space and U an open subset of E. We will consider classes **A** and **B** of maps.

Definition 1.1. We say $F \in A(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E and $\overline{U}$ denotes the closure of U in E.

<u>Fix</u> a $\Phi \in B(\overline{U}, E)$.

Definition 1.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 1.3. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_1 = F$, $H_0 = G$ and $\big\{x \in \overline{U} : \Phi(x) \cap H(x, t) \neq \emptyset$ for some $t \in [0, 1]\big\}$ is closed; here $H_t(x) = H(x, t)$.

Received May ?, 2017

Definition 1.4. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

In [6] we established the following result.

Theorem 1.5. *Let E be a normal topological space, U an open subset of E, $G, F \in A_{\partial U}(\overline{U}, E)$ and F is Φ-essential in $A_{\partial U}(\overline{U}, E)$. Suppose $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Then there exists a $x \in U$ with $\Phi(x) \cap F(x) \neq \emptyset$.*

Remark 1.6. Suppose we change Definition 1.4 as follows: Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$ (in this case we need to add an extra condition in Definition 1.3, namely: if $\mu : \overline{U} \to [0,1]$ is any continuous map with $\mu(\partial U) =$ then

$$\big\{x \in \overline{U} : \Phi(x) \cap H(x, t\mu(x)) \neq \emptyset \text{ for some } t \in [0,1]\big\}$$

is closed). Then once again Theorem 1.5 is true (see [3]).

In [6] we also discussed Φ-epi maps.

Definition 1.7. We say $F \in B_\Phi(\overline{U}, E)$ if $F \in B(\overline{U}, E)$ and $F(x) \subseteq \Phi(x)$ for $x \in \partial U$.

Definition 1.8. A map $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi in $A_{\partial U}(\overline{U}, E)$ if for every map $G \in B_\Phi(\overline{U}, E)$ there exists $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Theorem 1.9. *Let E be a normal topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi in $A_{\partial U}(\overline{U}, E)$, $G \in B(\overline{U}, E)$ and assume the following conditions hold:*

$$(1.1) \qquad \begin{cases} \mu(\cdot)G(\cdot) + (1-\mu(\cdot))\Phi(\cdot) \in B(\overline{U}, E) \text{ for any} \\ \text{continuous map } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \end{cases}$$

and

$$(1.2) \qquad \begin{cases} \{x \in \overline{U} : F(x) \cap [tG(x) + (1-t)\Phi(x)] \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{is closed and does not intersect } \partial U. \end{cases}$$

Then there exists $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Other results can be found in [6]. In fact we could consider more general classes of maps. Consider the classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of maps.

Definition 1.10. We say $F \in D(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$.

Definition 1.11. We say $F \in CB(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{B}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F.

Fix a $\Phi \in CB(\overline{U}, E)$.

Definition 1.12. We say $F \in CB_{\Phi}(\overline{U}, E)$ if $F \in CB(\overline{U}, E)$ and $F(x) \subseteq \Phi(x)$ for $x \in \partial U$.

Definition 1.13. Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is $C\Phi$-epi in $A_{\partial U}(\overline{U}, E)$ if for any map $G \in CB_{\Phi}(\overline{U}, E)$ and any selection $\Psi \in D(\overline{U}, E)$ of G there exists $x \in U$ with $F(x) \cap \Psi(x) \neq \emptyset$.

In [7] we established the following result (for other results see also [7]).

Theorem 1.14. *Let E be a normal topological vector space, U an open subset of E, $G \in CB(\overline{U}, E)$, $F \in A_{\partial U}(\overline{U}, E)$ is $C\Phi$-epi in $A_{\partial U}(\overline{U}, E)$ and suppose*

$$(1.3) \qquad \begin{cases} \mu(\cdot)G(\cdot) + (1-\mu(\cdot))\Phi(\cdot) \in CB(\overline{U}, E) \text{ for any} \\ \text{continuous map } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0. \end{cases}$$

For any selection $\Lambda \in D(\overline{U}, E)$ of G and any selection $\phi \in D(\overline{U}, E)$ of Φ assume

$$(1.4) \qquad \begin{cases} K = \{x \in \overline{U} : F(x) \cap [t\Lambda(x) + (1-t)\phi(x)] \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{is closed and } K \text{ does not intersect } \partial U \end{cases}$$

and

$$(1.5) \qquad \begin{cases} \mu(\cdot)\Lambda(\cdot) + (1-\mu(\cdot))\phi(\cdot) \in D(\overline{U}, E) \text{ for any continuous} \\ \text{map } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \text{ and } \mu(K) = 1. \end{cases}$$

Then there exists $x \in U$ with $F(x) \cap \Lambda(x) \neq \emptyset$ (so $\emptyset \neq F(x) \cap \Lambda(x) \subseteq F(x) \cap G(x)$).

Remark 1.15. It is also possible to consider Φ-essential maps using the classes **A**, **B** and **D**; we refer the reader to [8].

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall \alpha, \beta \in I, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [4] $\cap_{\alpha \in I} E_\alpha$).

2. COINCIDENCE THEORY IN FRÉCHET SPACES

We now present an approach to establishing coincidence points based on projective limits (see [4]). Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, \dots\}$. We assume that the family of seminorms satisfies

$$(2.1) \qquad |x|_1 \leq |x|_2 \leq |x|_3 \leq \cdots \text{ for every } x \in E.$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x,r) = \{y \in E : |x-y|_n \leq r \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \text{ iff } |x-y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\mu_{m,k} = \mu_{n,k}$ if $n \leq m \leq k$ and $\mu_n = \mu_{n,m}\mu_m$ if $n \leq m$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii). In many applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$\begin{cases} E_1 \supseteq E_2 \supseteq \cdots \text{ and for each } n \in N, \\ |j_n\mu_{n,n+1}j_{n+1}^{-1}x|_n \leq |x|_{n+1} \forall x \in E_{n+1} \end{cases} \tag{2.4}$$

(here we use the notation from [4] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [4]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\mu_{n,m}j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\mu_n(X)$, and we let $\overline{X_n}$, int X_n and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$\text{pseudo-int}(X) = \{x \in X : j_n\mu_n(x) \in \overline{X_n}\backslash\partial X_n \text{ for every } n \in N\}.$$

The set X is pseudo-open if $X = \text{pseudo-int}(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x,r) = \{y \in E_n : |x-y|_n \leq r\}$.

Remark 2.2. If X is pseudo-open then for every $n \in N$ we claim that X_n is an open subset of E_n. Fix $n \in N$. We show $X_n = \text{int } X_n$. To see this note $X_n \subseteq \overline{X_n} \backslash \partial X_n$ since if $y \in X_n$ then there exists $x \in X$ with $y = j_n \mu_n(x)$ and this together with $X = \text{pseudo-int } X$ yields $j_n \mu_n(x) \in \overline{X_n} \backslash \partial X_n$ i.e. $y \in \overline{X_n} \backslash \partial X_n$. In addition notice

$$\overline{X_n} \backslash \partial X_n = (\text{int } X_n \cup \partial X_n) \backslash \partial X_n = \text{int } X_n \backslash \partial X_n = \text{int } X_n$$

since $\text{int } X_n \cap \partial X_n = \emptyset$. Consequently

$$X_n \subseteq \overline{X_n} \backslash \partial X_n = \text{int } X_n, \quad \text{so } X_n = \text{int } X_n.$$

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume for each $n \in N$ and $x \in M$ that $j_n \mu_n F(x)$ is closed. Let $n \in N$ and $M_n = j_n \mu_n(M)$. Since we first consider Volterra type operators we assume (note this assumption is only needed in Theorem 2.3, Theorem 2.6 and Theorem 2.8)

$$\text{if } x, y \in M \text{ with } |x - y|_n = 0 \text{ then } H_n(Fx, Fy) = 0; \tag{2.5}$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume for $n \in N$ if $x, y \in M$ with $j_n \mu_n x = j_n \mu_n y$ then $j_n \mu_n F x = j_n \mu_n F y$ and of course here we do not need to assume that $j_n \mu_n F(x)$ is closed for each $n \in N$ and $x \in M$). Now (2.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n \mu_n(x)$ and we let

$$F_n y = j_n \mu_n F x$$

(we could of course call it Fy since it is clear in the situation we use it); note $F_n : M_n \to C(E_n)$ and note if there exists a $z \in M$ with $y = j_n \mu_n(z)$ then $j_n \mu_n F x = j_n \mu_n F z$ from (2.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In our next three results we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

Our first result is motivated by Volterra type operators.

Theorem 2.3. *Let E and E_n be as described above, U a pseudo-open subset of E and $F : U \to 2^E$, $G : U \to 2^E$ and $\Phi : U \to 2^E$. Also assume for each $n \in N$ and $x \in U$ that $j_n \mu_n F(x)$, $j_n \mu_n G(x)$ and $j_n \mu_n \Phi(x)$ are closed and in addition for each $n \in N$ that $F_n : \overline{U_n} \to 2^{E_n}$, $G_n : \overline{U_n} \to 2^{E_n}$ and $\Phi_n : \overline{U_n} \to 2^{E_n}$ are as described above. Suppose the following conditions are satisfied:*

$$\begin{cases} \text{for each } n \in N, F_n, G_n \in A_{\partial U_n}(\overline{U_n}, E_n), \Phi_n \in B(\overline{U_n}, E_n) \\ \text{and } G_n \text{ is } \Phi_n\text{-essential in } A_{\partial U_n}(\overline{U_n}, E_n) \end{cases} \tag{2.6}$$

$$\text{for each } n \in N, G_n \cong F_n \text{ in } A_{\partial U_n}(\overline{U_n}, E_n) \tag{2.7}$$

$$\text{(2.8)} \qquad \begin{cases} \textit{for each } n \in \{2,3,\dots\} \textit{ if } y \in U_n \textit{ is such} \\ \textit{that } F_n(y) \cap \Phi_n(y) \neq \emptyset \textit{ in } E_n \textit{ then} \\ j_k \mu_{k,n} j_n^{-1}(y) \in U_k \textit{ for } k \in \{1,\dots,n-1\} \end{cases}$$

and

$$\text{(2.9)} \qquad \begin{cases} \textit{for every } k \in N \textit{ and any sequence } \{y_n\}_{n \in N_{k-1}} \textit{ with } y_n \in U_n \\ \textit{and } F_k(j_k \mu_{k,n} j_n^{-1} y_n) \cap \Phi_k(j_k \mu_{k,n} j_n^{-1} y_n) \neq \emptyset \textit{ on } E_k \textit{ there} \\ \textit{exists a subsequence } N_k \subseteq \{k+1, k+2, \dots\}, N_k \subseteq N_{k-1} \\ \textit{for } k \in \{1,2,\dots\}, N_0 = N, \textit{ and a } z_k \in \overline{U_k} \textit{ with} \\ j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k \textit{ in } E_k \textit{ as } n \to \infty \textit{ in } N_k \textit{ and} \\ F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset \textit{ on } E_k. \end{cases}$$

Then there exists $x \in E$ with $F(x) \cap \Phi(x) \neq \emptyset$ in E; here $x = (z_k)$ where $z_k \in U_k$ for each $k \in N$.

Proof. For each $n \in N$, from Theorem 1.5 there exists $y_n \in U_n$ with $F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset$ in E_n. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in U_1$ and $j_1 \mu_{1,k} j_k^{-1}(y_k) \in U_1$ for $k \in N \backslash \{1\}$ from (2.8). Fix $n \in N$. There exists a $x \in E$ with $y_n = j_n \mu_n(x)$ so

$$j_n \mu_n F(x) \cap j_n \mu_n \Phi(x) \neq \emptyset \text{ on } E_n. \tag{2.10}$$

We now claim

$$F_1(j_1 \mu_{1,n} j_n^{-1} y_n) \cap \Phi_1(j_1 \mu_{1,n} j_n^{-1} y_n) \neq \emptyset \text{ on } E_1. \tag{2.11}$$

To see this note on E_1 that

$$\begin{aligned} F_1(j_1 \mu_{1,n} j_n^{-1} y_n) \cap \Phi_1(j_1 \mu_{1,n} j_n^{-1} y_n) &= F_1(j_1 \mu_{1,n} j_n^{-1} j_n \mu_n(x)) \\ &\cap \Phi_1(j_1 \mu_{1,n} j_n^{-1} j_n \mu_n(x)) \\ &= F_1(j_1 \mu_{1,n} \mu_n(x)) \\ &\cap \Phi_1(j_1 \mu_{1,n} \mu_n(x)) \\ &= F_1(j_1 \mu_1(x)) \cap \Phi_1(j_1 \mu_1(x)) \\ &= j_1 \mu_1 F(x) \cap j_1 \mu_1 \Phi(x) \\ &= j_1 \mu_{1,n} j_n^{-1} j_n \mu_n F(x) \\ &\cap j_1 \mu_{1,n} j_n^{-1} j_n \mu_n \Phi(x) \\ &\neq \emptyset \end{aligned}$$

from (2.10). We can do this for each $n \in N$ so (2.11) holds for each $n \in N$. Now (2.9) guarantees that there is a subsequence $N_1 \subseteq \{2,3,\dots\}$ and a $z_1 \in \overline{U_1}$ with $j_1 \mu_{1,n} j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in N_1 and $F_1(z_1) \cap \Phi_1(z_1) \neq \emptyset$ on E_1. Also note $z_1 \in U_1$ since $F_1 \in A_{\partial U_1}(\overline{U_1}, E_1)$.

Now $j_2\mu_{2,n}j_n^{-1}(y_n) \in U_2$ for $n \in N_1$ from (2.8). Note also (argument similar to the above) for $n \in N_1$ that

$$F_2(j_2\mu_{2,n}j_n^{-1}y_n) \cap \Phi_2(j_2\mu_{2,n}j_n^{-1}y_n) \neq \emptyset \text{ on } E_2.$$

Now (2.9) guarantees that there is a subsequence $N_2 \subseteq \{3, 4, \dots\}$ of N_1 and a $z_2 \in \overline{U_2}$ with $j_2\mu_{2,n}j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in N_2 and $F_2(z_2) \cap \Phi_2(z_2) \neq \emptyset$ on E_2. Also note $z_2 \in U_2$ since $F_2 \in A_{\partial U_2}(\overline{U_2}, E_2)$. Notice from (2.4) and the uniqueness of limits that $j_1\mu_{1,2}j_2^{-1}z_2 = z_1$ in E_1 since $N_2 \subseteq N_1$ (note $j_1\mu_{1,n}j_n^{-1}(y_n) = j_1\mu_{1,2}j_2^{-1}j_2\mu_{2,n}j_n^{-1}(y_n)$ for $n \in N_2$). Proceed inductively to obtain subsequences of integers

$$N_1 \supseteq N_2 \supseteq \cdots, \quad N_k \subseteq \{k+1, k+2, \dots\}$$

and $z_k \in \overline{U_k}$ with $j_k\mu_{k,n}j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in N_k and $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ on E_k. Also note $z_k \in U_k$ since $F_k \in A_{\partial U_k}(\overline{U_k}, E_k)$, and $j_k\mu_{k,k+1}j_{k+1}^{-1}z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$.

Fix $k \in N$. Now $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\mu_{k,k+1}j_{k+1}^{-1}z_{k+1} = j_k\mu_{k,k+1}j_{k+1}^{-1}j_{k+1}\mu_{k+1,k+2}j_{k+2}^{-1}z_{k+2} \\ &= j_k\mu_{k,k+2}j_{k+2}^{-1}z_{k+2} = \cdots = j_k\mu_{k,m}j_m^{-1}z_m = \pi_{k,m}z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in U_k$ for each $k \in N$. Now for each $k \in N$, $j_k\mu_k(y) = z_k$ in E_k, and $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ in E_k (i.e. $j_k\mu_kF(y) \cap j_k\mu_k\Phi(y) \neq \emptyset$ in E_k). Thus $F(y) \cap \Phi(y) \neq \emptyset$ in E. □

Remark 2.4. We can remove the map G and assumptions (2.6) and (2.7) in Theorem 2.3 if instead we assume:

$$(2.12) \qquad \begin{cases} \text{for each } n \in N, F_n \in A_{\partial U_n}(\overline{U_n}, E_n), \Phi_n \in B(\overline{U_n}, E_n) \text{ and} \\ \text{there exists } y_n \in U_n \text{ with } F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset \text{ in } E_n. \end{cases}$$

Remark 2.5. If we assume for each $n \in N$ that $F_n : \overline{U_n} \to 2^{E_n}$ and $\Phi_n : \overline{U_n} \to 2^{E_n}$ are upper semicontinuous with nonempty compact values then automatically $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ on E_k is true in (2.9). To see this let k, N_k, $\{y_n\}$ and z_k be as in (2.9). Let $w_n \in F_k(j_k\mu_{k,n}j_n^{-1}y_n)$ and $w_n \in \Phi_k(j_k\mu_{k,n}j_n^{-1}y_n)$ for $n \in N_k$. Now since F_k is upper semicontinuous with nonempty compact values then [9] guarantees that there exists $w_k^\star \in F_k(z_k)$ and a subsequence (w_m) of (w_n) with $w_m \to w_k^\star$. The upper semicontinuity of the map Φ_k together with $w_m \to w_k^\star$ and $w_m \in \Phi_k(j_k\mu_{k,n}j_n^{-1}y_m)$ implies $w_k^\star \in \Phi_k(z_k)$. Thus $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ on E_k.

Theorem 2.6. *Let E and E_n be as described above, U a pseudo-open subset of E and $F : U \to 2^E$, $G : U \to 2^E$ and $\Phi : U \to 2^E$. Also assume for each $n \in N$ and $x \in U$ that $j_n\mu_nF(x)$, $j_n\mu_nG(x)$ and $j_n\mu_n\Phi(x)$ are closed and in addition for each*

$n \in N$ that $F_n : \overline{U_n} \to 2^{E_n}$, $G_n : \overline{U_n} \to 2^{E_n}$ and $\Phi_n : \overline{U_n} \to 2^{E_n}$ are as described above. Suppose the following conditions are satisfied:

$$\left\{ \begin{array}{l} \text{for each } n \in N, F_n \in A_{\partial U_n}(\overline{U_n}, E_n), G_n \in B(\overline{U_n}, E_n), \\ \Phi_n \in B(\overline{U_n}, E_n) \text{ and } F_n \text{ is } \Phi_n\text{-epi in } A_{\partial U_n}(\overline{U_n}, E_n) \end{array} \right. \tag{2.13}$$

$$\left\{ \begin{array}{l} \text{for each } n \in N, \mu_n(\cdot)G_n(\cdot) + (1-\mu_n(\cdot))\Phi_n(\cdot) \in B(\overline{U_n}, E_n) \\ \text{for any continuous map } \mu_n : \overline{U_n} \to [0,1] \text{ with } \mu_n(\partial U_n) = 0 \end{array} \right. \tag{2.14}$$

$$\left\{ \begin{array}{l} \{x \in \overline{U_n} : F_n(x) \cap [tG_n(x) + (1-t)\Phi_n(x)] \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{is closed (in } E_n\text{) and does not intersect } \partial U_n \text{ (for each } n \in N\text{)} \end{array} \right. \tag{2.15}$$

$$\left\{ \begin{array}{l} \text{for each } n \in \{2,3,\dots\} \text{ if } y \in U_n \text{ is such} \\ \text{that } F_n(y) \cap G_n(y) \neq \emptyset \text{ in } E_n \text{ then} \\ j_k \mu_{k,n} j_n^{-1}(y) \in U_k \text{ for } k \in \{1,\dots,n-1\} \end{array} \right. \tag{2.16}$$

and

$$\left\{ \begin{array}{l} \text{for every } k \in N \text{ and any sequence } \{y_n\}_{n \in N_{k-1}} \text{ with } y_n \in U_n \\ \text{and } F_k(j_k \mu_{k,n} j_n^{-1} y_n) \cap G_k(j_k \mu_{k,n} j_n^{-1} y_n) \neq \emptyset \text{ on } E_k \text{ there} \\ \text{exists a subsequence } N_k \subseteq \{k+1, k+2, \dots\}, N_k \subseteq N_{k-1} \\ \text{for } k \in \{1,2,\dots\}, N_0 = N, \text{ and a } z_k \in \overline{U_k} \text{ with} \\ j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k \text{ and} \\ F_k(z_k) \cap G_k(z_k) \neq \emptyset \text{ on } E_k. \end{array} \right. \tag{2.17}$$

Then there exists $x \in E$ with $F(x) \cap G(x) \neq \emptyset$ in E; here $x = (z_k)$ where $z_k \in U_k$ for each $k \in N$.

Proof. For each $n \in N$, from Theorem 1.9 there exists $y_n \in U_n$ with $F_n(y_n) \cap G_n(y_n) \neq \emptyset$ in E_n. The same argument as in Theorem 2.3 guarantees the result. □

Remark 2.7. There is an analogue of Remark 2.5 for Theorem 2.6.

We can obtain a more general version of Theorem 2.6 if we use Theorem 1.14.

Theorem 2.8. *Let E and E_n be as described above, U a pseudo-open subset of E and $F : U \to 2^E$, $G : U \to 2^E$ and $\Phi : U \to 2^E$. Also assume for each $n \in N$ and $x \in U$ that $j_n \mu_n F(x)$, $j_n \mu_n G(x)$ and $j_n \mu_n \Phi(x)$ are closed and in addition for each $n \in N$ that $F_n : \overline{U_n} \to 2^{E_n}$, $G_n : \overline{U_n} \to 2^{E_n}$ and $\Phi_n : \overline{U_n} \to 2^{E_n}$ are as described above. Suppose the following conditions are satisfied:*

$$\left\{ \begin{array}{l} \text{for each } n \in N, F_n \in A_{\partial U_n}(\overline{U_n}, E_n), G_n \in CB(\overline{U_n}, E_n), \\ \Phi_n \in CB(\overline{U_n}, E_n) \text{ and } F_n \text{ is } C\Phi_n\text{-epi in } A_{\partial U_n}(\overline{U_n}, E_n) \end{array} \right. \tag{2.18}$$

$$\left\{ \begin{array}{l} \text{for each } n \in N, \mu_n(\cdot)G_n(\cdot) + (1-\mu_n(\cdot))\Phi_n(\cdot) \in CB(\overline{U_n}, E_n) \\ \text{for any continuous map } \mu_n : \overline{U_n} \to [0,1] \text{ with } \mu_n(\partial U_n) = 0 \end{array} \right. \tag{2.19}$$

and

$$(2.20)\quad \begin{cases} \text{for each } n \in N \text{ and any selection } \Lambda_n \in D(\overline{U_n}, E_n) \text{ of } G_n \\ \text{and any selection } \phi_n \in D(\overline{U_n}, E_n) \text{ of } \Phi_n \text{ assume} \\ K_n = \{x \in \overline{U_n} : F_n(x) \cap [t\Lambda_n(x) + (1-t)\phi_n(x)] \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{is closed (in } E_n\text{) and does not intersect } \partial U_n \text{ and} \\ \mu_n(\cdot)\Lambda_n(\cdot) + (1-\mu_n(\cdot))\phi_n(\cdot) \in D(\overline{U_n}, E_n) \text{ for any continuous} \\ \text{map } \mu_n : \overline{U_n} \to [0,1] \text{ with } \mu_n(\partial U_n) = 0 \text{ and } \mu_n(K_n) = 1. \end{cases}$$

Also suppose (2.16) *and* (2.17) *hold. Then there exists* $x \in E$ *with* $F(x) \cap G(x) \neq \emptyset$ *in* E*; here* $x = (z_k)$ *where* $z_k \in U_k$ *for each* $k \in N$*.*

Proof. For each $n \in N$, from Theorem 1.14 there exists $y_n \in U_n$ with $F_n(y_n) \cap G_n(y_n) \neq \emptyset$ in E_n. The same argument as in Theorem 2.3 guarantees the result. □

Remark 2.9. It is also possible to obtain a more general version of Theorem 2.3 using **A**, **B** and **D** maps via Remark 1.15.

Our next result is motivated by Urysohn type operators.

Theorem 2.10. *Let* E *and* E_n *be as described above,* U *a pseudo-open subset of* E *and* $F : Y \to 2^E$, $G : Y \to 2^E$ *and* $\Phi : Y \to 2^E$ *with* $U \subseteq Y$ *and* $\overline{U_n} \subseteq Y_n$ *for each* $n \in N$*. Also for each* $n \in N$ *assume there exist* $F_n : \overline{U_n} \to 2^{E_n}$, $G_n : \overline{U_n} \to 2^{E_n}$ *and* $\Phi_n : \overline{U_n} \to 2^{E_n}$ *and suppose* (2.6), (2.7) *and* (2.8) *hold. In addition assume the following conditions hold:*

$$(2.21)\quad \begin{cases} \text{for any sequence } \{y_n\}_{n\in N} \text{ with } y_n \in U_n \\ \text{and } F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset \text{ in } E_n \text{ for } n \in N \text{ and} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \dots\}, N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2, \dots\}, N_0 = N, \text{ and a } z_k \in \overline{U_k} \text{ with} \\ j_k\mu_{k,n}j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k \end{cases}$$

and

$$(2.22)\quad \begin{cases} \text{if there exists a } w \in Y \text{ and a sequence } \{y_n\}_{n\in N} \\ \text{with } y_n \in U_n \text{ and } F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence } S \subseteq \\ \{k+1, k+2, \dots\} \text{ of } N \text{ with } j_k\mu_{k,n}j_n^{-1}(y_n) \to j_k\mu_k(w) \\ \text{in } E_k \text{ as } n \to \infty \text{ in } S, \text{ then } F(w) \cap \Phi(w) \neq \emptyset \text{ in } E. \end{cases}$$

Then there exists $x \in E$ *with* $F(x) \cap \Phi(x) \neq \emptyset$ *in* E*; here* $x = (z_k)$ *where* $z_k \in \overline{U_k}$ *for each* $k \in N$*.*

Proof. For each $n \in N$, from Theorem 1.5 there exists $y_n \in U_n$ with $F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset$ in E_n. Lets look at $\{y_n\}_{n\in N}$. Notice $y_1 \in U_1$ and $j_1\mu_{1,k}j_k^{-1}(y_k) \in U_1$

for $k \in \{2, 3, \dots\}$ from (2.8). Now (2.21) with $k = 1$ guarantees that there exists a subsequence $N_1 \subseteq \{2, 3, \dots\}$ and a $z_1 \in \overline{U_1}$ with $j_1 \mu_{1,n} j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in N_1. Look at $\{y_n\}_{n \in N_1}$. Now $j_2 \mu_{2,n} j_n^{-1}(y_n) \in U_2$ for $k \in N_1$ from (2.8). Now (2.21) with $k = 2$ guarantees that there exists a subsequence $N_2 \subseteq \{3, 4, \dots\}$ of N_1 and a $z_2 \in \overline{U_2}$ with $j_2 \mu_{2,n} j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in N_2. Note from (2.4) and the uniqueness of limits that $j_1 \mu_{1,2} j_2^{-1} z_2 = z_1$ in E_1 since $N_2 \subseteq N_1$ (note $j_1 \mu_{1,n} j_n^{-1}(y_n) = j_1 \mu_{1,2} j_2^{-1} j_2 \mu_{2,n} j_n^{-1}(y_n)$ for $n \in N_2$). Proceed inductively to obtain subsequences of integers

$$N_1 \supseteq N_2 \supseteq \cdots, N_k \subseteq \{k+1, k+2, \dots\}$$

and $z_k \in \overline{U_k}$ with $j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in N_k. Note $j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$.

Fix $k \in N$. Note

$$\begin{aligned} z_k &= j_k \mu_{k,k+1} j_{k+1}^{-1} z_{k+1} = j_k \mu_{k,k+1} j_{k+1}^{-1} j_{k+1} \mu_{k+1,k+2} j_{k+2}^{-1} z_{k+2} \\ &= j_k \mu_{k,k+2} j_{k+2}^{-1} z_{k+2} = \cdots = j_k \mu_{k,m} j_m^{-1} z_m = \pi_{k,m} z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in \overline{U_k}$ for each $k \in N$. Also since $F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset$ in E_n for $n \in N_k$ and $j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k = j_k \mu_k(y)$ in E_k as $n \to \infty$ in N_k we have from (2.22) that $F(y) \cap \Phi(y) \neq \emptyset$ in E. □

Remark 2.11. If we replace (2.21) with

$$\begin{cases} \text{for any sequence } \{y_n\}_{n \in N} \text{ with } y_n \in U_n \\ \text{and } F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset \text{ in } E_n \text{ for } n \in N \text{ and} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \dots\}, N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2, \dots\}, N_0 = N, \text{ and a } z_k \in U_k \text{ with} \\ j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k, \end{cases}$$

then Y is the statement of Theorem 2.10 can be replaced by U.

Remark 2.12. There is an analogue of Theorem 2.10 if we replace (2.6), (2.7) and (2.8) with (2.13), (2.14), (2.15) and (2.16). Also Φ_n in (2.21) and (2.22) is replaced by G_n and we conclude that there exists $x \in E$ with $F(x) \cap G(x) \neq \emptyset$ in E; here $x = (z_k)$ where $z_k \in \overline{U_k}$ for each $k \in N$.

REFERENCES

[1] H. H. Alsulami and D. O'Regan, Coincidence points for multimaps defined on subsets of Fréchet spaces, *Dynamic Systems and Applications*, 25:393–408, 2016.

[2] G. Gabor, L. Gorniewicz and M. Slosarski, Generalized topological essentiality and coincidence points of multivalued maps, *Set-Valued Anal.*, 17,1–19, 2009.

[3] M. Jleli, D. O'Regan and B. Samet, Topological coincidence principles, *Journal of Nonlinear Science and Applications*, to appear.

[4] L. V. Kantorovich and G. P. Akilov, *Functional analysis in normed spaces*, Pergamon Press, Oxford, 1964.

[5] D. O'Regan, An essential map approach for multimaps defined on closed subsets of Fréchet spaces, *Applicable Analysis*, 85:503–513, 2006.

[6] D. O'Regan, Generalized coincidence theory for set valued maps, *Journal of Nonlinear Science and Applications*, 10:855–664, 2017.

[7] D. O'Regan, Generalized Φ-epi maps and topological coincidence principles, *Fixed Point Theory*, to appear.

[8] D. O'Regan, Essential maps and coincidence principles for general classes of maps, *Filomat*, to appear.

[9] C. H. Su and V. M. Sehgal, Some fixed point theorems for condensing multifunctions in locally convex spaces, *Proc. Amer. Math. Soc.*, 50:15–154, 1975.