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COINCIDENCE POINTS FOR MULTIMAPS DEFINED ON SUBSETS OF FRÉCHET SPACES

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ABSTRACT. We establish coincidence points for maps defined on Fréchet spaces. The proofs rely on the notion of a Φ-essential map and on viewing the Fréchet space as the projective limit of a sequence of Banach spaces.

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1. INTRODUCTION

This paper presents a number of coincidence point results for multivalued maps defined between Fréchet spaces. To establish these results we use recent results in Banach spaces (see [1, 6]) and we view the Fréchet space E as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (see [2, 4, 5] and the references therein); here $N = \{1, 2, \dots\}$. Our approach relies on constructing maps F_n and Φ_n defined on subsets of E_n whose coincidence points "converge" to a coincidence point of the original operators F and Φ.

Now we recall some coincidence results [6] which will be needed in Section 2. Let E be a normed space and U an open subset of E.

We will consider classes **A** and **B** of maps.

Definition 1.1. We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map; here $\overline{U}$ denotes the closure of U in E and $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 1.2. We say $F \in B(\overline{U}, E)$ if $F \in \mathbf{B}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map.

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$.

Definition 1.3. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

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Definition 1.4. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Psi_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $\Psi_1 = F$ and $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(x,t)$).

Definition 1.5. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to K(E)$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

In [6] we established the following result.

Theorem 1.6. *Let E be a normed space, U an open subset of E and $G, F \in A_{\partial U}(\overline{U}, E)$. Suppose G is Φ-essential in $A_{\partial U}(\overline{U}, E)$ and $G \cong F$ in $A_{\partial U}(\overline{U}, E)$. Then there exists a $x \in U$ with $\Phi(x) \cap F(x) \neq \emptyset$.*

Remark 1.7. In fact the result in Theorem 1.6 is true if we change Definition 1.5 as follows: Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to K(E)$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Remark 1.8. In this paper we could replace the Φ-essential maps by the Φ-epi maps [6] and obtain similar results as in Section 2 (we leave this to the interested reader).

The following concepts will be needed in Section 2. Let (X, d) be a metric space and S a nonempty subset of X. For $x \in X$ let $d(x, S) = \inf_{y \in S} d(x, y)$. Also *diam* $S = \sup\{d(x,y) : x, y \in S\}$. We let $B(x, r)$ denote the open ball in X centered at x of radius r and by $B(S, r)$ we denote $\cup_{x \in S} B(x, r)$. For two nonempty subsets S_1 and S_2 of X we define the generalized Hausdorff distance H to be

$$H(S_1, S_2) = \inf\{\epsilon > 0 : S_1 \subseteq B(S_2, \epsilon), S_2 \subseteq B(S_1, \epsilon)\}.$$

Now suppose $G : S \to 2^X$ and $\Phi : S \to 2^X$. Then G is said to be Φ-hemicompact if each sequence $\{x_n\}_{n \in N}$ in S has a convergent subsequence whenever $G(x_n) \cap \Phi(x_n) \neq \emptyset$ (here $N = \{1, 2, \dots\}$).

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta) \ \forall \ \alpha, \beta \in I, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [3] $\cap_{\alpha \in I} E_\alpha$.)

2. COINCIDENCE THEORY IN FRÉCHET SPACES

We now present an approach to establishing coincidence points based on projective limits (see [3]). Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, \dots\}$. We assume that the family of seminorms satisfies

$$(2.1) \qquad |x|_1 \le |x|_2 \le |x|_3 \le \cdots \quad \text{for every } x \in E.$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \le r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x, r) = \{y \in E : |x - y|_n \le r \ \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$(2.2) \qquad x \sim_n y \text{ iff } |x - y|_n = 0.$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \ge n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\mu_{m,k} = \mu_{n,k}$ if $n \le m \le k$ and $\mu_n = \mu_{n,m}\mu_m$ if $n \le m$. We now assume the following condition holds:

$$(2.3) \qquad \begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases}$$

Remark 2.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii). In many applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0, \infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally we assume

$$(2.4) \qquad \begin{cases} E_1 \supseteq E_2 \supseteq \cdots \text{ and for each } n \in N, \\ |j_n\mu_{n,n+1}j_{n+1}^{-1}x|_n \le |x|_{n+1} \forall x \in E_{n+1} \end{cases}$$

(here we use the notation from [3] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [3]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\mu_{n,m}j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\mu_n(X)$, and we let $\overline{X_n}$, $intX_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$\text{pseudo-int } (X) = \{x \in X : j_n\mu_n(x) \in \overline{X_n} \backslash \partial X_n \text{ for every } n \in N\}.$$

The set X is pseudo-open if $X = \text{pseudo-int}(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x, r) = \{y \in E_n : |x - y|_n \leq r\}$.

Remark 2.2. If X is pseudo-open then for every $n \in N$ we claim that X_n is an open subset of E_n. Fix $n \in N$. We show $X_n = \text{int } X_n$. To see this note $X_n \subseteq \overline{X_n} \backslash \partial X_n$ since if $y \in X_n$ then there exists $x \in X$ with $y = j_n\mu_n(x)$ and this together with $X = \text{pseudo-int } X$ yields $j_n\mu_n(x) \in \overline{X_n} \backslash \partial X_n$ i.e. $y \in \overline{X_n} \backslash \partial X_n$. In addition notice

$$\overline{X_n} \backslash \partial X_n = (\text{int } X_n \cup \partial X_n) \backslash \partial X_n = \text{int } X_n \backslash \partial X_n = \text{int } X_n$$

since $\text{int } X_n \cap \partial X_n = \emptyset$. Consequently

$$X_n \subseteq \overline{X_n} \backslash \partial X_n = \text{int } X_n, \text{ so } X_n = \text{int } X_n.$$

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume for each $n \in N$ and $x \in M$ that $j_n\mu_n F(x)$ is closed. Let $n \in N$ and $M_n = j_n\mu_n(M)$. Since we first consider Volterra type operators we assume (note this assumption is only needed in Theorems 2.3)

$$\text{if } x, y \in E \text{ with } |x - y|_n = 0 \text{ then } H_n(Fx, Fy) = 0; \tag{2.5}$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume $\forall n \in N, \forall x, y \in M$ if $j_n\mu_n x = j_n\mu_n y$ then $j_n\mu_n Fx = j_n\mu_n Fy$ and of course here we do not need to assume that $j_n\mu_n F(x)$ is closed for each $n \in N$ and $x \in M$). Now (2.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n\mu_n(x)$ and we let

$$F_n y = j_n\mu_n F x$$

(we could of course call it Fy since it is clear in the situation we use it); note $F_n : M_n \to C(E_n)$ and note if there exists a $z \in M$ with $y = j_n\mu_n(z)$ then $j_n\mu_n Fx = j_n\mu_n Fz$ from (2.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In our next result we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

Now we present some results in Fréchet spaces. Our first result is motivated by Volterra type operators.

Theorem 2.3. *Let E and E_n be as described above, U a pseudo-open subset of E and $F : U \to 2^E$, $G : U \to 2^E$ and $\Phi : U \to 2^E$. Also assume for each $n \in N$ and $x \in U$ that $j_n\mu_n F(x)$, $j_n\mu_n G(x)$ and $j_n\mu_n\Phi(x)$ are closed and in addition for each $n \in N$ that $F_n : \overline{U_n} \to 2^{E_n}$, $G_n : \overline{U_n} \to 2^{E_n}$ and $\Phi_n : \overline{U_n} \to 2^{E_n}$ are as described above. Suppose the following conditions are satisfied:*

$$\left\{ \begin{array}{l} \text{for each } n \in N, G_n \in A_{\partial U_n}(\overline{U_n}, E_n), \Phi_n \in B(\overline{U_n}, E_n) \\ \text{and } G_n \text{ is } \Phi_n\text{-essential in } A_{\partial U_n}(\overline{U_n}, E_n) \end{array} \right. \tag{2.6}$$

$$\text{for each } n \in N, F_n \in A_{\partial U_n}(\overline{U_n}, E_n) \text{ is } \Phi_n\text{-hemicompact} \tag{2.7}$$

$$\text{for each } n \in N, G_n \cong F_n \text{ in } A_{\partial U_n}(\overline{U_n}, E_n) \tag{2.8}$$

and

$$\left\{ \begin{array}{l} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in U_n \text{ is such} \\ \text{that } F_n(y) \cap \Phi_n(y) \neq \emptyset \text{ in } E_n \text{ then} \\ j_k\mu_{k,n} j_n^{-1}(y) \in U_k \text{ for } k \in \{1, \dots, n-1\}. \end{array} \right. \tag{2.9}$$

Then there exists $x \in E$ with $F(x) \cap \Phi(x) \neq \emptyset$ in E; here $x = (z_k)$ where $z_k \in U_k$ for each $k \in N$.

Proof. For each $n \in N$, from Theorem 1.6 there exists $y_n \in U_n$ with $F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset$ in E_n. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in U_1$ and $j_1\mu_{1,k} j_k^{-1}(y_k) \in U_1$ for $k \in N\backslash\{1\}$ from (2.9). Fix $n \in N$. There exists a $x \in E$ with $y_n = j_n\mu_n(x)$ so

$$j_n\mu_n F(x) \cap j_n\mu_n \Phi(x) \neq \emptyset \text{ on } E_n. \tag{2.10}$$

We now claim

$$F_1(j_1\mu_{1,n} j_n^{-1} y_n) \cap \Phi_1(j_1\mu_{1,n} j_n^{-1} y_n) \neq \emptyset \text{ on } E_1. \tag{2.11}$$

To see this note on E_1 that

$$\begin{aligned} F_1(j_1\mu_{1,n} j_n^{-1} y_n) \cap \Phi_1(j_1\mu_{1,n} j_n^{-1} y_n) &= F_1(j_1\mu_{1,n} j_n^{-1} j_n\mu_n(x)) \\ &\cap \Phi_1(j_1\mu_{1,n} j_n^{-1} j_n\mu_n(x)) \\ &= F_1(j_1\mu_{1,n}\mu_n(x)) \\ &\cap \Phi_1(j_1\mu_{1,n}\mu_n(x)) \\ &= F_1(j_1\mu_1(x)) \cap \Phi_1(j_1\mu_1(x)) \\ &= j_1\mu_1 F(x) \cap j_1\mu_1\Phi(x) \\ &= j_1\mu_{1,n} j_n^{-1} j_n\mu_n F(x) \\ &\cap j_1\mu_{1,n} j_n^{-1} j_n\mu_n \Phi(x) \\ &\neq \emptyset \end{aligned}$$

from (2.10). We can do this for each $n \in N$ so (2.11) holds for each $n \in N$. Now (2.7) guarantees that there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{U_1}$ with $j_1\mu_{1,n}j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $w_n \in F_1(j_1\mu_{1,n}j_n^{-1}y_n)$ and $w_n \in \Phi_1(j_1\mu_{1,n}j_n^{-1}y_n)$. Now since F_1 is upper semicontinuous then [7] there exists a $w_1 \in F_1(z_1)$ and a subsequence (w_m) of (w_n) with $w_m \to w_1$. The upper semicontinuity of the map Φ_1 together with $w_m \to w_1$ and $w_m \in \Phi_1(j_1\mu_{1,m}j_m^{-1}y_m)$ implies $w_1 \in \Phi_1(z_1)$. Thus $F_1(z_1) \cap \Phi_1(z_1) \neq \emptyset$ on E_1. Also note $z_1 \in U_1$ since $F_1 \in A_{\partial U_1}(\overline{U_1}, E_1)$.

Let $N_1 = N_1^\star \backslash \{1\}$. Now $j_2\mu_{2,n}j_n^{-1}(y_n) \in U_2$ for $n \in N_1$. Note also (argument similar to the above) for $n \in N_1$ that

$$F_2(j_2\mu_{2,n}j_n^{-1}y_n) \cap \Phi_2(j_2\mu_{2,n}j_n^{-1}y_n) \neq \emptyset \text{ on } E_2.$$

Now (2.7) guarantees that there is a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{U_2}$ with $j_2\mu_{2,n}j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Similar reasoning as above yields $F_2(z_2) \cap \Phi_2(z_2) \neq \emptyset$ on E_2. Also note $z_2 \in U_2$ since $F_2 \in A_{\partial U_2}(\overline{U_2}, E_2)$. Note from (2.4) and the uniqueness of limits that $j_1\mu_{1,2}j_2^{-1}z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\mu_{1,n}j_n^{-1}(y_n) = j_1\mu_{1,2}j_2^{-1}j_2\mu_{2,n}j_n^{-1}(y_n)$ for $n \in N_2^\star$). Let $N_2 = N_2^\star \backslash \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \cdots N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in \overline{U_k}$ with $j_k\mu_{k,n}j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Also note $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ on E_k, $z_k \in U_k$ since $F_k \in A_{\partial U_k}(\overline{U_k}, E_k)$, and $j_k\mu_{k,k+1}j_{k+1}^{-1}z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^\star \backslash \{k\}$.

Fix $k \in N$. Now $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\mu_{k,k+1}j_{k+1}^{-1}z_{k+1} = j_k\mu_{k,k+1}j_{k+1}^{-1}j_{k+1}\mu_{k+1,k+2}j_{k+2}^{-1}z_{k+2} \\ &= j_k\mu_{k,k+2}j_{k+2}^{-1}z_{k+2} = \cdots = j_k\mu_{k,m}j_m^{-1}z_m = \pi_{k,m}z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in U_k$ for each $k \in N$. Now for each $k \in N$, $j_k\mu_k(y) = z_k$ in E_k, and $F_k(z_k) \cap \Phi_k(z_k) \neq \emptyset$ in E_k (i.e. $j_k\mu_k F(y) \cap j_k\mu_k\Phi(y) \neq \emptyset$ in E_k). Thus $F(y) \cap \Phi(y) \neq \emptyset$ in E. □

Our next result is motivated by Urysohn type operators. In this case the maps F_n, Φ_n will be related to F, Φ by the closure property (2.16). For the convenience of the reader we write the hemicompact condition in an easy verifiable form (see (2.15) and Remark 2.5).

Theorem 2.4. *Let E and E_n be as described above, U a pseudo-open subset of E and $F : Y \to 2^E$, $G : Y \to 2^E$ and $\Phi : Y \to 2^E$ with $U \subseteq Y$ and $\overline{U_n} \subseteq Y_n$ for each $n \in N$. Also for each $n \in N$ assume there exist $F_n : \overline{U_n} \to 2^{E_n}$, $G_n : \overline{U_n} \to 2^{E_n}$ and*

$\Phi_n : \overline{U_n} \to 2^{E_n}$ *and suppose the following conditions hold:*

$$(2.12) \quad \begin{cases} \textit{for each } n \in N, F_n, G_n \in A_{\partial U_n}(\overline{U_n}, E_n), \Phi_n \in B(\overline{U_n}, E_n) \\ \textit{and } G_n \textit{ is } \Phi_n\textit{-essential in } A_{\partial U_n}(\overline{U_n}, E_n) \end{cases}$$

$$(2.13) \quad \textit{for each } n \in N, G_n \cong F_n \textit{ in } A_{\partial U_n}(\overline{U_n}, E_n)$$

$$(2.14) \quad \begin{cases} \textit{for each } n \in \{2, 3, \dots\} \textit{ if } y \in U_n \textit{ is such} \\ \textit{that } F_n(y) \cap \Phi_n(y) \neq \emptyset \textit{ in } E_n \textit{ then} \\ j_k \mu_{k,n} j_n^{-1}(y) \in U_k \textit{ for } k \in \{1, \dots, n-1\}. \end{cases}$$

$$(2.15) \quad \begin{cases} \textit{for any sequence } \{y_n\}_{n \in N} \textit{ with } y_n \in U_n \\ \textit{and } F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset \textit{ in } E_n \textit{ for } n \in N \textit{ and} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \dots\}, N_k \subseteq N_{k-1} \textit{ for} \\ k \in \{1, 2, \dots\}, N_0 = N, \textit{ and a } z_k \in \overline{U_k} \textit{ with} \\ j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k \textit{ in } E_k \textit{ as } n \to \infty \textit{ in } N_k \end{cases}$$

and

$$(2.16) \quad \begin{cases} \textit{if there exists a } w \in Y \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in U_n \textit{ and } F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence } S \subseteq \\ \{k+1, k+2, \dots\} \textit{ of } N \textit{ with } j_k \mu_{k,n} j_n^{-1}(y_n) \to j_k \mu_k(w) \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S, \textit{ then } F(w) \cap \Phi(w) \neq \emptyset \textit{ in } E. \end{cases}$$

Then there exists $x \in E$ *with* $F(x) \cap \Phi(x) \neq \emptyset$ *in* E*; here* $x = (z_k)$ *where* $z_k \in \overline{U_k}$ *for each* $k \in N$*.*

Remark 2.5. Notice to check (2.15) we need to show that for each $k \in N$, $\{j_k \mu_{k,n} j_n^{-1}(y_n)\}_{n \in N_{k-1}} \subseteq \overline{U_k}$ is sequentially compact.

Remark 2.6. If we replace (2.15) with

$$\begin{cases} \text{for any sequence } \{y_n\}_{n \in N} \text{ with } y_n \in U_n \\ \text{and } F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset \text{ in } E_n \text{ for } n \in N \text{ and} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ N_k \subseteq \{k+1, k+2, \dots\}, N_k \subseteq N_{k-1} \text{ for} \\ k \in \{1, 2, \dots\}, N_0 = N, \text{ and a } z_k \in U_k \text{ with} \\ j_k \mu_{k,n} j_n^{-1}(y_n) \to z_k \text{ in } E_k \text{ as } n \to \infty \text{ in } N_k, \end{cases}$$

then Y is the statement of Theorem 2.4 can be replaced by U.

Proof. For each $n \in N$ there exists $y_n \in U_n$ with $F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset$ in E_n. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in U_1$ and $j_1 \mu_{1,k} j_k^{-1}(y_k) \in U_1$ for $k \in \{2, 3, \dots\}$. Now (2.15) with $k = 1$ guarantees that there exists a subsequence $N_1 \subseteq \{2, 3, \dots\}$ and a

$z_1 \in \overline{U_1}$ with $j_1\mu_{1,n}j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in N_1. Look at $\{y_n\}_{n\in N_1}$. Now $j_2\mu_{2,n}j_n^{-1}(y_n) \in U_2$ for $k \in N_1$. Now (2.15) with $k = 2$ guarantees that there exists a subsequence $N_2 \subseteq \{3, 4, \dots\}$ of N_1 and a $z_2 \in \overline{U_2}$ with $j_2\mu_{2,n}j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in N_2. Note from (2.4) and the uniqueness of limits that $j_1\mu_{1,2}j_2^{-1}z_2 = z_1$ in E_1 since $N_2 \subseteq N_1$ (note $j_1\mu_{1,n}j_n^{-1}(y_n) = j_1\mu_{1,2}j_2^{-1}j_2\mu_{2,n}j_n^{-1}(y_n)$ for $n \in N_2$). Proceed inductively to obtain subsequences of integers

$$N_1 \supseteq N_2 \supseteq \cdots N_k \subseteq \{k+1, k+2, \dots\}$$

and $z_k \in \overline{U_k}$ with $j_k\mu_{k,n}j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in N_k. Note $j_k\mu_{k,k+1}j_{k+1}^{-1}z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$.

Fix $k \in N$. Note

$$\begin{aligned} z_k &= j_k\mu_{k,k+1}j_{k+1}^{-1}z_{k+1} = j_k\mu_{k,k+1}j_{k+1}^{-1}j_{k+1}\mu_{k+1,k+2}j_{k+2}^{-1}z_{k+2} \\ &= j_k\mu_{k,k+2}j_{k+2}^{-1}z_{k+2} = \cdots = j_k\mu_{k,m}j_m^{-1}z_m = \pi_{k,m}z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in \overline{U_k}$ for each $k \in N$. Also since $F_n(y_n) \cap \Phi_n(y_n) \neq \emptyset$ in E_n for $n \in N_k$ and $j_k\mu_{k,n}j_n^{-1}(y_n) \to z_k = y$ in E_k as $n \to \infty$ in N_k we have from (2.16) that $F(y) \cap \Phi(y) \neq \emptyset$ in E. □

Remark 2.7. From the proof we see that condition (2.14) can be removed from the statement of Theorem 2.2. We include it only to explain condition (2.15) (see Remark 2.5).

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