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EXISTENCE AND MONOTONE ITERATION OF CONCAVE POSITIVE SYMMETRIC SOLUTIONS FOR A THREE-POINT SECOND-ORDER BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this article, we make use of the monotone iterative technique to verify the existence of concave symmetric positive solutions of a second-order three-point boundary value problem with integral boundary conditions. The interesting point here is that the nonlinear term f depends on the first-order derivative explicitly. An example which supports our result is also indicated.

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1. INTRODUCTION

The multi-point boundary value problems for ordinary differential equations arise in variety of different areas of applied mathematics and physics. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated Il'in and Moiseev [5]. Since then, nonlinear multi-point boundary value problems have been studied by many authors. We refer the reader to [2–4,12] and their references.

At the same time, boundary value problems with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems. For an overview of the literature on integral boundary value problems, see [1,6,11,14].

In [14], J. Tariboon and T. Sitthiwirattham considered the second-order three-point differential equation

$$\begin{cases} u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \alpha \int_0^\eta u(s)ds. \end{cases}$$

They showed the existence of at least one positive solutions if f is either superlinear or sublinear by applying the fixed point theorem in cones.

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In [11], H. Pang and Y. Tong considered second-order boundary value problem

$$\begin{cases} u''(x) + f(x, u(x), u'(x)) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds. \end{cases}$$

They investigated the existence of concave symmetric positive solutions and established corresponding iterative schemes for a second-order boundary value problem with integral boundary conditions.

Motivated by the results above, in this paper, we are interested in the existence of the concave symmetric positive solutions for the following second-order three-point boundary value problems with integral boundary conditions

$$\begin{cases} u''(x) + f(x, u(x), u'(x)) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = \alpha \int_0^\eta u(s)ds, \end{cases} \tag{1.1}$$

where $\eta \in (0, 1)$, $0 < \alpha < \frac{1}{\eta}$, and $f \in \mathcal{C}((0, 1) \times [0, +\infty) \times \mathbb{R}, [0, +\infty))$.

The organization of the paper is as follows. In Section 2, we present definitions and some necessary lemmas that will be used to prove our main result. In Section 3, we apply the monotone iterative technique to obtain the existence of concave symmetric positive solutions for BVP (1.1). Monotone iterative technique has been successfully used to prove to existence of a positive solutions of boundary value problems, see [7-11,13,15,16]. In Section 4, we give example to illustrate our result.

2. PRELIMINARIES

Definition 2.1. Let E be a real Banach Space. A nonempty closed convex set $P \subset E$ is called a cone if it provides the following two conditions:
(i) $u \in P$, $\lambda \geq 0$ implies $\lambda u \in P$;
(ii) $u \in P$, $-u \in P$ implies $u = 0$.

Definition 2.2. Let E be a real Banach Space. A function $u \in E$ is said to be symmetric on $[0, 1]$ if

$$u(x) = u(1 - x), \quad x \in [0, 1].$$

Definition 2.3. Let $(E, \leq)$ be an ordered real Banach Space. An operator $\alpha : E \to E$ is said to be nondecreasing provided that $\alpha(u) \leq \alpha(v)$ for all $u, v \in E$ with $u \leq v$. If the inequality is strict, then α is said to be strictly nondecreasing.

Definition 2.4. Let E be a real Banach Space, $u \in E$ is said to be concave on $[0, 1]$ if

$$u(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda u(x_1) + (1 - \lambda)u(x_2)$$

for any $x_1, x_2 \in [0, 1]$ and $\lambda \in [0, 1]$.

We consider the Banach space $E = \mathcal{C}^2[0,1]$ equipped with norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u'\|_\infty = \max_{x\in[0,1]} |u'(x)|$. Throughout this paper, we always assume that the following assumptions are satisfied:

(H1) $f \in \mathcal{C}((0,1) \times [0,+\infty) \times \mathbb{R}, [0,+\infty))$, $f(x,u,v) = f(1-x,u,-v)$ for $x \in (0, \frac{1}{2}]$, and $f(x,u,v) \geq 0$ for all $(x,u,v) \in (0,1) \times [0,+\infty) \times \mathbb{R}$.

(H2) $f(x,.,v)$ is nondecreasing for each $(x,v) \in (0,\frac{1}{2}] \times \mathbb{R}$, $f(x,u,.)$ is nondecreasing for $(x,u) \in (0,\frac{1}{2}] \times [0,+\infty)$.

Define the cone $P \subset E$ by

$$P = \{u \in E : u(x) \geq 0 \ \text{ is concave and } u(x) = u(1-x), x \in [0,1]\}.$$

Lemma 2.5. *For any $u \in \mathcal{C}^2[0,1]$, suppose that u is the solution of the following BVP*

$$\begin{cases} u''(x) + f(x,u(x),u'(x)) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = \alpha \displaystyle\int_0^\eta u(s)ds. \end{cases}$$

Then we can easily get the solution

$$u(x) = \int_0^1 (H(s) + G(x,s)) f(s,u(s),u'(s))ds, \tag{2.1}$$

where

$$V(s) = \begin{cases} (\eta - s)^2, & s \leq \eta; \\ 0, & \eta \leq s. \end{cases} \quad H(s) = \frac{\alpha\eta^2}{2(1-\alpha\eta)}(1-s) - \frac{\alpha}{2(1-\alpha\eta)} V(s), \tag{2.2}$$

$$G(x,s) = \begin{cases} s(1-x), & 0 \leq s \leq x \leq 1; \\ x(1-s), & 0 \leq x \leq s \leq 1. \end{cases} \tag{2.3}$$

Proof. Suppose that $u \in \mathcal{C}^2[0,1]$ is a solution of problem (1.1). Then we have

$$u''(x) = -f(x,u(x),u'(x)).$$

For $x \in [0,1]$, by integration from 0 to 1, we have

$$u'(x) = u'(0) - \int_0^x f(s,u(s),u'(s))ds.$$

For $x \in [0,1]$, by integration again from 0 to 1, we have

$$u(x) = u'(0)x - \int_0^x \left(\int_0^\tau f(s,u(s),u'(s))ds \right) d\tau.$$

That is,

$$u(x) = u(0) + u'(0)x - \int_0^x (x-s) f(s,u(s),u'(s))ds, \tag{2.4}$$

therefore,

$$u(1) = u(0) + u'(0) - \int_0^1 (1-s) f(s,u(s),u'(s))ds.$$

From condition (1.1), we have

$$u'(0) = \int_0^1 (1-s)f(s,u(s),u'(s))ds.$$

By integrating (2.4) from 0 to η, where $\eta \in (0,1)$, we have

$$\begin{aligned}\int_0^\eta u(s)ds &= u(0)\eta + u'(0)\frac{\eta^2}{2} - \int_0^\eta \left(\int_0^\tau (\tau - s)f(s,u(s),u'(s))ds\right)d\tau \\ &= u(0)\eta + u'(0)\frac{\eta^2}{2} - \frac{1}{2}\int_0^\eta (\eta-s)^2 f(s,u(s),u'(s))ds,\end{aligned}$$

and from $u(0) = \alpha \int_0^\eta u(s)ds$, we have

$$u(0) = \frac{\alpha\eta^2}{2(1-\alpha\eta)}u'(0) - \frac{\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 f(s,u(s),u'(s))ds.$$

Therefore, (1.1) has a unique solution

$$\begin{aligned}u(x) = &\frac{\alpha\eta^2}{2(1-\alpha\eta)}\int_0^1 (1-s)f(s,u(s),u'(s))ds \\ &- \frac{\alpha}{2(1-\alpha\eta)}\int_0^\eta (\eta-s)^2 f(s,u(s),u'(s))ds \\ &+ x\int_0^1 (1-s)f(s,u(s),u'(s))ds - \int_0^x (x-s)f(s,u(s),u'(s))ds.\end{aligned}$$

From (2.2) and (2.3), we obtain

$$u(x) = \int_0^1 (H(s) + G(x,s))f(s,u(s),u'(s))ds.$$

The proof is complete. □

The functions H and G have the following properties.

Lemma 2.6. *If $\eta \in (0,1)$ and $0 < \alpha < \frac{1}{\eta}$, then we have $H(s) \geq 0$, for $s \in [0,1]$.*

Proof. From the definition of $H(s)$, $s \in (0,1)$, $\eta \in (0,1)$, and $0 < \alpha < \frac{1}{\eta}$, we have $H(s) \geq 0$. □

Lemma 2.7. *$G(1-x,1-s) = G(x,s)$, $0 \leq G(x,s) \leq G(s,s)$ for $x,s \in [0,1]$.*

Proof. From the definition of $G(x,s)$, we get $G(1-x,1-s) = G(x,s)$ and $0 \leq G(x,s) \leq G(s,s)$ for $x,s \in [0,1]$. □

Lemma 2.8. *Let $\eta \in (0,1)$ and $0 < \alpha < \frac{1}{\eta}$. If $f(x,u(x),u'(x)) \in \mathcal{C}((0,1)\times[0,+\infty)\times\mathbb{R},[0,+\infty))$, then the unique solution u of BVP (1.1) satisfies $u(x) \geq 0$ for $x \in [0,1]$.*

Proof. From the definition of $u(x)$, $f(x,u(x),u'(x)) \in \mathcal{C}((0,1)\times[0,+\infty)\times\mathbb{R},[0,+\infty))$, Lemma 2.6, and Lemma 2.7, we have $u(x) \geq 0$. □

Lemma 2.9. *Let $\alpha\eta > 1$. If $f(x, u(x), u'(x)) \in \mathcal{C}((0,1) \times [0, +\infty) \times \mathbb{R}, [0, +\infty))$ then BVP* (1.1) *has no positive solution.*

Proof. Suppose that problem (1.1) has a positive solutions u satisfying $u(x) > 0$, $x \in (0,1)$. If $u(0) = u(1) > 0$, by the concavity of u

$$u(s) \geq u(1) \quad \text{for } s \in (0,1), \tag{2.5}$$

by integrating (2.5) from 0 to η, where $\eta \in (0,1)$, we have

$$\int_0^\eta u(s)ds \geq \eta u(1),$$

and from $u(1) = \alpha \int_0^\eta u(s)ds$, we have

$$u(1)(1 - \alpha\eta) \geq 0,$$

which is a contradiction to the $u(1) > 0$ and $(1 - \alpha\eta) < 0$. So, no positive solutions exist. □

For any $u \in \mathcal{C}^2[0,1]$, $T : P \to E$ is defined

$$(Tu)(x) = \int_0^1 (H(s) + G(x,s)) f(s, u(s), u'(s)) ds, \text{ for } x \in [0,1]. \tag{2.6}$$

Clearly, u is the solution of BVP (1.1) if and only if u is fixed point of T.

Lemma 2.10. *Assume that* $(H1)$ *and* $(H2)$ *are satisfied, and let $\eta \in (0,1)$, $0 < \alpha < \frac{1}{\eta}$. Then the operator T is completely continuous in $\mathcal{C}^2[0,1]$ and T is nondecreasing.*

Proof. For any $u \in P$, from the expression of Tu, we know

$$\begin{cases} (Tu)''(x) + f(x, u(x), u'(x)) = 0, \quad x \in (0,1), \\ (Tu)(0) = (Tu)(1) = \alpha \displaystyle\int_0^\eta (Tu)(s)ds. \end{cases}$$

Clearly, Tu is concave. From the definition of Tu, Lemma 2.6, and Lemma 2.7 we see that Tu is nonnegative on $[0,1]$. We now show that Tu is symmetric about $\frac{1}{2}$. From Lemma 2.7 and (H1), for $x \in [0,1]$, we have

$$\begin{aligned} (Tu)(1-x) &= \int_0^1 (H(s) + G(1-x,s)) f(s, u(s), u'(s)) ds \\ &= \int_0^1 H(s) f(s, u(s), u'(s)) ds + \int_0^1 G(1-x,s) f(s, u(s), u'(s)) ds \\ &= \int_0^1 H(s) f(s, u(s), u'(s)) ds \\ &\quad - \int_1^0 G(1-x, 1-s) f(1-s, u(1-s), u'(1-s)) ds \\ &= \int_0^1 H(s) f(s, u(s), u'(s)) ds + \int_0^1 G(x,s) f(1-s, u(s), -u'(s)) ds \end{aligned}$$

$$= \int_0^1 H(s)f(s,u(s),u'(s))ds + \int_0^1 G(x,s)f(s,u(s),u'(s))ds$$
$$= (Tu)(x).$$

Therefore, $TP \subset P$.

The continuity of T with respect to $u(x) \in \mathcal{C}^2[0,1]$ is clear. We now show that T is compact. Suppose that $D \subset P$ is a bounded set. Then there exists r such that

$$D = \{u \in P \mid \|\|u\| \leq r\}.$$

For any $u \in D$, we have

$$0 \leq f(s,u(s),u'(s)) \leq \max\{f(s,u,u') \mid |s \in [0,1], u \in [0,r], u' \in [-r,r]\} =: M.$$

So, we have from (2.6)

$$\begin{aligned} \|(Tu)(x)\|_\infty &= \max_{x\in[0,1]} \left| \int_0^1 (H(s)+G(x,s))f(s,u(s),u'(s))ds \right| \\ &\leq M\int_0^1 H(s)ds + M\max_{x\in[0,1]}\int_0^1 G(x,s)ds =: L \end{aligned}$$

and

$$\begin{aligned} \|(Tu)'(x)\|_\infty &= \max_{x\in[0,1]} \left| \int_0^1 (1-s)f(s,u(s),u'(s))ds - \int_0^x f(s,u(s),u'(s))ds \right| \\ &\leq \frac{M}{2} + M. \end{aligned}$$

These equations imply that the operator T is uniformly bounded. Now we show that Tu is equi-continuous. We separate these three conditions:

Case (i). $0 \leq x_1 \leq x_2 \leq \frac{1}{2}$;
Case (ii). $\frac{1}{2} \leq x_1 \leq x_2 \leq 1$;
Case (iii). $0 \leq x_1 \leq \frac{1}{2} \leq x_2 \leq 1$.

We solely need to think Case (i) since the proofs of the other two are like. For $0 \leq x_1 \leq x_2 \leq \dfrac{1}{2}$, we have

$$\begin{aligned} &|(Tu)(x_2)-(Tu)(x_1)| \\ &\quad= \left| \int_0^1 (G(x_2,s)-G(x_1,s))f(s,u(s),u'(s))ds \right| \\ &\quad\leq \begin{cases} \displaystyle\int_0^1 |(x_2-x_1)(1-s)|f(s,u(s),u'(s))ds, & 0 \leq x_1 \leq x_2 \leq s \leq \dfrac{1}{2}, \\ \displaystyle\int_0^1 |s(x_1-x_2)|f(s,u(s),u'(s))ds, & 0 \leq s \leq x_1 \leq x_2 \leq \dfrac{1}{2}, \\ \displaystyle\int_0^1 |s(1-x_2)-x_1(1-s)|f(s,u(s),u'(s))ds, & 0 \leq x_1 \leq s \leq x_2 \leq \dfrac{1}{2}. \end{cases} \end{aligned}$$

$$\leq \begin{cases} \frac{M}{2}|x_2 - x_1|, \\ \frac{M}{2}|x_2 - x_1|, \\ \frac{3M}{2}|x_2 - x_1|. \end{cases}$$

In addition

$$|(Tu)'(x_2) - (Tu)'(x_1)| = \left| \int_{x_2}^{x_1} f(s, u(s), u'(s))ds \right| \leq M|x_2 - x_1|.$$

By applying the Arzela-Ascoli theorem, we can guarantee that $T(D)$ is relatively compact, which means T is compact. Then we have T is completely continuous.

Finally, we show T is noncecreasing with respect to $u(x) \in \mathcal{C}^2[0, 1]$.

Let $u_i(x) \in P$ $(i = 1, 2)$ and $u_1(x) \leq u_2(x)$ then, we have $u_2(x) - u_1(x) \in P$ and $u_2(x) - u_1(x) \geq 0$ is concave, symmetric about $\frac{1}{2}$. Therefore

$$\begin{cases} u_2'(x) \geq u_1'(x) & \text{for } x \in [0, \frac{1}{2}], \\ u_2'(x) \leq u_1'(x) & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

So, for $x \in [0, 1]$, by applying $(H1)$, $(H2)$, and the definition of Tu, we have

$$\begin{aligned} (Tu_2)(x) - (Tu_1)(x) &= \int_0^1 (H(s) + G(x, s))f(s, u_2(s), u_2'(s))ds \\ &\quad - \int_0^1 (H(s) + G(x, s))f(s, u_1(s), u_1'(s))ds \\ &= \int_0^1 (H(s) + G(x, s))\Big(f(s, u_2(s), u_2'(s)) \\ &\quad - f(s, u_1(s), u_1'(s))\Big)ds \\ &\geq 0. \end{aligned}$$

Thus T is nondecreasing. These complete the proof. □

3. EXISTENCE OF TWO CONCAVE SYMMETRIC POSITIVE SOLUTIONS FOR BVP (1.1)

Now we find the existence of two concave symmetric positive solutions and corresponding iterative scheme for BVP (1.1).

Theorem 3.1. *Suppose that* $(H1)$ *and* $(H2)$ *are provided, and let* $\eta \in (0, 1)$, $0 < \alpha < \frac{1}{\eta}$. *If there exist two positive number* $b_1 < b$ *such that*

$$\sup_{x \in [0, \frac{1}{2}]} f(x, b, b) \leq b_1, \tag{3.1}$$

where b and b_1 satisfy,

$$b \geq \max\left\{\frac{\alpha\eta^2(3-2\eta)}{3(1-\alpha\eta)}, \frac{\alpha\eta^2}{4(1-\alpha\eta)} - \frac{\alpha\eta^3}{6(1-\alpha\eta)} + \frac{1}{8}\right\} b_1, \tag{3.2}$$

then BVP (1.1) *has a concave symmetric positive solutions $w^*, v^* \in P$ with*

$$\|w^*\| \leq b \quad \textit{and} \quad \lim_{n\to\infty} T^n w_0 = w^*, \quad \textit{where} \quad w_0(x) = bx(1-x) + \frac{b}{4},$$
$$\|v^*\| \leq b \quad \textit{and} \quad \lim_{n\to\infty} T^n v_0 = v^*, \quad \textit{where} \quad v_0(x) = 0.$$

Proof. We show $P_b = \{w \in P : \|w\| \leq b\}$. In what follows, we now show $TP_b \subset P_b$. Let $w \in P_b$, then $0 \leq w(x) \leq \max_{x\in[0,1]} w(x) = \|w\|_\infty \leq b$. On the other hand, $\max_{x\in[0,1]} |w'(x)| = w'(0) \leq b$. By using (3.1) and (H_2), for $x \in [0, \frac{1}{2}]$, we have

$$0 \leq f(x, w(x), w'(x)) \leq f(x, b, b) \leq \sup_{x\in[0,\frac{1}{2}]} f(x, b, b) \leq b_1.$$

Let $x \in [\frac{1}{2}, 1]$, then $(1-x) \in [0, \frac{1}{2}]$, by using (H_1), (H_2), and (3.1), we have

$$\begin{aligned} 0 \leq f(x, w(x), w'(x)) &= f(1-x, w(1-x), w'(1-x)) = f(1-x, w(x), -w'(x)) \\ &= f(x, w(x), w'(x)) \leq f(x, b, b) \\ &\leq \sup_{x\in[0,\frac{1}{2}]} f(x, b, b) \leq b_1. \end{aligned}$$

Then

$$f(x, w(x), w'(x)) \leq b_1, \quad \text{for } x \in [0, 1]. \tag{3.3}$$

For any $w(x) \in P_b$, from Lemma 2.10, we obtain $T(w) \in P$ and, thus

$$\begin{aligned} \|Tw\|_\infty &= (Tw)(\frac{1}{2}) \\ &= \int_0^1 (H(s) + G(\frac{1}{2}, s))f(s, w(s), w^{'}(s))ds \\ &\leq \frac{\alpha\eta^2}{4(1-\alpha\eta)}b_1 - \frac{\alpha\eta^3}{6(1-\alpha\eta)}b_1 + \frac{1}{8}b_1 \\ &\leq b, \end{aligned}$$

and

$$\|(Tw)'\|_\infty = (Tw)'(0) = \int_0^1 (1-s)f(s, w(s), w'(s))ds \leq \frac{b_1}{2} < b.$$

So, $\|Tw\| \leq b$. Then, we obtain $TP_b \subset P_b$. Let $w_0(x) = bx(1-x) + \frac{b}{4}$ for $x \in [0, 1]$, then $\|w_0\| = b$ and $w_0(x) \in P_b$. Let $w_1 = Tw_0$, then $w_1 \in P_b$. We denote

$$w_{n+1} = Tw_n = T^{n+1}w_0 \quad (n = 0, 1, 2, \dots). \tag{3.4}$$

Because $TP_b \subset P_b$, we have $w_n \in P_b$ $(n = 0, 1, 2, \dots)$. According to Lemma 2.10, T is compact, we claim that $\{w_n\}_{n=1}^\infty$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^\infty$ and there exist $w^* \in P_b$ such that $w_{n_k} \longrightarrow w^*$. From the definition of T, (3.2), and (3.3), we have

$$\begin{aligned}
w_1(x) &= (Tw_0)(x) \\
&= \int_0^1 (H(s) + G(x,s))f(s, w_0(s), {w_0}'(s))ds \\
&= \int_0^1 H(s)f(s, w_0(s), w_0'(s))ds + \int_0^1 G(x,s)f(s, w_0(s), w_0'(s))ds \\
&\le \frac{\alpha\eta^2(3-2\eta)}{12(1-\alpha\eta)}b_1 + \frac{b_1}{2}x(1-x) \\
&\le \frac{b}{4} + bx(1-x) = w_0(x).
\end{aligned}$$

Thus, $w_0(x) - w_1(x) \in P_b$. By using Lemma 2.10, we obtain $Tw_1 \le Tw_0$, which means $w_2 \le w_1$, $x \in [0,1]$. By induction, $w_{n+1} \le w_n$, $x \in [0,1]$, $(n = 0,1,2,\dots)$.

Now we show that $|w'_{n+1}(x)| \le |w'_n(x)|$, $x \in [0,1]$. We separate these two conditions:

Case (i). Let $x \in [0, \frac{1}{2}]$, then $w'_n(x) \ge 0$.

$$\begin{aligned}
w_1'(x) &= (Tw_0)'(x) \\
&= \int_0^1 (1-s)f(s, w_0(s), {w_0}'(s))ds - \int_0^x f(s, w_0(s), w_0'(s))ds \\
&\le \frac{b_1}{2} - b_1 x = b_1(\frac{1}{2} - x) \\
&\le b - 2bx = w_0'(x).
\end{aligned}$$

Then, $|w_1'(x)| \le |w_0'(x)|$, by using Lemma 2.10, we obtain $|Tw_1'(x)| \le |Tw_0'(x)|$, which means $|w_2'(x)| \le |w_1'(x)|$, $x \in [0, \frac{1}{2}]$. By the induction $|w'_{n+1}(x)| \le |w'_n(x)|$, $x \in [0, \frac{1}{2}]$.

Case (ii). Let $x \in [\frac{1}{2}, 1]$, then $w'_n(x) \le 0$.

$$\begin{aligned}
-w_1'(x) &= -(Tw_0')(x) \\
&= -\left(\int_0^1 (1-s)f(s, w_0(s), w_0'(s))ds - \int_0^x f(s, w_0(s), w_0'(s))ds\right) \\
&\le b_1(x - \frac{1}{2}) \\
&\le 2bx - b = -w_0'(x).
\end{aligned}$$

Then, $|w_1'(x)| \le |w_0'(x)|$, by using Lemma 2.10, we obtain $|Tw_1'(x)| \le |Tw_0'(x)|$, which means $|w_2'(x)| \le |w_1'(x)|$, $x \in [\frac{1}{2}, 1]$. By the induction $|w'_{n+1}(x)| \le |w'_n(x)|$, $x \in [\frac{1}{2}, 1]$. Consequently, $|w'_{n+1}(x)| \le |w'_n(x)|$, $x \in [0,1]$.

So, we claim that $w_n \longrightarrow w^*$ in norm $\|\cdot\|$. Let $n \longrightarrow \infty$ in (3.4) to get $Tw^* = w^*$ because T is continuous. It is clear that the fixed point of the operator T is the solution of BVP (1.1). Hence, w^* is concave symmetric positive solution (1.1). And since $w^* \in P_b$, we have $\|w^*\| \leq b$.

Let $v_0(x) = 0$, $0 \leq x \leq 1$, then $v_0 \in P_b$. Let $v_1 = Tv_0$, then $v_1 \in P_b$, we denote

$$v_{n+1} = Tv_n = T^{n+1}v_0 \quad (n = 0, 1, 2, \dots). \tag{3.5}$$

Likely to $\{v_n\}_{n=1}^{\infty}$, we claim that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exist $v^* \in P_b$ such that $v_{n_k} \longrightarrow v^*$. Because $v_1 \geq v_0$, by using Lemma 2.10, we obtain $Tv_1 \geq Tv_0$, which means $v_2 \geq v_1$, $x \in [0, 1]$. By induction, $v_{n+1} \geq v_n$, $x \in [0, 1]$ $(n = 0, 1, 2, \dots)$. And $|v_1'(x)| \geq |v_0'(x)|$, by using Lemma 2.10, we obtain $|Tv_1'(x)| \geq |Tv_0'(x)|$, which means $|{v_2}'(x)| \geq |v_1'(x)|$, $x \in [0, 1]$. By the induction, $|v_{n+1}'| \geq |v_n'|$, $x \in [0, 1]$ $(n = 0, 1, 2, \dots)$. So we claim that $v_n \longrightarrow v^*$ in norm $\| \cdot \|$ and then $Tv^* = v^*$ and $v^* \geq 0$, $0 \leq x \leq 1$. Hence, v^* is concave symmetric positive solution of BVP (1.1). And since $v^* \in P_b$, we have $\|v^*\| \leq b$. Therefore, our proof is complete. □

4. EXAMPLE

Example 4.1. We consider the following three-point second-order boundary value problem with integral boundary conditions:

$$\begin{cases} u''(x) + \dfrac{1}{3}e^{x(1-x)}\dfrac{((u')^2 + sgn(u+1) + 2)}{80} = 0, \quad 0 < x < 1, \\ u(0) = u(1) = 3\displaystyle\int_0^{\frac{1}{4}} u(s)ds, \end{cases} \tag{4.1}$$

where

$$f(x, u, v) = \frac{1}{3}e^{x(1-x)}\frac{(v^2 + sgn(u+1) + 2)}{80}, \quad \eta = \frac{1}{4}, \quad \alpha = 3.$$

It is not difficult to check that the assumptions $(H1)$ and $(H2)$ hold. Let $b = 33$ and $b_1 = 32$, then conditions (3.1) and (3.2) are confirmed. Then applying Theorem 3.1, BVP (4.1) has a concave symmetric positive solutions $w^*, v^* \in P$ with

$$\|w^*\| \leq 33 \quad \text{and} \quad \lim_{n\to\infty} T^n w_0 = w^*, \quad \text{where} \quad w_0(x) = 33x(1-x) + \frac{33}{4},$$
$$\|v^*\| \leq 33 \quad \text{and} \lim_{n\to\infty} T^n v_0 = v^*, \quad \text{where} \quad v_0(x) = 0.$$

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