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QUALITATIVE ANALYSIS OF SECOND ORDER FUNCTIONAL EVOLUTION INCLUSIONS

ABDESSLAM BALIKI[a], MOUFFAK BENCHOHRA[a,b], AND JUAN J. NIETO[c,b]

[a]Laboratory of Mathematics, University of Sidi Bel-Abbes
PO Box 89, Sidi Bel-Abbes 22000, Algeria
[b]Department of Mathematics, Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
[c]Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain

ABSTRACT. In this paper we investigate the existence and attractivity of mild solutions on infinite intervals to second order semilinear evolution inclusion with infinite delay in a Banach space. The proofs of the main results are based on Bohnenblust-Karlin's fixed point theorem and the theory of evolution system.

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1. INTRODUCTION

Functional evolution equations arise in various mathematical models to describe many phenomena of physics, mechanics, biology and other fields; For more details on this theory and on its applications we refer to the monographs of Abbas and Benchohra [1], Hale and Verduyn Lunel [11], Kolmanovskii and Myshkis [17], and Wu [22] and the reference therein.

Motivated by the papers by Balachandran *et al.* [2, 3], Fattorini [8], Hernández [12], Henríquez *et al.* [13, 14], Travis and Webb [21] concerning the existence of mild solutions of abstract second order differential equations on bounded intervals, the authors in [4] have studied existence and attractivity of solution of following second-order evolution equation

$$\begin{cases} y''(t) - A(t)y(t) = f(t, y_t), \quad t \in J := [0, \infty), \\ y_0 = \phi, \ y'(0) = \tilde{y}. \end{cases}$$

In this paper, we extend the obtained results to the following evolution inclusions

$$\begin{cases} y''(t) - A(t)y(t) \in F(t, y_t), \quad t \in J := [0, \infty), \\ y_0 = \phi, \ y'(0) = \tilde{y}, \end{cases} \tag{1.1}$$

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where $\{A(t)\}_{0\leq t<+\infty}$ is a family of linear closed operators from E into E that generate an evolution system of operators $\{\mathcal{U}(t,s)\}_{(t,s)\in J\times J}$ for $0\leq s\leq t<+\infty$, $F: J\times\mathcal{B}\to P(E)$ is a multivalued map with nonempty compact convexe values and $\phi\in\mathcal{B}$. $\mathcal{B}$ is an abstract phase space to be specified later, $\tilde{y}\in E$, $\phi\in\mathcal{B}$ and $(E,|\cdot|)$ a real separable Banach space.

For any continuous function y and any $t\geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta)=y(t+\theta)$ for $\theta\in(-\infty,0]$. Here $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

Our first primary concern is the question of existence and uniqueness of solution of the homogenous equation

$$y''(t)=A(t)y(t),\quad \text{for}\quad t\geq 0,$$

which is related to the existence of an evolution system noted by $\mathcal{U}(t,s)$. For this purpose there are many techniques to show the existence of $\mathcal{U}(t,s)$ which has been developed by Kozak [18].

The organization of this work is as follows. In the next section we give necessary preliminaries from the theory of evolution systems and the theory of multivalued maps. In Section 3, we give the existence of mild solutions of the problem (1.1). In Section 4, we show the attractivity of the mild solution and an illustrating example is given to show the applicability of our results in Section 5.

2. PRELIMINARIES

Let E a Banach space with norm $|\cdot|$ and $BC(J,E)$ the Banach space of bounded and continuous functions y mapping J into E with the usual supremum norm

$$\|y\|=\sup_{t\in J}|y(t)|.$$

Let $\mathcal{X}$ is the space defined by

$$\mathcal{X}=\{y:\mathbb{R}\to E \text{ such that } y|_J\in BC(J,E) \text{ and } y|_{(-\infty,0]}\in\mathcal{B}\},$$

we denote by $y|_J$ (resp. $y|_{(-\infty,0]}$) the restriction of y to J (resp. $(-\infty,0]$).

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [10] and follow the terminology used in [15]. Thus, $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into E, and satisfying the following axioms :

(A_1) If $y:(-\infty,b)\to E, b>0$, is continuous on $[0,b]$ and $y_0\in\mathcal{B}$, then for every $t\in[0,b)$ the following conditions hold:

(i) $y_t\in\mathcal{B}$;

(ii) There exists a positive constant H such that $|y(t)|\leq H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that:

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Remark 2.1. In the sequel we assume that K and M are bounded on J and

$$\gamma := \max\left\{\sup_{t\in\mathbb{R}_+}\{K(t)\}, \sup_{t\in\mathbb{R}_+}\{M(t)\}\right\}.$$

For other details we refer, for instance to the book by Hino *et al* [15].

In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ dense in E and independent of t.

In this work the existence of solution the problem (1.1) is related to the existence of an evolution operator $\mathcal{U}(t, s)$ for the following homogeneous problem

$$y''(t) = A(t)y(t) \qquad t \in J. \tag{2.1}$$

This concept of evolution operator has been developed by Kozak [18].

Definition 2.2. A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t, s) : E \to E$, $(t, s) \in \Delta := \{(t, s) \in J \times J : s \leq t\}$, is called an evolution operator of the equation (2.1) if de following conditions hold:

(D_1): For each $x \in E$ the map $(t, s) \longmapsto \mathcal{U}(t, s)x$ is of continuously differentiable and

(a): for each $t \in J$ $\mathcal{U}(t, t) = 0$.

(b): for all $(t, s) \in \Delta$ and for each $x \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t, s)x\big|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t, s)x\big|_{t=s} = -x$.

(D_2): For all $(t, s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t, s)x \in D(A(t))$, the map $(t, s) \longmapsto \mathcal{U}(t, s)x$ is of class C^2 and

(a): $\frac{\partial^2}{\partial t^2}\mathcal{U}(t, s)x = A(t)\mathcal{U}(t, s)x$.

(b): $\frac{\partial^2}{\partial s^2}\mathcal{U}(t, s)x = \mathcal{U}(t, s)A(s)x$

(c): $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t, s)x\big|_{t=s} = 0$.

(D_3): For all $(t, s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t, s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t, s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t, s)x$ and

(a): $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t, s)x = A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t, s)x$.

Moreover, the map $(t, s) \longmapsto A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t, s)x$ is continuous.

(b): $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t, s)x = \frac{\partial}{\partial t}\mathcal{U}(t, s)A(s)x$.

The following compactness criterion in $C(\mathbb{R}_+, E)$ is particularly useful.

Lemma 2.3 ([6] [Corduneanu]). *Let $C \subset BC(\mathbb{R}_+, E)$ be a set satisfying the following conditions:*

- **(i):** *C is bounded in $BC(\mathbb{R}_+, E)$;*
- **(ii):** *the functions belonging to C are equicontinuous on any compact interval of $\mathbb{R}_+$;*
- **(iii):** *the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of $\mathbb{R}_+$;*
- **(iv):** *the functions from C are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $y \in C$.*

Then C is relatively compact in $BC(\mathbb{R}_+, E)$.

2.1. **Some properties of set-valued maps.** Let (X, d) be a metric space and Y be a subset of X. We denote:

$$P(X) = \{Y \subset X : Y \neq \emptyset\},\ P_b(X) = \{Y \subset X : Y \text{ bounded}\},$$

$$P_{cl}(X) = \{Y \subset X : Y \text{ closed}\},\ P_{cp}(X) = \{Y \subset X : Y \text{ compact}\},$$

$$P_{cv}(X) = \{Y \subset X : Y \text{ convex}\},\ P_{cv,cp}(X) = P_{cv}(X) \cap P_{cp}(X).$$

Let $A, B \in P(X)$, consider $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ the Hausdorff distance between A and B given by:

$$H_d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ and $d(A, b) = \inf\{d(a, b) : a \in A\}$.

Then, $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space.

Definition 2.4. A multivalued map $F : J \to P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \to X$ defined by

$$Y(t) = d(x, F(t)) = \inf\{d(x, z) : z \in F(t)\},$$

is measurable.

Definition 2.5. Let $\Gamma_F \subset X \times Y$, defined by

$$\Gamma_F = \{(x, y) : x \in X, y \in F(x)\}$$

is called th graph of F. We say that F is has a closed graph, if Γ_F is closed in $X \times Y$.

Definition 2.6. A multifunction $F : X \to P(Y)$ is said to be

1. compact, if its range $F(X)$ is relatively compact in Y;
2. locally compact, if every point $x \in X$ has a neighborhood V_x such that the restriction of F to V_x is compact.

Proposition 2.7 ([9])**.** *Assume $F : X \longrightarrow P(Y)$ is a multivalued map such that $F(X) \subset K$ and the graph Γ_F of F is closed, where K is a compact set. Then F is u.s.c.*

Definition 2.8. Let E be a Banach space. A multivalued map $F : J \times E \to E$ is said to be L^1−Carathéodory if

(i): $t \longmapsto F(t, y)$ is measurable for all $y \in E$,
(ii): $y \longmapsto F(t, y)$ is upper semicontinuous for almost each $t \in J$,
(iii): for each $\rho > 0$, there exists $\psi_\rho \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\|_P \leq \psi_\rho(t), \quad \text{for all } |y| \leq \rho \text{ and almost each } \ t \in J,$$

such that $\|F(t, y)\|_P = \sup\{|f| : f \in F(t, y)\}$.

Definition 2.9. Let X, Y be nonempty sets and $F : X \to P(Y)$. The single-valued operator $f : X \to Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$. The set of all selection functions for F is denoted by S_F.

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the books of Deimling [7], Gorniewicz [9] and Hu and Papageorgiou [16].

The following lemmas are useful to prove our result.

Lemma 2.10 (Lasota and Opial [19])**.** *Let E be a Banach space and I bounded closed interval, and F be an L^1-Caratheodory multivalued map with compact convex values, and let $\mathcal{L} : L^1(I, E) \to C(J, E)$ be a linear continuous mapping. Then the operator*

$$\mathcal{L} \circ S_F : C(I, E) \to P_{cp,cv}(C(I, E)),$$

is a closed graph operator in $C(I, E) \times C(I, E)$.

Lemma 2.11 ([9])**.** *Let X be a separable metric space. Then every measurable multivalued map $F : X \to P_{cl}(X)$ has a measurable selection.*

Definition 2.12. Let $T : X \to P(X)$ be a multi-valued map. An element $x \in X$ is said to be a fixed point of T if $x \in T(x)$.

Lemma 2.13 (Bohnenblust-Karlin [5])**.** *Let E be a Banach space and $D \in P_{cl,c}(E)$. Suppose that the operator $T : D \to P_{cl,c}(D)$ is upper semicontinuous and the set $T(D)$ is relatively compact in E. Then T has a fixed point in D.*

3. MAIN RESULT

Definition 3.1. A function $y \in \mathcal{X}$ is called a mild solution of the problem (1.1), if y is continuous and there exists a function $f \in L^1(J, E)$ such that $f(t) \in F(t, y_t)$ a.e.

on J, and

$$y(t)=\begin{cases}\phi(t), & \text{if } t\leq 0\\ -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0)+\mathcal{U}(t,0)\tilde{y}+\displaystyle\int_0^t \mathcal{U}(t,s)f(s)ds, & \text{if } t\in J.\end{cases} \tag{3.1}$$

To prove our results we introduce the following conditions:

(H_1)**:** There exist a constant $\widehat{M}\geq 1$ and $\omega>0$ such that

$$\|\mathcal{U}(t,s)\|_{B(E)}\leq \widehat{M}e^{-\omega(t-s)} \quad \text{for every } (t,s)\in\Delta.$$

(H_2)**:** There exists a constant $\tilde{M}\geq 0$ such that

$$\left\|\frac{\partial}{\partial s}\mathcal{U}(t,s)\right\|_{B(E)}\leq \tilde{M}.$$

(H_3)**:** There exist a function $p\in L^1(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+\to(0,\infty)$ such that:

$$\|F(t,u)\|_P\leq p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B}.$$

(H_4)**:** For each $(t,s)\in\Delta$ we have

$$\lim_{t\to+\infty}\int_0^t e^{-w(t-s)}p(s)ds=0.$$

Theorem 3.2. *If assumptions* (H_1)–(H_4) *are satisfied, and there exists* $\eta>0$ *such that*

$$\eta\geq \widehat{M}\psi(\delta_\eta)\|p\|_{L^1}, \tag{3.2}$$

where

$$\delta_\eta:=\gamma\eta+\gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1)+\gamma\hat{M}|\tilde{y}|,$$

then the problem (1.1) *has at least one mild solution.*

Proof. To commence, we consider the operator $T:\mathcal{X}\longrightarrow P(\mathcal{X})$ defined by:

$$Ty(t)=\left\{g\in\mathcal{X}: g(t)=\begin{cases}\phi(t), & \text{if } t\in(-\infty,0]\\ -\dfrac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0)+\mathcal{U}(t,0)\tilde{y} & \\ +\displaystyle\int_0^t\mathcal{U}(t,s)f(s)ds, & \text{if } t\in J,\end{cases}\right\} \tag{3.3}$$

where $f\in S_{F(\cdot,y_\cdot)}=\{v\in L^1(J,E): v(t)\in F(t,y_t) \text{ a.e. } t\in J\}$. From Lemma 2.11 $S_{F(\cdot,y_\cdot)}$ is nonempty.

For $\phi\in\mathcal{B}$, Let $x:(-\infty,+\infty)\longrightarrow E$ be the function defined by

$$x(t)=\begin{cases}\phi(t), & \text{if } t\in(-\infty,0]\\ -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0)+\mathcal{U}(t,0)\tilde{y} & \text{if } t\in J.\end{cases}$$

Then $x_0=\phi$. For each function $z\in\mathcal{X}$, we denote

$$y(t)=x(t)+z(t).$$

It is obvious that y satisfies (3.3) if and only if z satisfies $z_0 = 0$ and for all $t \in J$

$$z(t) = \int_0^t \mathcal{U}(t,s)f(s)ds. \tag{3.4}$$

where $f(t) \in F(t, z_t + x_t)$ for each $t \in J$.

In the sequel, we always denote $\mathcal{X}_0$ as the Banach space defined by

$$\mathcal{X}_0 = \{z \in \mathcal{X} : z_0 = 0\},$$

endowed with the norm

$$\|z\|_{\mathcal{X}_0} = \sup_{t\in J} |z(t)| + \|z_0\|_{\mathcal{B}} = \sup_{t\in J} |z(t)|.$$

Now, we consider the operator $\widetilde{T} : \mathcal{X}_0 \longrightarrow P(\mathcal{X}_0)$ given by

$$\widetilde{T}z(t) = \left\{ h(t) \in \mathcal{X}_0 : \ h(t) = \int_0^t \mathcal{U}(t,s)f(s)ds, \ \text{for } t \in J \right\},$$

where $f \in S_{F(\cdot,z_\cdot)} = \{v \in L^1(J,E) : v(t) \in F(t, z_t + x_t) \text{ a.e. } t \in J\}$.

Obviously the problem (1.1) has a solution is equivalent to $\widetilde{T}$ has a fixed point. To prove this end, we start with the following estimate.

For each $z \in \mathcal{X}_0$ and $t \in J$, we have

$$\begin{aligned} \|z_t + x_t\|_{\mathcal{B}} &\leq \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}} \\ &\leq K(t)|z(t)| + K(t)\|\frac{\partial}{\partial s}\mathcal{U}(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\ &+ K(t)\|\mathcal{U}(t,0)\|_{B(E)}|\tilde{y}| + M(t)\|\phi\|_{\mathcal{B}} \\ &\leq \gamma\|z\|_{\mathcal{X}_0} + \gamma\tilde{M}\|\phi\|_{\mathcal{B}} + \gamma\hat{M}e^{-\omega t}|\tilde{y}| + \gamma\|\phi\|_{\mathcal{B}} \\ &\leq \gamma\|z\|_{\mathcal{X}_0} + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma\hat{M}|\tilde{y}|. \end{aligned} \tag{3.5}$$

Let y a possible solution of the problem (1.1). Then, for $t \in J$ there exist $f(t) \in F(t, z_t + x_t)$ and by using (H_1), (H_3) and (3.2) and (3.5) we have

$$\begin{aligned} |z(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}|f(s)|ds \\ &\leq \widehat{M}\int_0^t e^{-\omega(t-s)}p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})ds \\ &\leq \widehat{M}\|p\|_{L^1}\psi\left(\gamma\|z\|_{\mathcal{X}_0} + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma\hat{M}|\tilde{y}|\right), \\ &\leq \widehat{M}\psi(\delta_\eta)\|p\|_{L^1} \leq \eta. \end{aligned}$$

Now, for η satisfying (3.2) we consider the set

$$D_R = \{z \in \mathcal{X}_0 : \|z\|_{\mathcal{X}_0} \leq R\}.$$

Consequently, the operator $\widetilde{T}(D_R) \subset D_R$.

Now, we will show that the operator $\widetilde{T}$ satisfied all conditions of Theorem 2.13. The proof will be given in several steps.

Step 1. $\widetilde{T}$ convex.

Let $h_1, h_2 \in \widetilde{T}z(t)$ then, there exist $f_1, f_2 \in S_{F(.,z.)}$ such that for each $t \in J$ and $\alpha \in [0,1]$ we have

$$\alpha h_1(t) + (1-\alpha)h_2(t) = \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}(\alpha f_1(s) + (1-\alpha) f_2(s))\, ds$$

Since $S_{F(.,z.)}$ is convex (F has convex values), thus

$$\alpha h_1(t) + (1-\alpha)h_2(t) \in \widetilde{T}z(t).$$

Step 2. $\widetilde{T}(D_\eta)$ **relatively compact**.

Let D_η is a bounded subset of $\mathcal{X}_0$. To show that $\widetilde{T}(D_\eta)$ is relatively compact we will use Lemma 2.3.

♦ $\widetilde{T}(D_\eta)$ is **equicontinuous**.

Let $s, t \in [0,b]$ with $t > s$ and $h(\cdot) \in \widetilde{T}z(\cdot)$ for $z \in D_\eta$. Then, there is $f \in S_{F(\cdot,z.)}$, such that

$$\begin{aligned} |h(t) - h(s)| &= \left|\int_0^t \Big(\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)\Big) f(\tau)d\tau + \int_s^t \mathcal{U}(t,\tau)f(\tau)d\tau\right| \\ &\leq \int_0^s \|\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)\|_{B(E)} p(\tau)\psi(\|z_\tau + x_\tau\|_{\mathcal{B}})d\tau \\ &\quad + \hat{M}\int_s^t e^{-\omega(t-\tau)}p(\tau)\psi(\|z_\tau + x_\tau\|_{\mathcal{B}})d\tau. \end{aligned}$$

Now, by the inequality (3.5) we get

$$|h(t) - h(s)| \leq \psi(\delta_\eta)\int_0^s \|\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)\|_{B(E)}\, p(\tau)\, d\tau + \hat{M}\psi(\delta_\eta)\int_s^t p(\tau)d\tau.$$

The right-hand side of the above inequality tends to zero as $t - s \longrightarrow 0$, which implies that $\widetilde{T}(D_\eta)$ is equicontinuous.

♦ $\Lambda := \{(\widetilde{T}z)(t) : z \in D_\eta\}$ **is relatively compact in** E.

Let $t \in J$ be a fixed and let $0 < \varepsilon < t \leq b$. For $z \in D_\eta$ we define

$$\widetilde{T}_\varepsilon(z)(t) = \left\{h_\varepsilon \in \mathcal{X} : h_\varepsilon(t) = \mathcal{U}(t, t-\varepsilon)\int_0^{t-\varepsilon} \mathcal{U}(t-\varepsilon, s)f(s)ds\right\}.$$

Since $\mathcal{U}(t,s)$ is a compact operator for $t > s$, and the set $\Lambda_\varepsilon := \{(\widetilde{T}_\varepsilon z)(t) : z \in D_\eta\}$ is the image of bounded set of E by $\mathcal{U}(t,s)$ then Λ_ε is precompact in E. Furthermore, for $z \in D_\eta$ and $h_\varepsilon(t) \in \widetilde{T}_\varepsilon(z)(t)$, we have

$$\begin{aligned} |h(t) - h_\varepsilon(t)| &\leq \int_{t-\varepsilon}^t \|\mathcal{U}(t,s)\|_{B(E)}\Big|f(s, z_s + x_s)\Big|ds \\ &\leq \int_{t-\varepsilon}^t \|\mathcal{U}(t,s)\|_{B(E)} p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})ds \\ &\leq \hat{M}\psi(\delta_\eta)\int_{b-\varepsilon}^b e^{-\omega(b-s)}p(s)ds. \end{aligned}$$

The right-hand side tends to zero as $\varepsilon \longrightarrow 0$, then $\widetilde{T}_\varepsilon(z)$ converge uniformly to $\widetilde{T}(z)$ which implies that $D_\eta(t)$ is precompact in E.

♦ $\widetilde{T}(D_\eta)$ **is equiconvergent**.

Let $z \in D_\eta$ and $h(t) \in \widetilde{T}(z)(t)$ for $t \in J$, then from the assumptions (H_1)–(H_3) and (3.5) we have

$$|h(t)| \leq \hat{M}\psi(\delta_\eta)\int_0^t e^{-\omega(t-s)}p(s)ds,$$

it follows immediately by (3.5) that $|h(t)| \longrightarrow 0$ as $t \longrightarrow +\infty$. Then

$$\lim_{t\to+\infty}|h(t) - h(+\infty)| = 0$$

which implies that $\widetilde{T}$ is equiconvergent.

Hence the operator $\widetilde{T}$ is relatively compact in $\mathcal{X}_0$.

Step 3. $\widetilde{T}$ has closed graph.

Let $z_n \longrightarrow \hat{z}$, $(h_n)_n \subset \widetilde{T}(z_n)$ with $h_n \longrightarrow \widehat{h}$. We shall prove that $\hat{h} \in \widetilde{T}(\hat{z})$. Since $h_n \in \widetilde{T}(z_n)$, there exist $f_n \in S_{F(.,z.n)}$ such that

$$h_n(t) = \int_0^t \mathcal{U}(t,s)f_n(s)ds, \quad t \in J.$$

We must show that there exists $f \in S_{F(\cdot,\hat{z}.)}$ such that

$$\hat{h}(t) = \int_0^t \mathcal{U}(t,s)f(s)ds, \quad t \in J.$$

Consider the linear and continuous operator $N : L^1(J,E) \longrightarrow \mathcal{X}_0$ defined by

$$N(f)(t) = \int_0^t \mathcal{U}(t,s)f(s)ds, \text{ for } t \in J.$$

From Lemma 2.10 it follows that $N \circ S_{F(\cdot z.)}$ is a closed graph operator. Then, there exists $f \in S_{F(\cdot,\hat{z}.)}$ such that $\hat{h} \in \widetilde{T}(\hat{z})$. Therefore $\widetilde{T}$ is a completely continuous multivalued map, u.sc. with convex closed values.

As a consequence of Lemma 2.13 we conclude that T has a fixed point which is a mild solution of (1.1). □

4. ATTRACTIVITY OF SOLUTIONS

In this section we study the local attractivity of solutions the problem (1.1).

Definition 4.1 ([20])**.** We say that solutions of (1.1) are locally attractive if there exists a closed ball $\bar{B}(z^*,\sigma)$ in the space $\mathcal{X}_0$ for some $z^* \in \mathcal{X}$ such that for arbitrary solutions z and $\tilde{z}$ of (1.1) belonging to $\bar{B}(z^*,\sigma)$ we have

$$\lim_{t\to+\infty}(z(t) - \tilde{z}(t)) = 0.$$

Let y^* be a solution of (1.1) such that

$$y^*(t) = \mathcal{U}(t,0)\phi(0) + \int_0^t \mathcal{U}(t,s)f^*(s)ds,$$

where $f^* \in S_{F(\cdot,y^*_\cdot)}$.

For any $t \in J$ we have

$$\begin{aligned} \|y^*_t\|_{\mathcal{B}} &\leq K(t)\|z\|_{\mathcal{X}} + M(t)\|y_0\|_{\mathcal{B}} \\ &\leq \gamma(\|y\|_{\mathcal{X}} + \|\phi\|_{\mathcal{B}}). \end{aligned} \tag{4.1}$$

Let $\bar{B}(y^*,\rho)$ be a closed ball in $\mathcal{X}$ which ρ satisfies the following inequality

$$2\widehat{M}\|p\|_{L^1}\psi\Big(\gamma(\rho + \|\phi\|_{\mathcal{B}})\Big) \leq \rho.$$

Then, for $y \in \bar{B}(y^*,\rho)$ defined by

$$y(t) = \mathcal{U}(t,0)\phi(0) + \int_0^t \mathcal{U}(t,s)f(s)ds,$$

for some $f \in S_{F(\cdot,y_\cdot)}$ and by using assumptions (H_1)–(H_2) and (4.1) we get

$$\begin{aligned} H_d((Ty)(t), y^*(t)) &= H_d((Ty)(t), (Ty^*)(t)) \\ &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}|f(s) - f^*(s)|ds \\ &\leq \widehat{M}\int_0^t e^{-\omega(t-s)}p(s)\left[\psi(\|y_s\|_{\mathcal{B}}) + \psi(\|y^*_s\|_{\mathcal{B}})\right]ds, \\ &\leq \widehat{M}\int_0^t e^{-\omega(t-s)}p(s)\left[\psi\left(\gamma(\|y\|_{\mathcal{X}} + \|\phi\|_{\mathcal{B}})\right) + \psi\left(\gamma(\|y^*\|_{\mathcal{X}} + \|\phi\|_{\mathcal{B}})\right)\right]ds, \\ &\leq 2\widehat{M}\psi\left(\gamma(\rho + \|\phi\|_{\mathcal{B}})\right)\int_0^t e^{-\omega(t-s)}p(s)ds, \\ &\leq 2\widehat{M}\psi\left(\gamma(\rho + \|\phi\|_{\mathcal{B}})\right)\|p\|_{L^1} \leq \rho. \end{aligned}$$

Therefore, we get $T(\bar{B}(y^*,\rho)) \subset \bar{B}(y^*,\rho)$. So, for each $y, \tilde{y} \in \bar{B}(y^*,\rho)$ solution of problem (1.1) and $t \in J$, we have

$$\begin{aligned} |y(t) - \tilde{y}(t)| &\leq H_d(Tz(t), T\tilde{z}(t)) \\ &\leq 2\widehat{M}\psi\Big(\gamma(\rho + \|\phi\|_{\mathcal{B}})\Big)\int_0^t e^{-\omega(t-s)}p(s)ds. \end{aligned}$$

Hence, from (H_3), we conclude that

$$\lim_{t\to\infty} |y(t) - \widetilde{y}(t)| = 0.$$

Consequently, the solutions of problem (1.1) are locally attractive.

5. AN EXAMPLE

Consider the second order Cauchy problem

$$
(5.1) \qquad \begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,\xi) \in & \dfrac{\partial^2}{\partial \xi^2}u(t,\xi) + a(t)\dfrac{\partial}{\partial t}u(t,\xi) \\ \qquad\qquad + & \displaystyle\int_{-\infty}^{0} G(\tau, u(t+\tau,\xi))d\tau \qquad t \in J,\ \xi \in [0,2\pi], \\ u(t,0) = & u(t,2\pi) = 0 \qquad t \in J, \\ u(\theta,\xi) = & \phi(\theta,\xi),\ \theta \in (-\infty,0], \qquad \xi \in [0,2\pi] \\ \dfrac{\partial}{\partial t}u(0,\xi) = & \psi(\xi). \end{cases}
$$

where $J := [0,\infty)$, $G : J \times \mathbb{R} \to P(\mathbb{R})$ is a Carathéodory multivalued maps with compact convex values, $a : J \to \mathbb{R}$ is a continuous function, $\phi(\theta,\cdot) \in \mathcal{B}$.

Let $X = L^2(\mathbb{R},\mathbb{C})$ the space of 2π-periodic square-integrable functions from $\mathbb{R}$ into $\mathbb{C}$, and $H^2(\mathbb{R},\mathbb{C})$ denotes the Sobolev space of 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{R},\mathbb{C})$.

We consider the operator $A_1u(\xi) = u''(\xi)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$ infinitesimal generator of strongly continuous cosine function $C(t)$ on X. Moreover, we take $A_2(t)u(s) = a(t)u'(s)$ defined on $H^1(\mathbb{R},\mathbb{C})$, and considered the closed linear operator $A(t) = A_1 + A_2(t)$ where is generator of evolution operator $\mathcal{U}$ defined by

$$
\mathcal{U}(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,
$$

with z_n the solution of the following scalar initial value problem

$$
(5.2) \qquad \begin{cases} z''(t) = -n^2 z(t) + in\ a(t)z(t) \\ z(s) = 0, \qquad z'(s) = z_1. \end{cases}
$$

where $n \in \mathbb{Z}$ and i is the imaginary unit. In addition, assume that there exist an integrable function $\nu : J \longrightarrow [0,\infty)$ and $\sigma : J \longrightarrow (0,\infty)$ continuous and nondecreasing such that

$$
\|G(t, u_t(\cdot,\xi))\|_P \le \nu(t)\sigma(|u_t(\cdot,\xi)|).
$$

Set

$$
y(t)(\xi) = u(t,\xi),\ t \ge 0,\ \xi \in [0,2\pi],
$$

$$
\phi(s)(\xi) = u(s,\xi),\ -\infty < s \le 0,\ \xi \in [0,2\pi],
$$

and

$$
\frac{d}{dt}y(0)(\xi) = \frac{\partial}{\partial t}u(0,\xi),\ t \ge 0,\ \xi \in [0,2\pi].
$$

Consequently, (5.1) can be written in the abstract form (1.1) with A and f defined above. Now, the existence and attractivity of a mild solutions can be deduced from an application of results obtained in this paper.

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