Dynamic Systems and Applications 26 (2017) 183-200

INVERSE PROBLEM FOR A CLASS OF DIRAC OPERATORS BY THE WEYL FUNCTION

KHANLAR R. MAMEDOV AND OZGE AKCAY

Mathematics Department, Mersin University, 33343, Mersin, Turkey; Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ1141, Bakü, Azerbaijan.
Mathematics Department, Mersin University, 33343, Mersin, Turkey.

ABSTRACT. This paper is related to an inverse problem for a class of Dirac operators with discontinuous coefficient and eigenvalue parameter contained in boundary conditions. The asymptotic formula of eigenvalues of this problem is examined. Weyl solution and Weyl function are constructed. Uniqueness theorem of the inverse problem respect to the Weyl function is proved.

AMS (MOS) Subject Classification. 34A55, 34L40, 34L10.

1. INTRODUCTION

Let

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

be the well-known Pauli-matrices which has these properties: $\sigma_i^2 = I$, (I is 2×2 identity matrix) $\sigma_i^* = \sigma_i$ (self-adjointness) $i = 1, 2, 3$ and for $i \neq j$, $\sigma_i \sigma_j = -\sigma_j \sigma_i$ (anticommutativity).

We consider the following boundary value problem generated by the canonical Dirac system

$$By' + \Omega(x)\, y = \lambda \rho(x)\, y, \quad 0 < x < \pi \tag{1.1}$$

with boundary conditions

$$\begin{aligned} U_1(y) &:= b_1 y_2(0) + b_2 y_1(0) - \lambda \left(b_3 y_2(0) + b_4 y_1(0)\right) = 0, \\ U_2(y) &:= c_1 y_2(\pi) + c_2 y_1(\pi) + \lambda \left(c_3 y_2(\pi) + c_4 y_1(\pi)\right) = 0, \end{aligned} \tag{1.2}$$

where

$$B = \frac{1}{i}\sigma_1, \quad \Omega(x) = \sigma_2 p(x) + \sigma_3 q(x), \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real measurable functions, $p(x) \in L_2(0, \pi)$, $q(x) \in L_2(0, \pi)$, λ is a spectral parameter,

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha, & a < x \le \pi, \end{cases}$$

Received April 11, 2016

and $1 \neq \alpha > 0$. Let us define $k_1 = b_1 b_4 - b_2 b_3 > 0$ and $k_2 = c_1 c_4 - c_2 c_3 > 0$. The main aim of this paper is to solve the inverse problem for the boundary value problem (1.1), (1.2) by Weyl function on a finite interval.

The inverse problem and the spectral properties of Dirac operators were investigated in detail by many authors [1, 2, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 25, 26, 27, 28, 29, 30, 31, 32]. The inverse spectral problems according to two spectra was solved in [10]. Using Weyl-Titchmarsh function, direct and inverse problems for Dirac type-system were studied in [8, 9, 26]. Solution of the inverse quasiperiodic problem for Dirac system was given in [25]. For weighted Dirac system, inverse spectral problems was examined in [28]. Reconstruction of Dirac operator from nodal data was carried out in [30]. Necessary and sufficient conditions for the solution of Dirac operators with discontinuous coefficient was obtained in [20]. Inverse problem for interior spectral data of the Dirac operator was given in [23]. For Dirac operator, Ambarzumian-type theorems were proved in [14, 31]. On a positive half line, inverse scattering problem for a system of Dirac equations of order $2n$ was completely solved in [12] and when boundary condition contained spectral parameter, for Dirac operator, inverse scattering problem was worked in [6, 21]. Spectral boundary value problem in a 3 dimensional bounded domain for the Dirac system was studied in [2]. The applications of Dirac differential equations system has been widespread in various areas of physics, such as [3, 4, 24, 27].

This paper is organized as follows: in section 2, the operator formulation of problem (1.1), (1.2) and some spectral properties of the operator are given. In section 3, asymptotic formula of eigenvalues of the problem (1.1), (1.2) is examined. In section 4, Weyl solution, Weyl function are defined and uniqueness theorem for inverse problem according to Weyl function is proved.

2. OPERATOR FORMULATION AND SOME SPECTRAL PROPERTIES

An inner product in Hilbert space $H_\rho = L_{2,\rho}(0,\pi;\mathbb{C}^2) \oplus \mathbb{C}^2$ is given by

$$\langle Y, Z\rangle = \int_0^\pi \left\{ y_1(x)\overline{z_1(x)} + y_2(x)\overline{z_2(x)} \right\} \rho(x)\,dx + \frac{1}{k_1} y_3 \overline{z_3} + \frac{1}{k_2} y_4 \overline{z_4}, \tag{2.1}$$

where

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3 \\ y_4 \end{pmatrix} \in H_\rho, \quad Z = \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3 \\ z_4 \end{pmatrix} \in H_\rho.$$

Let us define the operator L:

$$L(Y) := \begin{pmatrix} l(y) \\ b_1 y_2(0) + b_2 y_1(0) \\ -\left(c_1 y_2(\pi) + c_2 y_1(\pi)\right) \end{pmatrix}$$

with domain

$$D(L) := \big\{ Y \mid Y = (y_1(x), y_2(x), y_3, y_4)^T \in H_\rho, y_1(x), y_2(x) \in AC[0,\pi], \\ y_3 = b_3 y_2(0) + b_4 y_1(0), y_4 = c_3 y_2(\pi) + c_2 y_1(\pi), l(y) \in L_{2,\rho}(0,\pi;\mathbb{C}^2) \big\}$$

where

$$l(y) = \frac{1}{\rho(x)} \left\{ By' + \Omega(x) y \right\}.$$

Consequently, the boundary value problem (1.1), (1.2) is equivalent to the operator equation $LY = \lambda Y$.

Lemma 2.1. *(i) The eigenvector functions corresponding to different eigenvalues are orthogonal.*

(ii) The eigenvalues of the operator L are real valued.

Let $\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$ and $\psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$ be solutions of the system (1.1) satisfying the initial conditions

$$\varphi(0,\lambda) = \begin{pmatrix} \lambda b_3 - b_1 \\ b_2 - \lambda b_4 \end{pmatrix}, \quad \psi(\pi,\lambda) = \begin{pmatrix} -c_1 - \lambda c_3 \\ c_2 + \lambda c_4 \end{pmatrix}.$$

The characteristic function of the problem (1.1), (1.2) is defined by

$$\Delta(\lambda) = W[\varphi(x,\lambda), \psi(x,\lambda)] = \varphi_2(x,\lambda)\psi_1(x,\lambda) - \varphi_1(x,\lambda)\psi_2(x,\lambda), \tag{2.2}$$

where $W[\varphi(x,\lambda), \psi(x,\lambda)]$ is Wronskian of the vector solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$. The Wronskian does not depend on x. It follows from (2.2) that

$$\Delta(\lambda) = b_2\psi_1(0,\lambda) + b_1\psi_2(0,\lambda) - \lambda\left(b_4\psi_1(0,\lambda) + b_3\psi_2(0,\lambda)\right) = U_1(\psi)$$

or

$$\Delta(\lambda) = -c_1\varphi_2(\pi,\lambda) - c_2\varphi_1(\pi,\lambda) - \lambda\left(c_3\varphi_2(\pi,\lambda) + c_4\varphi_1(\pi,\lambda)\right) = -U_2(\varphi).$$

Lemma 2.2. *The zeros λ_n of characteristic function coincide with the eigenvalues of the boundary value problem* (1.1), (1.2). *The function $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions and there exist a sequence β_n such that*

$$\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n), \quad \beta_n \neq 0.$$

Proof. This lemma is proved by a similar way in [7] (see Theorem 1.1.1). ∎

Norming constants are defined as follows:

$$\alpha_n := \int_0^\pi \left\{\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\right\}\rho(x)dx + \frac{1}{k_1}\left[b_3\varphi_2(0,\lambda_n)+b_4\varphi_1(0,\lambda_n)\right]^2 + \frac{1}{k_2}\left[c_3\varphi_2(\pi,\lambda_n)+c_4\varphi_1(\pi,\lambda_n)\right]^2. \tag{2.3}$$

Lemma 2.3. *The following relation is valid:*

$$\alpha_n\beta_n = \dot{\Delta}(\lambda_n),$$

where $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.

Proof. Since $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of this problem, we have

$$\begin{aligned}
\psi_2'(x,\lambda)+p(x)\psi_1(x,\lambda)+q(x)\psi_2(x,\lambda) &= \lambda\rho(x)\psi_1(x,\lambda),\\
-\psi_1'(x,\lambda)+q(x)\psi_1(x,\lambda)-p(x)\psi_2(x,\lambda) &= \lambda\rho(x)\psi_2(x,\lambda),\\
\varphi_2'(x,\lambda_n)+p(x)\varphi_1(x,\lambda_n)+q(x)\varphi_2(x,\lambda_n) &= \lambda_n\rho(x)\varphi_1(x,\lambda_n),\\
-\varphi_1'(x,\lambda_n)+q(x)\varphi_1(x,\lambda_n)-p(x)\varphi_2(x,\lambda_n) &= \lambda_n\rho(x)\varphi_2(x,\lambda_n).
\end{aligned}$$

Multiplying the equations by $\varphi_1'(x,\lambda_n)$, $\varphi_2'(x,\lambda_n)$, $-\psi_1'(x,\lambda)$, $-\psi_2'(x,\lambda)$ respectively and adding them together, we get

$$\begin{aligned}
&\frac{d}{dx}\left\{\varphi_1(x,\lambda_n)\psi_2(x,\lambda)-\psi_1(x,\lambda)\varphi_2(x,\lambda_n)\right\}\\
&\quad=(\lambda-\lambda_n)\rho(x)\left\{\varphi_1(x,\lambda_n)\psi_1(x,\lambda)+\varphi_2(x,\lambda_n)\psi_2(x,\lambda)\right\}.
\end{aligned}$$

Integrating it from 0 to π,

$$\begin{aligned}
&(\lambda-\lambda_n)\int_0^\pi\left\{\varphi_1(x,\lambda_n)\psi_1(x,\lambda)+\varphi_2(x,\lambda_n)\psi_2(x,\lambda)\right\}\rho(x)dx\\
&\quad=\varphi_1(\pi,\lambda_n)\psi_2(\pi,\lambda)-\varphi_2(\pi,\lambda_n)\psi_1(\pi,\lambda)-\varphi_1(0,\lambda_n)\psi_2(0,\lambda)+\varphi_2(0,\lambda_n)\psi_1(0,\lambda)
\end{aligned}$$

is found. Now, we add

$$\begin{aligned}
(\lambda-\lambda_n)\Bigg\{&\frac{1}{k_1}\left[b_3\psi_2(0,\lambda)+b_4\psi_1(0,\lambda)\right]\left[b_3\varphi_2(0,\lambda_n)+b_4\varphi_1(0,\lambda_n)\right]\\
&+\frac{1}{k_2}\left[c_3\psi_2(\pi,\lambda)+c_4\psi_1(\pi,\lambda)\right]\left[c_3\varphi_2(\pi,\lambda_n)+c_4\varphi_1(\pi,\lambda_n)\right]\Bigg\}
\end{aligned}$$

in the both sides of last equation and use the boundary condition (1.2). It follows that

$$\begin{aligned}
&\int_0^\pi\left\{\varphi_1(x,\lambda_n)\psi_1(x,\lambda)+\varphi_2(x,\lambda_n)\psi_2(x,\lambda)\right\}\rho(x)dx\\
&\quad+\frac{1}{k_1}\left[b_3\psi_2(0,\lambda)+b_4\psi_1(0,\lambda)\right]\left[b_3\varphi_2(0,\lambda_n)+b_4\varphi_1(0,\lambda_n)\right]\\
&\quad+\frac{1}{k_2}\left[c_3\psi_2(\pi,\lambda)+c_4\psi_1(\pi,\lambda)\right]\left[c_3\varphi_2(\pi,\lambda_n)+c_4\varphi_1(\pi,\lambda_n)\right]=\frac{\Delta(\lambda)-\Delta(\lambda_n)}{\lambda-\lambda_n}.
\end{aligned}$$

According to Lemma 2.2, since $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$, as $\lambda\to\lambda_n$, we obtain

$$\beta_n\alpha_n=\dot{\Delta}(\lambda_n).$$

$\square$

3. ASYMPTOTIC FORMULA OF EIGENVALUES

Lemma 3.1. *The solution* $\varphi(x,\lambda)=\begin{pmatrix}\varphi_1(x,\lambda)\\ \varphi_2(x,\lambda)\end{pmatrix}$ *has the following integral representation*

$$\begin{aligned}\varphi_1(x,\lambda)&=(\lambda b_3-b_1)\cos\lambda\mu(x)+(\lambda b_4-b_2)\sin\lambda\mu(x)\\&+(\lambda b_3-b_1)\int_0^{\mu(x)}\left[\tilde{A}_{11}(x,t)\cos\lambda t+\tilde{\tilde{A}}_{12}(x,t)\sin\lambda t\right]dt\\&+(\lambda b_4-b_2)\int_0^{\mu(x)}\left[\tilde{\tilde{A}}_{11}(x,t)\sin\lambda t-\tilde{A}_{12}(x,t)\cos\lambda t\right]dt,\end{aligned}\tag{3.1}$$

$$\begin{aligned}\varphi_2(x,\lambda)&=(\lambda b_3-b_1)\sin\lambda\mu(x)+(b_2-\lambda b_4)\cos\lambda\mu(x)\\&+(\lambda b_3-b_1)\int_0^{\mu(x)}\left[\tilde{A}_{21}(x,t)\cos\lambda t+\tilde{\tilde{A}}_{22}(x,t)\sin\lambda t\right]dt\\&+(\lambda b_4-b_2)\int_0^{\mu(x)}\left[\tilde{\tilde{A}}_{21}(x,t)\sin\lambda t-\tilde{A}_{22}(x,t)\cos\lambda t\right]dt,\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}\tilde{A}_{1j}(x,t)&=K_{1j}(x,-t)+K_{1j}(x,t),\\ \tilde{\tilde{A}}_{1j}(x,t)&=K_{1j}(x,t)-K_{1j}(x,-t),\\ \tilde{A}_{2j}(x,t)&=K_{2j}(x,-t)+K_{2j}(x,t),\\ \tilde{\tilde{A}}_{2j}(x,t)&=K_{2j}(x,t)-K_{2j}(x,-t),\end{aligned}$$

and $\tilde{A}_{1j}(x,.)\in L_2(0,\pi)$, $\tilde{\tilde{A}}_{1j}(x,.)\in L_2(0,\pi)$, $\tilde{A}_{2j}(x,.)\in L_2(0,\pi)$, $\tilde{\tilde{A}}_{2j}(x,.)\in L_2(0,\pi)$, $j=1,2$.

Proof. To obtain the form of $\varphi(x,\lambda)$, we use the integral representation for the solution of equation (1.1) [17]. This representation is not operator transformation and as follows: Assume that

$$\int_0^{\pi}\|\Omega(x)\|\,dx<+\infty$$

is satisfied for Euclidean norm of matrix function $\Omega(x)$. Then the integral representation of the solution of equation (1.1) satisfying the initial condition $E(0,\lambda)=I$, (I is unite matrix) can be represented

$$E(x,\lambda)=e^{-\lambda B\mu(x)}+\int_{-\mu(x)}^{\mu(x)}K(x,t)e^{-\lambda Bt}dt,$$

where

$$\mu(x)=\begin{cases} x, & 0\le x\le a,\\ \alpha x-\alpha a+a, & a<x\le\pi,\end{cases}$$

and for a kernel $K(x,t)$ the inequality

$$\int_{-\mu(x)}^{\mu(x)}\|K(x,t)\|\,dt\le e^{\sigma(x)}-1,$$

$$\sigma(x)=\int_0^x\|\Omega(s)\|\,ds$$

holds. Moreover, if $\Omega(x)$ is differentiable, then $K(x,t)$ satisfy the following relations

$$BK_x+\Omega(x)K+\rho(x)K_tB=0,$$

$$\rho(x)\left[K(x,\mu(x))B-BK(x,\mu(x))\right]=\Omega(x),$$

$$BK(x,-\mu(x))=0.$$

Now, to find $\varphi(x,\lambda)$, we will use $\varphi(x,\lambda)=E(x,\lambda)\begin{pmatrix}\lambda b_3-b_1\\ b_2-\lambda b_4\end{pmatrix}$. From the expression of $E(x,\lambda)$

$$\varphi(x,\lambda)=e^{-\lambda B\mu(x)}\begin{pmatrix}\lambda b_3-b_1\\ b_2-\lambda b_4\end{pmatrix}+\int_{-\mu(x)}^{\mu(x)}K(x,t)e^{-\lambda Bt}\begin{pmatrix}\lambda b_3-b_1\\ b_2-\lambda b_4\end{pmatrix}dt \tag{3.3}$$

can be written. Then

$$\begin{aligned} e^{-\lambda B\mu(x)}\begin{pmatrix}\lambda b_3-b_1\\ b_2-\lambda b_4\end{pmatrix}&=\left(I-\lambda B\mu(x)+\frac{\left(-\lambda B\mu(x)\right)^2}{2!}+\cdots\right)\begin{pmatrix}\lambda b_3-b_1\\ b_2-\lambda b_4\end{pmatrix}\\ &=\begin{pmatrix}\lambda b_3-b_1\\ b_2-\lambda b_4\end{pmatrix}-\lambda\begin{pmatrix}b_2-\lambda b_4\\ b_1-\lambda b_3\end{pmatrix}\mu(x)\\ &\quad+\frac{\lambda^2}{2!}\begin{pmatrix}b_1-\lambda b_3\\ \lambda b_4-b_2\end{pmatrix}\mu^2(x)+\cdots\\ &=\begin{pmatrix}(\lambda b_3-b_1)\cos\lambda\mu(x)+(\lambda b_4-b_2)\sin\lambda\mu(x)\\ (b_2-\lambda b_4)\cos\lambda\mu(x)+(\lambda b_3-b_1)\sin\lambda\mu(x)\end{pmatrix}.\end{aligned}$$

Similar to

$$e^{-\lambda Bt}\begin{pmatrix}\lambda b_3-b_1\\ b_2-\lambda b_4\end{pmatrix}=\begin{pmatrix}(\lambda b_3-b_1)\cos\lambda t+(\lambda b_4-b_2)\sin\lambda t\\ (b_2-\lambda b_4)\cos\lambda t+(\lambda b_3-b_1)\sin\lambda t\end{pmatrix}.$$

Putting these equalities into (3.3), we obtain (3.1) and (3.2). Moreover, as $|\lambda|\to\infty$ uniformly in $x\in[0,\pi]$, the following asymptotic formulas hold:

$$\varphi_1(x,\lambda)=\lambda\left(b_3\cos\lambda\mu(x)+b_4\sin\lambda\mu(x)\right)+O\left(e^{|Im\lambda|\mu(x)}\right), \tag{3.4}$$

$$\varphi_2(x,\lambda)=\lambda\left(b_3\sin\lambda\mu(x)-b_4\cos\lambda\mu(x)\right)+O\left(e^{|Im\lambda|\mu(x)}\right). \tag{3.5}$$

In fact, integrating by parts the integrals involved in (3.1) and (3.2) and also from $|\sin\lambda\mu(x)| \leq e^{|Im\lambda|\mu(x)}$ and $|\cos\lambda\mu(x)| \leq e^{|Im\lambda|\mu(x)}$, the asymptotic formulas (3.4) and (3.5) are found. ☐

Lemma 3.2. *The eigenvalues λ_n, $(n \in \mathbb{Z})$ of the boundary value problem* (1.1), (1.2) *are in the form*

$$\lambda_n = \tilde{\lambda}_n + \epsilon_n,$$

where

$$\tilde{\lambda}_n = \left[n + \frac{1}{\pi}\arctan\left(\frac{c_3b_4 - c_4b_3}{b_3c_3 + c_4b_4}\right)\right]\frac{\pi}{\mu(\pi)}$$

and $\{\epsilon_n\} \in l_2$. Moreover, the eigenvalues are simple.

Proof. Substituting asymptotic formulas (3.4) and (3.5) into the expression (2.2), we have

$$\Delta(\lambda) = \lambda^2\chi(\lambda) + O\left(|\lambda|\, e^{|Im\lambda|\mu(\pi)}\right), \tag{3.6}$$

where

$$\chi(\lambda) = c_3b_4\cos\lambda\mu(\pi) - b_3c_3\sin\lambda\mu(\pi) - c_4b_3\cos\lambda\mu(\pi) - b_4c_4\sin\lambda\mu(\pi).$$

Denote

$$G_\delta := \left\{\lambda : \left|\lambda - \tilde{\lambda}_n\right| \geq \delta, n = 0, \pm1, \pm2, \dots\right\},$$

where δ is a sufficiently small positive number. For $\lambda \in G_\delta$,

$$|\chi(\lambda)| \geq C_\delta \exp\left(|Im\lambda|\,\mu(\pi)\right) \tag{3.7}$$

is valid, where C_δ is a positive number. This inequality is similarly obtained as in [22, Lemma 1.3.2]. On the other hand, there exists a constant $C > 0$ such that

$$\left|\Delta(\lambda) - \lambda^2\chi(\lambda)\right| \leq C\,|\lambda|\, e^{|Im\lambda|\mu(\pi)}. \tag{3.8}$$

Therefore on infinitely expanding contours

$$\Gamma_n := \left\{\lambda : |\lambda| = \tilde{\lambda}_n + \frac{\pi}{2\mu(\pi)}, n = 0, \pm1, \pm2, \dots\right\},$$

for sufficiently large n, using (3.7) and (3.8) we get

$$\left|\Delta(\lambda) - \lambda^2\chi(\lambda)\right| < |\lambda|^2\,|\chi(\lambda)|, \quad \lambda \in \Gamma_n.$$

Applying the Rouche theorem, it is obtained that the number of zeros of the function $\{\Delta(\lambda) - \lambda^2\chi(\lambda)\} + \lambda^2\chi(\lambda) = \Delta(\lambda)$ inside the counter Γ_n coincides with the number of zeros of function $\lambda^2\chi(\lambda)$. Moreover, using the Rouche theorem, there exist only one zero λ_n of the function $\Delta(\lambda)$ in the circle $\gamma_n(\delta) = \left\{\lambda : \left|\lambda - \tilde{\lambda}_n\right| < \delta\right\}$ is concluded. Since $\delta > 0$ is arbitrary, we have

$$\lambda_n = \left[n + \frac{1}{\pi}\arctan\left(\frac{c_3b_4 - c_4b_3}{b_3c_3 + c_4b_4}\right)\right]\frac{\pi}{\mu(\pi)} + \epsilon_n, \quad \lim_{n\to\pm\infty}\epsilon_n = 0. \tag{3.9}$$

Substituting (3.9) into (3.6), we get $\sin \epsilon_n \mu(\pi) = O(\frac{1}{n})$. It follows that $\epsilon_n = O(\frac{1}{n})$. Thus $\epsilon_n \in l_2$ is found. Moreover, the eigenvalues are simple. In fact, since $\alpha_n \beta_n = \dot{\Delta}(\lambda_n)$ and $\alpha_n \neq 0$, $\beta_n \neq 0$, we get $\dot{\Delta}(\lambda_n) \neq 0$. □

4. UNIQUENESS THEOREM BY WEYL FUNCTION

In this section, we define Weyl function and Weyl solution. Uniqueness theorem for inverse problem according to Weyl function is proved.

Denote by $\Phi(x,\lambda) = \begin{pmatrix} \Phi_1(x,\lambda) \\ \Phi_2(x,\lambda) \end{pmatrix}$ the solution of the system (1.1), satisfying the conditions

$$\begin{aligned} b_1\Phi_2(0,\lambda) + b_2\Phi_1(0,\lambda) - \lambda\left(b_3\Phi_2(0,\lambda) + b_4\Phi_1(0,\lambda)\right) &= 1, \\ c_1\Phi_2(\pi,\lambda) + c_2\Phi_1(\pi,\lambda) + \lambda\left(c_3\Phi_2(\pi,\lambda) + c_4\Phi_1(\pi,\lambda)\right) &= 0. \end{aligned}$$

The function $\Phi(x,\lambda)$ is called Weyl solution of the problem (1.1), (1.2). Let the function $C(x,\lambda) = \begin{pmatrix} C_1(x,\lambda) \\ C_2(x,\lambda) \end{pmatrix}$ be the solution of system (1.1), satisfying the initial condition

$$C_1(0,\lambda) = -\frac{b_3}{k_1}, \quad C_2(0,\lambda) = \frac{b_4}{k_1}.$$

As in Lemma 3.1, it is obtained that $C(x,\lambda) = \begin{pmatrix} C_1(x,\lambda) \\ C_2(x,\lambda) \end{pmatrix}$ has the following integral representation

$$\begin{aligned} C_1(x,\lambda) = &-\frac{b_3}{k_1}\cos\lambda\mu(x) - \frac{b_4}{k_1}\sin\lambda\mu(x) \\ &-\frac{b_3}{k_1}\int_0^{\mu(x)}\left[\tilde{B}_{11}(x,t)\cos\lambda t + \tilde{\tilde{B}}_{12}(x,t)\sin\lambda t\right]dt \\ &-\frac{b_4}{k_1}\int_0^{\mu(x)}\left[\tilde{\tilde{B}}_{11}(x,t)\sin\lambda t - \tilde{B}_{12}(x,t)\cos\lambda t\right]dt, \\ C_2(x,\lambda) = &\frac{b_4}{k_1}\cos\lambda\mu(x) - \frac{b_3}{k_1}\sin\lambda\mu(x) \\ &-\frac{b_3}{k_1}\int_0^{\mu(x)}\left[\tilde{B}_{21}(x,t)\cos\lambda t + \tilde{\tilde{B}}_{22}(x,t)\sin\lambda t\right]dt \\ &-\frac{b_4}{k_1}\int_0^{\mu(x)}\left[\tilde{\tilde{B}}_{21}(x,t)\sin\lambda t - \tilde{B}_{22}(x,t)\cos\lambda t\right]dt, \end{aligned}$$

where $\tilde{B}_{ij}(x,.) \in L_2(0,\pi)$, $\tilde{\tilde{B}}_{ij}(x,.) \in L_2(0,\pi)$, $i,j = 1,2$. The solution $\psi(x,\lambda)$ can be shown that

$$\frac{\psi(x,\lambda)}{\Delta(\lambda)} = C(x,\lambda) - \frac{\left(b_4\psi_1(0,\lambda) + b_3\psi_2(0,\lambda)\right)}{k_1\Delta(\lambda)}\varphi(x,\lambda). \tag{4.1}$$

Denote

$$M(\lambda) := -\frac{(b_4\psi_1(0,\lambda) + b_3\psi_2(0,\lambda))}{k_1\Delta(\lambda)}. \tag{4.2}$$

It is obvious that

$$\Phi(x,\lambda) = C(x,\lambda) + M(\lambda)\varphi(x,\lambda). \tag{4.3}$$

The function

$$M(\lambda) = -\frac{(b_4\Phi_1(0,\lambda) + b_3\Phi_2(0,\lambda))}{k_1}$$

is called the Weyl function of the boundary value problem (1.1), (1.2). The Weyl solution and Weyl function are meromorphic functions having simple poles at points λ_n eigenvalues of problem (1.1), (1.2). It is obtained from (4.1) and (4.3) that

$$\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{4.4}$$

Theorem 4.1. *For the Weyl function $M(\lambda)$, the following representation holds:*

$$M(\lambda) = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{4.5}$$

Proof. Since

$$\begin{aligned} W[C(x,\lambda),\psi(x,\lambda)] &= C_2(x,\lambda)\psi_1(x,\lambda) - C_1(x,\lambda)\psi_2(x,\lambda) \\ &= \frac{b_4\psi_1(0,\lambda) + b_3\psi(0,\lambda)}{k_1} \\ &= -\left[(c_1+\lambda c_3)\, C_2(\pi,\lambda) + (c_2+\lambda c_4)\, C_1(\pi,\lambda)\right], \end{aligned}$$

we can rewrite the Weyl function (4.2) as follows

$$M(\lambda) = \frac{(c_1+\lambda c_3)\, C_2(\pi,\lambda) + (c_2+\lambda c_4)\, C_1(\pi,\lambda)}{\Delta(\lambda)}.$$

Using the expression of solution $C(x,\lambda)$ and taking into account

$$|\Delta(\lambda)| \geq |\lambda|^2\, C_\delta \exp(|Im\lambda|\, \mu(\pi)), \tag{4.6}$$

we have

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in G_\delta}} |M(\lambda)| = 0. \tag{4.7}$$

Since $\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n)$,

$$\beta_n = -\frac{b_4\psi_1(0,\lambda_n) + b_3\psi_2(0,\lambda_n)}{k_1}.$$

Then, we get

$$\underset{\lambda=\lambda_n}{Res}\, M(\lambda) = -\frac{b_4\psi_1(0,\lambda_n) + b_3\psi_2(0,\lambda_n)}{k_1\dot{\Delta}(\lambda_n)} = \frac{1}{\alpha_n}. \tag{4.8}$$

Consider the following contour integral

$$I_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N(\lambda)} \frac{M(\xi)}{\xi-\lambda}d\xi, \quad \xi \in int\Gamma_N,$$

where

$$\Gamma_N = \left\{\lambda : |\lambda| = \left(N + \frac{1}{\pi}\arctan\left(\frac{c_3b_4 - c_4b_3}{b_3c_3 + c_4b_4}\right)\right)\frac{\pi}{\mu(\pi)} + \frac{\pi}{2\mu(\pi)}\right\}.$$

It follows from (4.7) that $\lim_{N\to\infty} I_N(\lambda) = 0$. On the other hand, applying Residue theorem and the residue (4.8),

$$I_N(\lambda) = M(\lambda) + \sum_{\lambda_n \in int\Gamma_N} \frac{1}{\alpha_n(\lambda_n - \lambda)}$$

is found. Thus, as $N \to \infty$

$$M(\lambda) = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}$$

is obtained. □

Now, we seek inverse problem of the reconstruction of the problem (1.1), (1.2) by Weyl function $M(\lambda)$ and spectral data $\{\lambda_n, \alpha_n\}, (n \in \mathbb{Z})$. Along with problem (1.1), (1.2), we consider a boundary value problem of the same form, but with another potential function $\tilde{\Omega}(x)$. Let's agree to that if some symbol s denotes an object relating to the problem (1.1), (1.2), then $\tilde{s}$ will denote an object, relating to the boundary value problem with the function $\tilde{\Omega}(x)$.

Theorem 4.2. *If $M(\lambda) = \tilde{M}(\lambda)$, then $\Omega(x) = \tilde{\Omega}(x)$, i.e. the boundary value problem* (1.1), (1.2) *is uniquely determined by the Weyl function.*

Proof. We describe the matrix $P(x,\lambda) = [P_{ij}(x,\lambda)]_{i,j=1,2}$ with the formula

$$P(x,\lambda)\begin{pmatrix} \tilde{\varphi}_1 & \tilde{\Phi}_1 \\ \tilde{\varphi}_2 & \tilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 & \Phi_1 \\ \varphi_2 & \Phi_2 \end{pmatrix}. \tag{4.9}$$

The Wronskian of the solutions $\tilde{\varphi}(x,\lambda)$ and $\tilde{\Phi}(x,\lambda)$ is

$$W[\tilde{\varphi}(x,\lambda), \tilde{\Phi}(x,\lambda)] = \tilde{\varphi}_2(x,\lambda)\tilde{\Phi}_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\tilde{\Phi}_2(x,\lambda) = 1. \tag{4.10}$$

Using (4.9) and (4.10), we calculate

$$\begin{aligned} P_{11}(x,\lambda) &= \Phi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda) - \varphi_1(x,\lambda)\tilde{\Phi}_2(x,\lambda), \\ P_{12}(x,\lambda) &= \varphi_1(x,\lambda)\tilde{\Phi}_1(x,\lambda) - \Phi_1(x,\lambda)\tilde{\varphi}_1(x,\lambda), \\ P_{21}(x,\lambda) &= \Phi_2(x,\lambda)\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)\tilde{\Phi}_2(x,\lambda), \\ P_{22}(x,\lambda) &= \varphi_2(x,\lambda)\tilde{\Phi}_1(x,\lambda) - \Phi_2(x,\lambda)\tilde{\varphi}_1(x,\lambda) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\varphi_1(x,\lambda) &= P_{11}(x,\lambda)\tilde{\varphi}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}_2(x,\lambda),\\
\varphi_2(x,\lambda) &= P_{21}(x,\lambda)\tilde{\varphi}_1(x,\lambda) + P_{22}(x,\lambda)\tilde{\varphi}_2(x,\lambda),\\
\Phi_1(x,\lambda) &= P_{11}(x,\lambda)\tilde{\Phi}_1(x,\lambda) + P_{12}(x,\lambda)\tilde{\Phi}_2(x,\lambda),\\
\Phi_2(x,\lambda) &= P_{21}(x,\lambda)\tilde{\Phi}_1(x,\lambda) + P_{22}(x,\lambda)\tilde{\Phi}_2(x,\lambda).
\end{aligned} \tag{4.12}$$

Taking into account (4.4), (4.10) and (4.11),

$$P_{11}(x,\lambda) - 1 = \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}\{\tilde{\varphi}_1(x,\lambda) - \varphi_1(x,\lambda)\} - \tilde{\varphi}_2(x,\lambda)\left\{\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right\}$$

$$P_{12}(x,\lambda) = \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\{\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\} + \varphi_1(x,\lambda)\left\{\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right\}$$

$$P_{21}(x,\lambda) = \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\{\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)\} + \varphi_2(x,\lambda)\left\{\frac{\psi_2(x,\lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)}\right\}$$

$$P_{22}(x,\lambda) - 1 = \tilde{\varphi}_1(x,\lambda)\left\{\frac{\tilde{\psi}_2(x,\lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\right\} - \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{\Delta}(\lambda)}\{\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)\}$$

are found. Using (4.6), we obtain

$$\begin{aligned}
&\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{11}(x,\lambda) - 1| = 0,\\
&\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{22}(x,\lambda) - 1| = 0,\\
&\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{12}(x,\lambda)| = 0,\\
&\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}} \max_{0\le x\le\pi} |P_{21}(x,\lambda)| = 0.
\end{aligned} \tag{4.13}$$

Substituting (4.3) into (4.11), we have

$$P_{11}(x,\lambda) = C_1(x,\lambda)\tilde{\varphi}_2(x,\lambda) - \varphi_1(x,\lambda)\tilde{C}_2(x,\lambda) + \varphi_1(x,\lambda)\tilde{\varphi}_2(x,\lambda)\left[M(\lambda) - \tilde{M}(\lambda)\right],$$

$$P_{12}(x,\lambda) = \varphi_1(x,\lambda)\tilde{C}_1(x,\lambda) - C_1(x,\lambda)\tilde{\varphi}_1(x,\lambda) + \varphi_1(x,\lambda)\tilde{\varphi}_1(x,\lambda)\left[\tilde{M}(\lambda) - M(\lambda)\right],$$

$$P_{21}(x,\lambda) = C_2(x,\lambda)\tilde{\varphi}_2(x,\lambda) - \varphi_2(x,\lambda)\tilde{C}_2(x,\lambda) + \varphi_2(x,\lambda)\tilde{\varphi}_2(x,\lambda)\left[M(\lambda) - \tilde{M}(\lambda)\right],$$

$$P_{22}(x,\lambda) = \varphi_2(x,\lambda)\tilde{C}_1(x,\lambda) - C_2(x,\lambda)\tilde{\varphi}_1(x,\lambda) + \varphi_2(x,\lambda)\tilde{\varphi}_1(x,\lambda)\left[\tilde{M}(\lambda) - M(\lambda)\right].$$

Hence, if $M(\lambda) \equiv \tilde{M}(\lambda)$, $P_{ij}(x,\lambda)_{i,j=1,2}$ are entire functions with respect to λ for every fixed x. Then from (4.13), we find

$$P_{11}(x,\lambda) \equiv 1, \quad P_{12}(x,\lambda) \equiv 0,$$

$$P_{21}(x,\lambda) \equiv 0, \quad P_{22}(x,\lambda) \equiv 1.$$

Substituting these identities into (4.12),

$$\varphi_1(x,\lambda) \equiv \tilde{\varphi}_1(x,\lambda), \quad \varphi_2(x,\lambda) \equiv \tilde{\varphi}_2(x,\lambda),$$

$$\Phi_1(x,\lambda) \equiv \tilde{\Phi}_1(x,\lambda), \quad \Phi_2(x,\lambda) \equiv \tilde{\Phi}_2(x,\lambda)$$

are obtained for all x and λ, so $\Omega(x) \equiv \tilde{\Omega}(x)$. □

According to (4.5), the specification of the Weyl function $M(\lambda)$ is equivalent to the specification of the spectral data $\{\lambda_n, \alpha_n\}$, $n \in \mathbb{Z}$. That is, if $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$ for all $n \in \mathbb{Z}$, $M(\lambda) = \tilde{M}(\lambda)$ is obtained. It follows from Theorem 4.2 that $\Omega(x) = \tilde{\Omega}(x)$. We have thus proved the following theorem:

Theorem 4.3. *The boundary value problem* (1.1), (1.2) *is uniquely determined by spectral data* $\{\lambda_n, \alpha_n\}$, $n \in \mathbb{Z}$.

REFERENCES

[1] M. J. Ablowitz, H. Segur, *Solutions and Inverse Scattering Transform*, SIAM Stud. Appl. Math. 4, Society for Industrial and Applied Mathematics, Philadelphia, 1981.

[2] M. S. Agranovich, Spectral problems for the Dirac system with spectral parameter in local boundary conditions, *Functional Analysis and Applications*, 35(3):161–175, 2001.

[3] O. Aydoğdu, E. Maghsoodi, H. Hassanabadi, Dirac equation for the Hulthen potential within the Yukawa-type tensor interaction, *Chin. Phys. B*, 22(1): 010302, 2013.

[4] O. Aydoğdu, R. Sever, Pseudospin and spin symmetry in Dirac-Morse problem with a tensor potential, *Physics Letters B*, 703(3):379–385, 2011.

[5] S. Clark, F. Gesztesy, Weyl-Titchmarsh M-function asymptotics, local uniqueness results, trace formulas and Borg-type theorems for Dirac operators, *Trans. Amer. Math. Soc.*, 354(9):3475–3534, 2002.

[6] A. Çöl, Kh. R. Mamedov, On an inverse scattering problem for a class of Dirac operators with spectral parameter in the boundary condition, *J. Math. Anal. Appl.*, 393:470–478, 2012.

[7] G. Freiling, V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science Publisher, Huntington, New York, 2001.

[8] B. Fritzsche, B. Kirstein, A. L. Sakhnovich, Semiseparable integral operators and explicit solution of an inverse problem for a skew-self-adjoint Dirac-type system, *Integral Equations and Operator Theory*, 66(2):231–251, 2010.

[9] B. Fritzsche, B. Kirstein, I. Ya. Roitberg, A. Sakhnovich, Skew-self-adjoint Dirac system with a rectangular matrix potential: Weyl theory, direct and inverse problems, *Integral Equations and Operator Theory*, 74(2):163–187, 2012.

[10] M. G. Gasymov, T. T. Dzabiev, Solution of inverse problem by two spectra for the Dirac equation on a finite interval, *Akad. Nauk Azerbaidzan SSR Dokl.*, 22(7):3–6, 1966 (Russian).

[11] M. G. Gasymov, B. M. Levitan, The inverse problem for the Dirac system, *Dokl. Akad. Nauk SSSR*, 167:967–970, 1966 (Russian).

[12] M. G. Gasymov, The inverse scattering problem for a system of Dirac equations of order $2n$, *Trans. Moskow Math. Soc.*, 19:41–119, 1968.

[13] M. Horvath, On the inverse spectral theory of Schrödinger and Dirac operators, *Trans. Amer. Math. Soc.*, 353:4155–4171, 2001.

[14] M. Kiss, An n-dimensional Ambarzumian type theorem for Dirac operators, *Inverse Problems*, 20(5):1593–1597, 2004.

[15] V. M. Kurbanov, A. I. Ismailova, Absolute and uniform convergence of expansions in the root vector functions of the Dirac operator, *Doklady Mathematics*, 86(2):663–666, 2012.

[16] V. M. Kurbanov, A. I. Ismailova, Riesz inequality for systems of root vector functions of the Dirac operator, *Differential Equations*, 48(3):336–342, 2012.

[17] A. R. Latifova, On the representation of solution with initial conditions for Dirac equations system with discontinuous coefficient, *Proceeding of IMM of NAS of Azerbaijan*, 16(24):64–68, 2002.

[18] B. M. Levitan, I. S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Kluwer Academic Publishers Group, Dordrecht, 1991.

[19] X. Li, On the inverse problem for the Dirac operator, *Inverse Problems*, 23(3):919–932, 2007.

[20] Kh. R. Mamedov, O. Akcay, Inverse eigenvalue problem for a class of Dirac operators with discontinuous coefficient, *Boundary Value Problems*, 2014:110, doi:10.1186/1687-2770-2014-110

[21] Kh. R. Mamedov, A. Çöl, On an inverse scattering problem for a class Dirac operator with discontinuous coefficient and nonlinear dependence on the spectral parameter in the boundary condition, *Mathematical Methods in the Applied Sciences*, 35(14):1712–1720, 2012.

[22] V. A. Marchenko, *Sturm-Liouville Operators and Their Applications*, AMS Chelsea Publishing, Providence, Rhode Island, 2011.

[23] K. Mochizuki, I. Trooshin, Inverse problem for interior spectral data of the Dirac operator on a finite interval, *Publ. RIMS Kyoto Univ.*, 38:387–395, 2002.

[24] A. B. de Monvel, D. Shepelsky, Initial boundary value problem for the mKdV equation on a finite interval, *Ann. Inst. Fourier, Grenoble*, 54(5):1477–1495, 2004.

[25] I. M. Nabiev, Solution of the inverse quasiperiodic problem for the Dirac system, *Math. Notes*, 89(5-6):845–852, 2011.

[26] A. Sakhnovich, Skew-self-adjoint discrete and continuous Dirac-type systems: inverse problems and Borg-Marchenko theorems, *Inverse Problems*, 22(6):2083–2101, 2006.

[27] B. Thaller, *The Dirac Equation*, Springer, Berlin, 1992.

[28] B. A. Watson, Inverse spectral problems for weighted Dirac system, *Inverse Problems*, 15(3):793–805, 1999.

[29] C. F. Yang, Hochstadt Lieberman theorem for Dirac operator with eigenparameter dependent boundary conditions, *Nonlinear Analysis*, 74:2475–2484, 2011.

[30] C. F. Yang, Z. Y. Huang, Reconstruction of the Dirac operator from nodal data, *Integral Equations and Operator Theory*, 66(4):539–551, 2010.

[31] C. F. Yang, A. P. Yang, Some Ambarzumian-type theorems for Dirac operators, *Inverse Problems*, 25(9):095012 (13pp), 2009.

[32] V. A. Yurko, *Inverse Spectral Problems for Differential Operators and Their Applications*, Gordon and Breach Science Publishers, 2000.