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IDENTIFICATION OF UNKNOWN COEFFICIENT IN TIME FRACTIONAL PARABOLIC EQUATION WITH MIXED BOUNDARY CONDITIONS VIA SEMIGROUP APPROACH

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ABSTRACT. This article presents a semigroup approach for the mathematical analysis of the inverse coefficient problem of identifying the unknown coefficient $k(x)$ in the linear time fractional parabolic equation $D_t^{\alpha} u(x,t) = (k(x)u_x)_x$, $0 < \alpha \leq 1$, with mixed boundary conditions $u(0,t) = \psi_0(t)$, $u_x(1,t) = \psi_1(t)$. Our aim is the investigation of the distinguishability of the input-output mapping $\Phi[\cdot] : \mathcal{K} \to C[0,T]$, via semigroup theory. This work shows that if the null space of the semigroup $T_{\alpha,\alpha}(t)$ consists of only zero function, then the input-output mapping $\Phi[\cdot]$ has distinguishability property. Also, the value $k(0)$ of the unknown function $k(x)$ is determined explicitly. In addition to these the boundary observation $f(t)$ can be shown as an integral representation. This also implies that the mapping $\Phi[\cdot] : \mathcal{K} \to C[0,T]$ can be described in terms of the semigroup.

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1. PRELIMINARIES

The inverse problem of determining unknown coefficient in a linear time fractional parabolic equation by using over measured data have attracted considerable interest from engineers and scientist recently. The generalizations of ordinary and partial differential equations are called fractional differential equations and they are used for modeling various processes such as fluid mechanics, viscoelasticity and electromagnetic polymerheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium. This kind of problems are very useful in engineering, physics and applied mathematics. They help us to model complex phenomena. Various numerical methods have been developed to overcome the determination of the unknown coefficient(s) [19, 20, 21, 23]. Luchko extended the classical maximum principle and uniqueness of solution for nonlinear fractional differential equation [15, 16].

In this study, we focus our attention on the inverse problem of determining unknown coefficient $k(x)$ in a one dimensional time fractional parabolic equation by

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semigroup approach. In our analysis we will make extensive use of semigroup and noisy free measured output data. We first obtain the unique solution of this problem, written in terms of semigroup with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem under certain conditions. As the next step, the noisy free measured output data is used to introduce the input-output mapping $\Phi[\cdot] : \mathcal{K} \to C[0,T]$. Finally we investigate the distinguishability of the unknown coefficient via the above input-output mapping $\Phi[\cdot]$.

Semigroup approach is an analytical approach for inverse problems of identifying unknown coefficients in parabolic problems. The inverse problem of unknown coefficients in a quasi-linear parabolic equations was studied by Demir and Ozbilge [5, 6, 10, 12, 13, 14].

Consider now the following initial boundary value problem:

$$\left\{ \begin{array}{l} D_t^\alpha u(x,t) = (k(x)u_x)_x, \quad 0 < \alpha \le 1, \quad (x,t) \in \Omega_T, \\ u(x,0) = g(x), \quad 0 < x < 1, \\ u(0,t) = \psi_0(t),\ u_x(1,t) = \psi_1(t),\ 0 < t < T, \end{array} \right. \tag{1.1}$$

where $\Omega_T = \{(x,t) \in R^2 : 0 < x < 1,\ 0 < t \le T\}$ and the fractional derivative $D_t^\alpha u(x,t)$ is defined in the Caputo sense $D_t^\alpha u(x,t) = (I^{1-\alpha}u')(t)$, $0 < \alpha \le 1$, I^α being the Riemann-Liouville fractional integral

$$(I^\alpha f)(t) = \left\{ \begin{array}{ll} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, & 0 < \alpha \le 1 \\ f(t), & \alpha = 0. \end{array} \right.$$

The left and right boundary value functions $\psi_0(t)$ and $\psi_1(t)$ belong to $C[0,T]$. The functions $0 < c_0 \le k(x) < c_1$ and $g(x)$ satisfy the following conditions:

(C1) $k(x) \in C^1[0,1]$
(C2) $g(x) \in C^2[0,1]$, $g(0) = \psi_0(0)$, $g'(1) = \psi_1(0)$.

Under the conditions (C1) and (C2), the initial boundary value problem (1) has the unique solution $u(x,t)$ defined in the domain $\overline{\Omega}_T = \{(x,t) \in R^2 : 0 \le x \le 1, 0 \le t \le T\}$ which belongs to the space $C(\overline{\Omega}_T) \cap W_t^1(0,T] \cap C_x^2(0,1)$. The space $W_t^1(0,T]$ consist of $f \in C^1(0,T]$ such that $f'(x) \in L(0,T)$.

The problem (1.1) is the mathematical model of various physical and chemical events such as solute transport in a porous medium where the dependent variable $u(x,t)$ denotes a solute concentration depending continuously on independent variables x and t.

The Neumann type measured output data at the boundary $x = 0$ is given as $k(0)u_x(0,t) = f(t)$ in the determination of the unknown coefficient.

$u = u(x,t)$ is the solution of the parabolic problem (1.1) and $f(t)$ is assumed to be noisy free measured output data. The problem (1) is called a direct (forward) problem, with the inputs $g(x)$ and $k(x)$. It is also assumed that the function $f(t)$ belongs to $C^1[0,T]$ and satisfy the consistency condition $f(0) = k(0)g'(0)$.

Denote $\mathcal{K} := \{k(x) \in C^1[0,1] : c_1 > k(x) \geq c_0 > 0,\ x \in [0,1]\} \subset C[0,1]$, as a set of admissible coefficients $k(x)$, let us define the input-output mapping $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$ as follows:

$$\Phi[k] = k(x)u_x(x,t;k)|_{x=0}, \quad k \in \mathcal{K}$$

Then the inverse problem with the measured output data $f(t)$ can be formulated as the following operator equation:

$$\Phi[k] = f, \qquad f \in C^1[0,1]$$

The purpose of this paper is to study the distinguishability of the unknown coefficient via the above input-output mapping. We say that the mapping $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$ have the distinguishability if $\Phi[k_1] \neq \Phi[k_2]$ implies $k_1(x) \neq k_2(x)$ which means the injectivity of the inverse mapping Φ^{-1}. Neumann type measured output data at $x = 0$ is used in the identification of the unknown coefficient. In addition to this, analytical results are obtained.

The paper is organized as follows. In section 2, an analysis of the semigroup approach is given for the inverse problem with the single measured output data $f(t)$ at the boundary $x = 0$. Finally, some concluding remarks are given in the last section.

2. AN ANALYSIS OF THE INVERSE PROBLEM WITH GIVEN MEASURED DATA $f(t)$

Consider the inverse problem with one measured output data $f(t)$ at $x = 0$. In order to formulate the solution of the parabolic problem (1.1) in terms of semigroup, we have to introduce an auxiliary function $v(x,t)$ as follows:

$$v(x,t) = u(x,t) - \psi_0(t) - \psi_1(t)x,\ \ x \in [0,1].$$

$v(x,t)$ transforms the problem (1.1) into a problem with homogeneous boundary conditions. Hence the problem (1.1) can be rewritten in terms of $v(x,t)$ as in (2.1).

(2.1)

$$\begin{cases} D_t^\alpha v(x,t) + A[v(x,t)] = ((k(x)-1)v_x(x,t))_x - xD_t^\alpha\psi_1(t) - D_t^\alpha\psi_0(t) + k'(x)\psi_1(t), \\ v(x,0) = g(x) - \psi_0(0) - \psi_1(0)x, \quad 0 < x < 1, \\ v(0,t) = 0,\ v_x(1,t) = 0,\ 0 < t < T. \end{cases}$$

$A[\cdot] := -\frac{d^2[\cdot]}{dx^2}$ is a second order differential operator and its domain is $D_A = \{v(x) \in C^2(0,1) \cap C^1[0,1] : v(0) = v'(1) = 0\}$. Obviously, $g(x) \in D_A$, since the initial value function $g(x)$ belongs to $C^2[0,1]$. Let $T_{\alpha,\alpha}(t)$ be the semigroup of linear operators generated by the operator $-A$ [7, 8]. Note that eigenvalues and eigenfunctions of the differential operator A can easily be identified and the semigroup $T_{\alpha,\alpha}(t)$ can be constructed by using these eigenvalues and eigenfunctions of the infinitesimal generator A. First, the eigenvalue problem (2.2) must be considered:

$$A\phi(x) = \lambda\phi(x), \quad \phi(0) = 0, \quad \phi'(1) = 0. \tag{2.2}$$

This problem (2.2) is called the Sturm-Liouville problem. The eigenvalues are determined, with $\lambda_n = \frac{n^2\pi^2}{4}$, $\forall n = 1, 2, \ldots$ the corresponding eigenfunctions as $\phi_n(x) = \sqrt{2}\sin(\frac{n\pi x}{2})$. In this case, the semigroup $T_{\alpha,\alpha}(t)$ can be represented in the following way:

$$T_{\alpha,\alpha}(t)U(x,s) = \sum_{n=1}^{\infty} \langle \phi_n(\theta), U(\theta,s)\rangle E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\phi_n(x),$$

where $< \phi_n(\theta), U(\theta,s) >= \int_0^1 \phi_n(\theta)U(\theta,s)d\theta$ and $E_{\beta,\alpha}$ being the generalized Mittag-Leffler function, playing a special role in solving the fractional differential equation which is defined by

$$E_{\beta,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \alpha)}.$$

The Sturm-Liouville problem (2.2) generates a complete orthogonal family of eigenfunctions so that the null space of the semigroup $T(t)$ is trivial, i.e., $N(T_{\alpha,\alpha}) = \{0\}$. The null space of the semigroup $T_{\alpha,\alpha}(t)$ of the linear operators can be defined as follows:

$$N(T) = \{U(\theta,t) : \langle \phi_n(\theta), U(\theta,t)\rangle = 0, \forall n = 0, 1, 2, \ldots\}$$

The unique solution of the initial-boundary value problem (2.2) in terms of semigroup $T(t)$ can be represented in the following form:

$$\begin{aligned} v(x,t) = T_{\alpha,1}(t)v(x,0) + \int_0^t s^{\alpha-1} T_{\alpha,\alpha}(t-s)[((k(x)-1)v_x)_x + k'(x)\psi_1(s) \\ - D_t^{\alpha}\psi_0(s) - D_t^{\alpha}\psi_1(s)x]ds \end{aligned} \tag{2.3}$$

Hence, by using identity (2.3) and taking the initial value $u(x,0) = g(x)$ into account, the solution $u(x,t)$ of the parabolic problem (1.1) in terms of semigroup can be written in the following form:

$$\begin{aligned} u(x,t) = \psi_0(t) + \psi_1(t)x + T_{\alpha,1}(t)(g(x) - \psi_0(0) - \psi_1(0)x) \\ + \int_0^t s^{\alpha-1} T_{\alpha,\alpha}(t-s)[((k(x)-1)v_x)_x + k'(x)\psi_1(s) \\ - D_t^{\alpha}\psi_0(s) - D_t^{\alpha}\psi_1(s)x]ds \end{aligned}$$

In order to arrange the above solution representation, let us define the following:

$$\zeta(x) = g(x) - \psi_0(0) - \psi_1(0)x,$$

$$\xi(x,t) = ((k(x)-1)v_x(x,t))_x - xD_t^{\alpha}\psi_1(t) - D_t^{\alpha}\psi_0(t) + k'(x)\psi_1(t),$$

$$z(x,t) = \sum_{n=0}^{\infty} \langle \phi_n(\theta), \zeta(\theta) \rangle E_{\alpha,1}(-\lambda_n t^\alpha) \phi_n'(x), \tag{2.4}$$

$$w(x,t,s) = \sum_{n=0}^{\infty} \langle \phi_n(\theta), \xi(\theta,s) \rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha) \phi_n'(x).$$

The solution in terms of $\zeta(x)$ and $\xi(x,s)$ can be represented as follows:

$$u(x,t) = \psi_0(t) + \psi_1(t)x + T_{\alpha,1}(t)\zeta(x) + \int_0^t s^{\alpha-1} T_{\alpha,\alpha}(t-s)\xi(x,s)ds$$

Differentiating both sides of the above identity with respect to x and using semigroup properties at $x = 0$ yields:

$$u_x(0,t) = \psi_1(t) + z(0,t) + \int_0^t s^{\alpha-1} w(0, t-s, s) ds.$$

Taking into account the over-measured data $k(0)u_x(0,t) = f(t)$

$$f(t) = k(0)\left(\psi_1(t) + z(0,t) + \int_0^t s^{\alpha-1} w(0,t-s,s)ds\right), \tag{2.5}$$

is obtained which implies that $f(t)$ can be determined analytically. Substituting $t = 0$ into this yields

$$f(0) = k(0)g'(0).$$

Hence, the previous consistency condition is obtained. Using the measured output data $k(0)u_x(0,t) = f(t)$, we can write $k(0) = \frac{f(t)}{u_x(0,t)}$ $\forall t > 0$ which can be rewritten in terms of semigroup in the following form:

$$k(0) = \frac{f(t)}{\psi_1(t) + z(0,t) + \int_0^t s^{\alpha-1} w(0,t-s,s)ds},$$

Taking limit as $t \longrightarrow 0$ in the above identity, we obtain the following explicit formula for the value $k(0)$ of the unknown coefficient $k(x)$:

$$k(0) = \frac{f(0)}{\psi_1(0) + z(0,0)}.$$

Under the determined value $k(0)$, the set of admissible coefficients can be defined as follows:

$$\mathcal{K}_0 = \left\{ k(x) \in C^1[0,1] : \quad c_1 > k(x) \geq c_0 > 0, \quad x \in [0,1], \quad k(0) = \frac{f(0)}{\psi_1(0) + z(0,0)} \right\}$$

The right-hand side of identity (2.5) defines the semigroup representation of the input-output mapping $\Phi[k]$ on the set of admissible source functions $\mathcal{K}$:

$$\Phi[k](t) := k(0)\left(\psi_1(t) + z(0,t) + \int_0^t s^{\alpha-1} w(0,t-s,s)ds\right), \quad \forall t \in [0,T].$$

The following lemma implies the relation between the parameters $k_1(x), k_2(x) \in \mathcal{K}_0$ at $x = 0$ and the corresponding outputs $f_j(t) := k(0)u_x(0,t;k_j)$, $j = 1, 2$.

Lemma 2.1. *Let* $v_1(x,t) = v(x,t;k_1)$ *and* $v_2(x,t) = v(x,t;k_2)$ *be the solutions of the direct problem* (1)*, corresponding to the admissible parameters* $k_1(x), k_2(x) \in \mathcal{K}_0$*. Suppose that* $f_j(t) = k(0)u_x(0,t;k_j)$*,* $j = 1,2$*, are the corresponding outputs and* $k_1(0) = k_2(0) = k(0)$ *holds, then the outputs* $f_j(t)$*,* $j = 1,2$ *satisfy the following integral identity:*

$$\Delta f(t) = k(0)\int_0^t s^{\alpha-1}\Delta w(0,t-s,s)ds, \quad \forall t \in (0,T],$$

where $\Delta f(t) = f_1(t) - f_2(t)$*,* $\Delta w(x,t,s) = w^1(x,t,s) - w^2(x,t,s)$*.*

Proof. By using identity (2.5), the measured output data $f_j(t) := k(0)u_x(0,t;k_j)$, $j = 1,2$ can be written as follows:

$$f_1(t) = k(0)\left(\psi_1(t) + z^1(0,t) + \int_0^t s^{\alpha-1}w^1(0,t-s,s)ds\right),$$

$$f_2(t) = k(0)\left(\psi_1(t) + z^2(0,t) + \int_0^t s^{\alpha-1}w^2(0,t-s,s)ds\right),$$

respectively. From identity (2.4) it is obvious that $z^1(0,t) = z^2(0,t)$ for each $t \in (0,T]$. Hence the difference of these formulas implies the desired result. □

The lemma and the definitions enable us to reach the following conclusion:

Corollary 2.1. *Let the conditions of Lemma* 2.1 *hold. If in addition*

$$\begin{aligned}&\langle \phi_n(x), \xi^1(x,t) - \xi^2(x,t)\rangle \\ &= \langle \phi_n(x), ((k_1(x)-1)v_x)_x - ((k_2(x)-1)v_x)_x + (k_1'(x) + k_2'(x))\psi_1(s)\rangle = 0,\end{aligned}$$

$\forall t \in (0,T]$*,* $\forall n = 0,1,\dots$ *holds, then* $f_1(t) = f_2(t)$*,* $\forall t \in [0,T]$*.*

Note that if $\langle \phi_n(x), \xi^1(x,t) - \xi^2(x,t)\rangle \neq 0$ then the definition of $w(x,t,s)$ implies that $\Delta w(x,t,s) \neq 0$. Hence by Lemma 2.1 we conclude that $f_1(t) \neq f_2(t)$ $\forall t \in [0,T]$. Moreover, it leads us $\Phi[k]$ is distinguishable, i.e., $k_1(x) \neq k_2(x)$ implies $\Phi[k_1] \neq \Phi[k_2]$.

Theorem 2.1. *Let conditions* (C1), (C2) *hold. Assume that* $\Phi[\cdot] : \mathcal{K}_0 \to C^1[0,T]$ *is the input-output mapping corresponding to the measured output* $f(t) := k(0)u_x(0,t)$*. Then the mapping* $\Phi[k]$ *has the distinguishability property in the class of admissible parameters* $\mathcal{K}_0$*, i.e.,*

$$\Phi[k_1] \neq \Phi[k_2] \quad \forall k_1, k_2 \in \mathcal{K}_0 \Rightarrow k_1(x) \neq k_2(x).$$

3. CONCLUSION

In this paper, we have proved the distinguishability properties of the input-output mapping $\Phi[\cdot] : \mathcal{K} \to C^1[0,T]$ which is determined by the measured output data for the linear time fractional parabolic equations with mixed boundary conditions. It is shown that the semigroup with a trivial null space, i.e. $N(T_{\alpha,\alpha}) = \{0\}$ plays a crucial role in the distinguishability of the input-output mapping. This study also shows that boundary conditions and the region on which the problem is defined play an important role on the distinguishability of the input-output mapping $\Phi[\cdot]$ since these key elements determine the structure of the semigroup $T_{\alpha,\alpha}(t)$ of linear operators and its null space. This work advances our understanding of the use of semigroup and the input-output mapping in the investigation of inverse problems for fractional parabolic equations.

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