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SOLUTIONS TO THE BLASIUS AND SAKIADIS PROBLEMS VIA A NEW SINC-COLLOCATION APPROACH

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ABSTRACT. Two well-known nonlinear laminar boundary layer problems, the Blasius and the Sakiadis problems, are treated by a new Sinc-Collocation approach based on first derivative interpolation. Even in the presence of singularities or infinite domains, the Sinc-Collocation method is known to exhibit exponential convergence, resulting in highly accurate solutions. The new method is suggested over the customary Sinc approaches due to decreased sensitivity to numerical errors. It is shown that this approach is an accurate and efficient tool in solving these nonlinear boundary value problems.

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1. INTRODUCTION

Sinc Numerical methods have begun to be studied more closely as they show exponential convergence in the presence of singularities and on infinite domains. The typical strategy in using the Sinc-method to solve boundary value problems (BVPs) is to start with Sinc interpolation of the unknown function and to obtain its first and higher derivatives through successive differentiation. However, this approach has a basic drawback as it is well-known that numerical differentiation is highly sensitive to numerical errors [8].

As shown in [2], a new approach to solving linear boundary value problems shows very promising results in terms of high rate of convergence and decreased numerical errors. This alternative method is advantageous over the conventional method as it decreases the sensitivity to numerical errors of the solution present in numerical differentiation.

In this paper, we have applied this new approach to two nonlinear boundary value problems; the Blasius Equation and Sakiadis Equation. We interpolate the first derivative of each equation using Sinc numerical methods and obtain the desired solution through numerical integration of the interpolation. All higher order derivatives are

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found by differentiating the interpolation. Non-homogeneous boundary value conditions are met by means of a suitable transformation of the function that transform them into a homogeneous case.

2. LAMINAR BOUNDARY LAYER PROBLEMS

Blasius [6] flow is a boundary layer flow induced over a static, impermeable plate placed in a fluid stream moving with constant velocity. If the plate moves with constant velocity in a static fluid, then the Sakiadis flow [23] occurs.

Considering the thermal radiation term in the energy equation, the governing equations of motion and heat transfer for the classical Blasius flat-plate flow problem can be summarized by the following boundary value problem [10]

$$\text{continuity eq:}\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$\text{momentum eq:}\ \frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}=\nu\,\frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

$$\text{energy eq:}\ \frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\,\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho C_p}\frac{\partial q_r}{\partial y}, \tag{2.3}$$

subject to the boundary conditions

i) Blasius flat-plate flow problem:

$$u=v=0, \quad \text{at } y=0, \tag{2.4}$$

$$u\to U, \quad \text{as } y\to\infty. \tag{2.5}$$

ii) classical Sakiadis flat-plate flow problem:

$$u=U, \quad v=0, \quad \text{at } y=0, \tag{2.6}$$

$$u\to 0, \quad \text{as } y\to\infty, \tag{2.7}$$

$$T=T_w \quad \text{at } y=0, \tag{2.8}$$

$$T=T_\infty \quad \text{as } y\to\infty. \tag{2.9}$$

where u and v are the velocity components along the x-axis and y-axis, ν is the kinematic viscosity, k is the thermal conductivity, C_p is the specific heat capacity of the fluid at constant pressure, ρ is the density, q_r is the radiative heat flux in the y-direction, T_w the constant temperature of the wall, T_∞ the constant temperature of the fluid, and U is a constant velocity of free stream or that of a moving plate.

Using the Rosseland approximation for radiation [22], the radiative heat flux is simplified as

$$q_r=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{2.10}$$

where σ^*, k^*, and T are the Stefan-Boltzmann constant, the Rosseland mean absorption coefficient and the temperature differences within the flow, respectively. T^4 can be given by

$$T^4 \approx 4T_\infty^3 T - 3T_\infty^4, \tag{2.11}$$

Equations (2.10) and (2.11) reduce equation (2.3) to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \left(\alpha + \frac{16\sigma^* T_\infty^3}{3\rho C_p k^*}\right)\frac{\partial^2 T}{\partial y^2}, \tag{2.12}$$

where $\alpha = k/\rho C_p$ is the thermal diffusivity.

Defining the following transformations [10],

$$\begin{aligned} \eta &= y\sqrt{\frac{U}{\nu x}}, \quad u = U\frac{\partial f}{\partial \eta}, \\ v &= \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left[\eta\frac{df}{d\eta} - f\right] \end{aligned} \tag{2.13}$$

equation (2.1) is satisfied identically, while equations (2.2) and (2.3) reduce to the following coupled ordinary differential equations:

$$f'''(\eta) + \tfrac{1}{2}f(\eta)f''(\eta) = 0, \tag{2.14}$$

$$\theta''(\eta) + \tfrac{1}{2}Pr\, k_0\, f(\eta)\theta'(\eta) = 0, \tag{2.15}$$

where non-dimensional temperature $\theta(\eta)$ and the Prandtl number Pr are given by

$$\theta(\eta) = (T - T_\infty)/(T_w - T_\infty); \quad Pr = \nu/\alpha. \tag{2.16}$$

In this paper, we consider $k_0 = 1$ and $Pr = 0.7$.

The boundary conditions of the Blasius equation are transformed to

$$f = 0,\ f' = 0 \ \text{ at } \ \eta = 0, \tag{2.17}$$

$$f' \to 1 \ \text{ as } \ \eta \to \infty, \tag{2.18}$$

while the transformed boundary conditions of the Sakiadis equation are given by

$$f = 0,\ f' = 1 \ \text{ at } \ \eta = 0, \tag{2.19}$$

$$f' \to 0 \ \text{ as } \ \eta \to \infty, \tag{2.20}$$

$$\theta = 1 \ \text{ at } \ \eta = 0, \tag{2.21}$$

$$\theta \to 0 \ \text{ as } \ \eta \to \infty. \tag{2.22}$$

The Blasius equation defined by (2.14), (2.17), (2.18) is a very important boundary layer equation in fluid mechanics. Since the pioneering work of Blasius in 1908 [6], this problem has been an active subject of research [14, 9, 20, 12, 11], due to its key role in fluid mechanics. However, there is no closed-form solution for it.

Blasius [6] proposed the following solution in power series

$$f(\eta) = \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{2.23}$$

where

$$\mathrm{A_k} = \begin{cases} 1, & k = 0, 1, \\ \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1} & k \geq 2. \end{cases} \tag{2.24}$$

and $\sigma = f''(0)$. Note that this solution is not closed because σ is unknown and has to be determined numerically. Later, Weyl [29] claimed that this approximation may not be valid.

Solutions provided to the Blasius equation, thus far, fall into three classes of analytical, numerical and semi-analytical solutions. Perturbation method [13], homotopy analysis method (HAM) [15], and Adomian decomposition method (ADM) [27] are among the analytical solutions utilized to handle the Blasius equation. Recently, the fixed point method (FPM) is adopted to obtain the approximate semi-analytical solution to the Blasius problem [30]. Some of the numerical methods applied to the Blasius problem are; shooting method [4], variational iteration method (VIM) [28], and generalized iterative differential quadrature method [12]. Parand et al. [19] solved the Blasius equation using the Sinc-Collocation method and compared their results with Howarth's [14] and Asaithambi's [4] numerical solutions. A more comprehensive list of solution methods that have been used for the Blasius problem used may be found in [7].

The Sakiadis problem (2.14), (2.15), and (2.19) to (2.22) has also attracted significant attention [26, 25, 21, 18, 3, 10, 5, 12, 11, 30] since the pioneering work of Sakiadis in 1961 [23]. It has practical relevance in various extrusion processes as well as in canonical flow problems in the boundary layer theory of Newtonian and non-Newtonian fluid mechanics. Like the Blasius equation, a wide variety of solution methods have been used to solve the Sakiadis problem. However, the Sinc-Collocation method has not been applied to the Sakiadis problem yet.

In this paper, we apply the Sinc-Collocation approach proposed by Abdella ([2, 1]) to both the Blasius and Sakiadis equations. The approach we utilize has been recently applied in oceanography [16] and has led to efficient and accurate results when compared to other numerical solution methods including those in [6, 14, 4, 19].

3. SINC FUNCTION PRELIMINARIES

On the whole real line $\Re$ the Sinc function is defined as

$$\mathrm{sinc}(x) \equiv \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{3.1}$$

If f is a function defined on $\Re$, then for a step-size $h > 0$ the series

$$C(f,h)(x) \equiv \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(x), \tag{3.2}$$

where $S(k,h)(x)$ is the translated k^{th} Sinc function given by

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x-kh}{h}\right) \tag{3.3}$$

is called the Whittaker Cardinal expansion of f whenever the series converges. However, in practice, the infinite series defining these approximations are truncated as

$$C_N(f,h)(x) \equiv \sum_{k=-N}^{N} S(k,h)(x)f(kh), \tag{3.4}$$

for a given positive integer N. Note that $C_N(f,h)(x)$ defines an interpolation of $f(x)$ with $C_N(f,h)(x) = f(x)$ at all the Sinc grid points given by $x_k = kh$. For a class of functions which are analytic only on an infinite strip containing the real line and allowing specific growth restrictions, the Sinc interpolations provide approximation that exhibit exponentially decaying absolute errors as established by the theorem subsequent to the following definition [24].

Definition 3.1. Let D_d denote the infinite strip of width $2d$ $(d > 0)$ in the complex plane:

$$D_d = \left\{ z = x + iy \,\middle|\, |y| < d < \frac{\pi}{2} \right\}.$$

Then $H^1(D_d)$ is defined as the class of functions f that are analytic in D_d such that

$$N(f, D_d) \equiv \lim_{\epsilon \to 0} \int_{\partial D_d(\epsilon)} |f(z)||dz| < \infty$$

where

$$\partial D_d(\epsilon) = \left\{ z = x + iy \,\middle|\, |x| < \frac{1}{\epsilon}, |y| > d(1-\epsilon) \right\}.$$

Theorem 3.2. *If $f(x) \in H^1(D_d)$ and decays exponentially for $x \in \Re$ such that*

$$|f(x)| \le \alpha \exp\left(-\beta \exp(\gamma |x|)\right) \text{ for all } x \in \Re$$

where α, β and γ are positive constants, then the error of the Sinc approximation is bounded by:

$$\sup_{-\infty \le x \le \infty} \left| f(x) - \sum_{k=-N}^{N} S(k,h)(x) f(kh) \right| \le CE(h)$$

for some positive constant C and where

$$E(h) = \exp\left(\frac{-\pi d \gamma N}{\log(\pi d \gamma N/\beta)}\right)$$

and the mesh size h is taken as:

$$h = \frac{\log(\pi d \gamma N/\beta)}{\gamma N}.$$

In order to construct the approximation over the semi-infinite interval $[0, \infty]$, we use a variable transformation

$$\xi = \varphi(x) = \operatorname{arcsinh}\left(\frac{2}{\pi}\ln(x)\right) \tag{3.5}$$

with a corresponding inverse

$$x = \psi(\xi) = \exp\left(\frac{\pi}{2}\sinh(\xi)\right)$$

such that $x_k = \psi(kh)$, that transfers the interval $[0, \infty]$ onto $\Re$, and apply the above Sinc approximation on $\Re$ to the transformed function $f(\psi(\xi))$ so that:

$$f(x) \approx \sum_{k=-N}^{N} S(k,h)(\varphi(x))f(\psi(kh)), \ \ 0 \leq x < \infty, \tag{3.6}$$

where $\lim_{x\to\infty}\varphi(x) = -\infty$ and $\lim_{x\to\infty}\varphi(x) = \infty$.

Therefore, the corresponding error bound theorem will be as follows:

Theorem 3.3. *If $f(\psi(\xi)) \in H^1(D_d)$ and decays exponentially for $\xi \in \Re$ such that*

$$|f(\psi(\xi))\psi'(\xi)| \leq \alpha \exp\left(-\beta \exp(\gamma|\xi|)\right) \ \textit{for all}\ \xi \in \Re$$

where α, β and γ are positive constants and $x = \psi(\xi)$ is the inverse of the transformation $\xi = \varphi(x)$, then the error of the Sinc approximation is bounded by:

$$\sup_{a\leq x\leq b}\left|f(x) - \sum_{k=-N}^{N} S(k,h)(\varphi(x))f(\psi(kh))\right| \leq CE(h)$$

for some positive constant C and where

$$E(h) = \exp\left(\frac{-\pi d\gamma N}{\log(\pi d\gamma N/\beta)}\right)$$

and the mesh size h is taken as:

$$h = \frac{\log(\pi d\gamma N/\beta)}{\gamma N}.$$

4. THE DERIVATIVE INTERPOLATION METHOD

Note that the Sinc basis functions have unbounded derivative at zero. Therefore, we modify the Sinc basis functions as

$$\frac{S_k(x)}{\varphi'(x)} \tag{4.1}$$

Where $\varphi'(x)$ is the derivative of our transformation, equation (3.5), and the regular Sinc function $S_k(x)$ is given by

$$S_k(x) = \operatorname{Sinc}\left(\frac{\varphi(x)-kh}{h}\right) = \begin{cases} \frac{\sin\left(\frac{\pi}{h}(\varphi(x)-kh)\right)}{\frac{\pi}{h}(\varphi(x)-kh)} & \varphi(x) \neq kh \\ 1 & \varphi x = kh \end{cases} \tag{4.2}$$

Hence we interpolate the first derivative as

$$u'(x) = \sum_{k=-N}^{N} \frac{C_k S_k(x)}{\varphi'(x)}. \tag{4.3}$$

Then

$$u'(x_l) = \sum_{k=-N}^{N} \frac{C_k S_k(x_l)}{\varphi'(x_l)}. \tag{4.4}$$

However,

$$S_k(x_l) = \mathrm{Sinc}\left(\frac{\varphi(x_l) - kh}{h}\right) = \delta_{l,k}^{(0)} \tag{4.5}$$

where

$$\delta_{l,k}^{(0)} = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases} \tag{4.6}$$

Hence

$$C_l = \frac{u'(x_l)}{\varphi(x_l)}, \quad l = -N, \dots, N. \tag{4.7}$$

In order to get $u''(x_l)$ we differentiate (4.3) as follows:

$$\begin{aligned} u''(x) &= \sum_{k=-N}^{N} C_k \frac{d}{dx}\left(\frac{S_k(x)}{\varphi'(x)}\right) \\ &= \sum_{k=-N}^{N} C_k \left(\frac{\frac{d}{d\varphi} S_k(x) \cdot \frac{d\varphi}{dx}\varphi'(x) - S_k(x)\varphi''(x)}{(\varphi'(x))^2}\right) \\ u''(x) &= \sum_{k=-N}^{N} C_k \left(\frac{d}{d\varphi}(S_k(x)) - \frac{\varphi''(x)}{(\varphi'(x))^2} S_k(x).\right) \end{aligned} \tag{4.8}$$

Hence

$$u''(x_l) = \sum_{K=-N}^{N} C_k \left(\frac{1}{h}\delta_{l,k}^{(1)} \frac{\varphi''(x_l)}{\varphi'(x_l)^2}\delta_{l,k}^{(0)}\right). \tag{4.9}$$

where

$$\delta_{l,k}^{(1)} = \begin{cases} \frac{(-1)^{l-k}}{l-k}, & k \neq l, \\ 0, & k = l. \end{cases} \tag{4.10}$$

Similarly, differentiating (4.8) we get

$$u'''(x) = \sum_{k=-N}^{N} C_k \left[\frac{d^2}{d\varphi^2}(S_k(x))\varphi'(x) - \frac{\varphi''(x)}{\varphi'(x)^2}\frac{dS_k(x)}{d\varphi}\varphi'(x) - S_k(x)\left(\frac{\varphi'''(x)\varphi'(x)^2 - 2\varphi''(x)\varphi'(x)\varphi''(x)}{\varphi(x)^4}\right)\right]$$

(4.11)

$$= \sum_{k=-N}^{N} C_k \left[\frac{d^2}{d\varphi^2}(S_k(x))\varphi' - \frac{\varphi''}{\varphi'}\frac{d}{d\varphi}S - k(x) - \frac{S_k(x)\varphi'''}{\varphi'^2} + \frac{2S_k(x)\varphi''^2}{\varphi'^3}\right]$$

Hence

(4.12)

$$u'''(x_l) = \sum_{k=-N}^{N} C_k \left[\frac{1}{h^2}\delta_{l,k}^{(2)}\varphi'(x_l) - \frac{\varphi''(x_l)}{\varphi'(x_l)}\delta_{l,k}^{(1)}\frac{1}{h} + I_0(k,l)\left[\frac{\varphi'''(x_l)}{\varphi'(x_l)^3} + \frac{2\varphi''(x_l)^2}{\varphi(x_l)^3}\right]\right]$$

where

$$\delta_{l,k}^{(2)} = \begin{cases} \frac{-2(-1)^{l-k}}{(l-k)^2}, & k \neq l, \\ \frac{-\pi^2}{3}, & k = l. \end{cases} \tag{4.13}$$

In order to obtain $u(x)$ we integrate (4.3) as follows. On $(0,\infty)$ domain:

$$u(x) = \int_0^x u'(s)ds + u(0) = \int_0^x u'(s)ds \tag{4.14}$$

since $u(0) = 0$.

Using the change of variable $s = \psi(t)$ where t is in the transformed domain $(-\infty,\infty)$, we have

$$u(x) = \int_{\psi^{-1}(0)}^{\psi^{-1}(x)} u'(\psi(t))\psi'(t)dt \tag{4.15}$$

Then

$$u(x) = \int_{\varphi(0)}^{\varphi(x)} u'(\psi(t))\psi'(t)dt = \int_{-\infty}^{\varphi(x)} u'(\psi(t))\psi'(t)dt \tag{4.16}$$

We now use sinc interpolation to express $u'(\psi(t))\,(\psi'(t))^2$ in terms of the sinc bases:

$$u'(\psi(t))\psi'(t)^2 = \sum_{k=-N}^{N} u'(\psi(t_k))\,(\psi'(t_k))^2\,\frac{S_k(\psi(t))}{\varphi'(\psi(t))}. \tag{4.17}$$

Substituting (4.17) into (4.16) we get

$$
\begin{aligned}
u(x) &= \int_{-\infty}^{\varphi(x)} u'(\psi(t))\psi'(t)^2 \left(\frac{1}{\psi'(t)}\right) dt \\
&= \int_{-\infty}^{\varphi(x)} \sum_{k=-N}^{N} u'(\psi(t_k))\psi'(t_k)^2 \frac{S_k(\psi(t))}{\varphi'(\psi(t))\psi'(t)} dt \\
&= \sum_{k=-N}^{N} u'(\psi(t_k))\psi'(t_k)^2 H(x)
\end{aligned} \tag{4.18}
$$

where

$$
H(x) = \int_{-\infty}^{\varphi(x)} \frac{\sin(\frac{\pi}{h}(\varphi(\psi(t)) - kh)}{\frac{\pi}{h}(\varphi(\psi(t)) - kh)} dt.
$$

Hence

$$
H(x) = \int_{-\infty}^{\varphi(x)} \frac{\sin(\frac{\pi}{h}(t - kh))}{\frac{\pi}{h}(t - kh)} dt.
$$

Then using the substitution $z = \frac{\pi}{h}(t - kh)$ with $dz = \frac{\pi}{h} dt$ we get

$$
H(x) = \int_{-\infty}^{\frac{\pi}{h}(\varphi(x)-kh)} \frac{\sin(z)}{z} \left(\frac{h}{\pi}\right) dz \tag{4.19}
$$

Hence,

$$
\begin{aligned}
H(x) &= \frac{h}{\pi} \left[\int_{-\infty}^{0} \frac{\sin(z)}{z} dz + \int_{0}^{\frac{\pi}{h}(\varphi(x)-kh)} \frac{\sin(z)}{z} dz \right] \\
&= \frac{h}{\pi} \left[\frac{\pi}{2} + \int_{0}^{\frac{\pi}{h}(\varphi(x)-kh)} \frac{\sin(z)}{z} dz \right]
\end{aligned} \tag{4.20}
$$

Substituting (4.20) into (4.18) we get

$$
u(x_l) = \sum_{k=-N}^{N} \frac{u'(x_k)\psi}{\varphi'(x_k)^2} \frac{h}{\pi} \left[\frac{\pi}{2} + \int_{0}^{\frac{\pi}{h}(t_l - kh)} \frac{\sin(z)}{z} dz \right] \tag{4.21}
$$

where $t_l = \varphi'(x_l) = lh$. Then

$$
\begin{aligned}
u(x_l) &= \sum_{k=-N}^{N} \frac{hu'(x_k)}{\varphi'(x_k)^2} \left[\frac{1}{2} + \frac{1}{\pi} \int_{0}^{\pi(l-k)} \frac{\sin(z)}{z} dz \right] \\
&= \sum_{k=-N}^{N} \frac{hu'(x_k)}{\varphi'(x_k)^2} \left[\frac{1}{2} + \frac{1}{\pi} \text{Sinintegral}(\pi(l-k)) \right] \\
&= \sum_{k=-N}^{N} \frac{hu'(x_k)}{\varphi'(x_k)^2} \left[\frac{1}{2} + \delta_{l,k}^{(-1)} \right]
\end{aligned} \tag{4.22}
$$

where

$$
\delta_{l,k}^{(-1)} = \begin{cases} \frac{1}{2} + \int_0^{l-k} \frac{\sin(\pi t)}{\pi t}, & k \neq l, \\ \frac{1}{2}, & k = l, \end{cases} \tag{4.23}
$$

5. RESULTS AND DISCUSSION

5.1. SOLUTION TO THE BLASIUS EQUATION. In order to apply the Sinc-Collocation method to the Blasius equation (2.14) and its boundary conditions, we construct a function $P(x)$ that also satisfies (2.17) and (2.18). This function is given by

$$P(\eta) = \eta \tanh(a\eta) \tag{5.1}$$

where a is a constant to be determined. We define the approximate solution of equation (2.14) by

$$f(\eta) = u(\eta) + P(\eta) \tag{5.2}$$

in which $u(\eta)$ is given by (4.18). Note that the approximate solution $u(\eta)$ satisfies the homogeneous boundary conditions:

$$\lim_{\eta\to 0} u(\eta) = \lim_{\eta\to 0} u'(\eta) = 0,$$
$$\lim_{\eta\to\infty} u'(\eta) = 0.$$

We utilize equations (4.4), (4.9), (4.12) and (4.22) to construct $u(\eta)$ and its derivatives. Evaluating them at the sinc points

$$\eta_k = e^{\frac{\pi}{2}\sinh(kh)}; \quad k = -N, \dots, N.$$

and substituting the results into equation (2.14) we obtain

$$u'''(x) + \frac{1}{2}u(\eta_k)u''(\eta_k) + \frac{1}{2}P(\eta_k)u''(\eta_k)\frac{1}{2}u(\eta_k)P''(\eta_k) = -\frac{1}{2}P(\eta_k)P''(\eta_k); \quad k = -N-1, \dots, N \tag{5.3}$$

Equation (5.3) leads to $2N + 2$ nonlinear algebraic equations. Newton's method is utilized to solve this system for the unknown coefficients of the derivative interpolations, C_k, for $k = -N, \dots, N$ and the variable a. Note that from (5.2), it can be shown that $a = \frac{f''(0)}{2}$. Once equation (5.3) is solved, the coefficients are used to determine the values of the unknown functions $u(\eta)$ and its derivatives at the Sinc nodes. The original unknown, $f(\eta)$ and its derivatives are then determined from equation (5.2) and its derivatives.

Table 1 includes the values of $f(\eta)$ obtained by the present method and those in [19] which used the standard Sinc method as well as the numerical method of [14]. The table clearly shows that the current method is highly accurate.

Figure 1 shows the approximations of $f(\eta)$ and $f'(\eta)$ for the Blasius equation obtained by the present method for $N = 32$ against those suggested by Blasius [6]. The two solution curves are indistinguishable.

5.2. SOLUTION TO THE SAKIADIS EQUATION.

TABLE 1. The comparison of $f(\eta)$ between the present method when $N = 26$ and those in [19] and [14].

η	Current Method	results in [19]	results in [14]
0.2	0.0066458	0.0066926	0.00664
0.4	0.0265696	0.0268895	0.02656
0.6	0.0597392	0.0595069	0.05973
0.8	0.1061276	0.1068849	0.10611
1.0	0.1655957	0.1650097	0.16557
2.0	0.6500699	0.6503782	0.65002
3.0	1.3968696	1.3968501	1.39681
4.0	2.3058135	2.3058000	2.30575
5.0	3.2833419	3.2833981	3.28327
6.0	4.2796891	4.2797544	4.27962
7.0	5.2793099	5.2794705	5.27924
8.0	6.2792763	6.2793664	6.27921

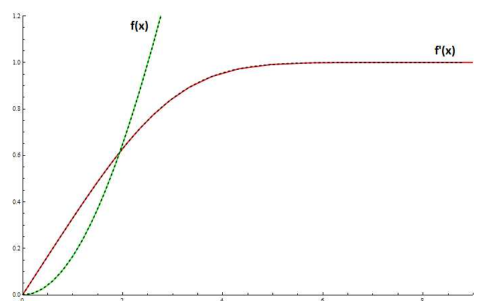

FIGURE 1. A Comparison between our approximate results for $f(\eta)$ and $f'(\eta)$ and those proposed by Blasius [6]

5.2.1. *THE MOMENTUM TRANSFER EQUATION.* Our approach to approximate the solution of equation (2.14) together with boundary conditions (2.19) and (2.20) is similar to that of the Blasius equation. However, we need to construct a new function $Q(\eta)$ that satisfies the boundary conditions (2.19) and (2.20). This function is given by

$$Q(\eta) = \eta e^{-\eta} \tag{5.4}$$

We define the approximate solution of equation (2.14) together with boundary conditions (2.19),(2.20) by

$$f(\eta) = u(\eta) + Q(\eta) \tag{5.5}$$

where $u(\eta)$ is given by (4.14). Note that the approximate solution $u(\eta)$ satisfies homogeneous boundary conditions.

The results of the current numerical solutions for $f(\eta)$, and $f'(\eta)$ are given in Table 2. Figure 2 shows the approximations of $f(\eta)$ and $f'(\eta)$ for the Sakiadis equation obtained by the present method for $N = 32$ against those reported in [10].

TABLE 2. Momentum transfer solutions using the current method

η	f	f'
0.1	0.09777856	0.9556268
0.2	0.1911326	0.9115064
0.3	0.2800938	0.8678045
0.4	0.3647137	0.8247095
0.5	0.4450617	0.7823923
0.6	0.5212232	0.7410052
0.7	0.5932982	0.7006818
0.8	0.6613988	0.6615361
0.9	0.7256478	0.6236627
1.0	0.7861763	0.5871380
1.5	1.0379819	0.4262347
2.0	1.2185192	0.3017807
3.0	1.4326971	0.1440172
4.0	1.5330501	0.06624525
5.0	1.5788152	0.02995112

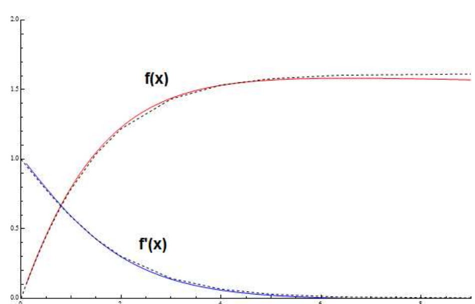

FIGURE 2. A Comparison between our approximate results for $f(\eta)$ and $f'(\eta)$ and those reported by Cortell [10].

Finally, the results of the current numerical solutions for $f''(\eta)$ are given in Table 3 along the solution obtained by Cortell [10]. The result shows the excellent agreement between the two methods.

5.2.2. *THE HEAT TRANSFER EQUATION.* As in the case of the momentum transfer equation we begin by defining

$$\theta(\eta) = u(\eta) + R(\eta) \tag{5.6}$$

TABLE 3. Comparison between the current method and those reported by Cortell [10].

η	$-f''$ (current method)	$-f''$ ([10])
0.1	0.44265570	0.4426395
0.2	0.43946170	0.4394406
0.3	0.43430680	0.4342870
0.4	0.42735390	0.4273341
0.5	0.41878160	0.4187607
0.6	0.40877870	0.4087565
0.7	0.39753900	0.3975208
0.8	0.38525610	0.3852365
0.9	0.37211950	0.3721032
1.0	0.35831140	0.3582943
1.5	0.28477490	0.2847647
2.0	0.21450470	0.2144988
3.0	0.10983430	0.1098329
4.0	0.05215941	0.0521597
5.0	0.02392260	0.02392326

where the function $R(\eta)$ is given by

$$R(\eta) = e^{-\eta} \tag{5.7}$$

With this definition, the unknown variable $u(\eta)$ satisfies the homogeneous boundary conditions

$$\lim_{\eta\to 0} u(\eta) = 0$$
$$\lim_{\eta\to\infty} u(\eta) = 0.$$

In order to obtain u we set up a separate Sinc-Collocation procedure for it. However, as there are no boundary conditions given for the first derivative, we must solve this in the typical manner of interpolating the unknown function and obtaining the first derivative values through differentiation of the result. Therefore, we have the following modified definitions of the Sinc approximations

$$u(\eta_l) = \sum_{k=-N}^{N} \delta_{l,k}^{(0)} u(\eta_k) \tag{5.8}$$

$$u'(\eta_l) = \sum_{k=-N}^{N} \delta_{i,k}^{(1)} \varphi'(\eta_l) \frac{u(\eta_k)}{h} \tag{5.9}$$

The rest of the procedure is identical to those described above.

The results of our numerical solutions for $\theta(\eta)$ and $\theta'(\eta)$ are given in Table 4. This result is consistent with [17] who found the numerical solution $\theta'(0) = -0.34924$.

TABLE 4. Heat transfer solutions with $Pr = 0.7$, $k_0 = 1$

η	$\theta(\eta)$	$-\theta'(\eta)$
0.0	1.0	0.3493033
0.1	0.9650821	0.3486259
0.2	0.9302986	0.3468609
0.3	0.8957465	0.3440083
0.4	0.8615310	0.3401437
0.5	0.8277489	0.3353524
0.6	0.7944885	0.3297241
0.7	0.7618287	0.3233533
0.8	0.7298398	0.3163258
0.9	0.6985822	0.3087377
1.0	0.6681081	0.3006722
1.5	0.5287246	0.2560152
2.0	0.4123072	0.2099533
3.0	0.2436348	0.1314129
4.0	0.1409237	0.0780297
5.0	0.08070539	0.04521677

6. **CONCLUSION**

In this paper, we have shown that first derivative interpolation using Sinc numerical methods can be used to efficiently solve nonlinear boundary value problems. Sinc numerical methods are preferable as they result in exponential convergence and tolerance of singularities. This was shown by solving a system of two nonlinear boundary value problems, the Blasius Equation and the Sakiadis Equation. It was found that the method gives comparable accuracy to other results [19] while using a lowered resolution, suggesting a higher efficiency in the proposed method.

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