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COINCIDENCE POINTS FOR MULTIVALUED MAPS BASED ON Φ-EPI AND Φ-ESSENTIAL MAPS

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ABSTRACT. Two approaches to establishing coincidence points for general classes of maps are presented. The first is based on the notion of a Φ-epi map and the second is based on the notion of a Φ-essential map.

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1. INTRODUCTION

The 0-epi maps were introduced by Furi, Martelli and Vignoli [1] and essential maps were introduced by Granas [3]. The notion of Φ-epi maps is presented in Section 2 and the notion of Φ-essential maps is presented in Section 3. Both approaches allow us to study coincidence points (i.e. $F(x) \cap \Phi(x) \neq \emptyset$) of the maps F and Φ. This new theory presents a unified theory for establishing coincidence points for general classes of maps. Our results are more general than those in the literature (see [1–8] and the references therein).

2. Φ-EPI MAPS

Let E be a Hausdorff topological space and U an open subset of E.

We will consider classes $\mathbf{A}$ and $\mathbf{B}$ of maps.

Definition 2.1. We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map; here $\overline{U}$ denotes the closure of U in E and $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 2.2. We say $F \in B(\overline{U}, E)$ if $F \in \mathbf{B}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map.

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$ in the first three results.

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Definition 2.3. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.4. We say $F \in B_{\Phi}(\overline{U}, E)$ if $F \in B(\overline{U}, E)$ and $F(x) \subseteq \Phi(x)$ for $x \in \partial U$.

Definition 2.5. A map $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi if for every map $G \in B_{\Phi}(\overline{U}, E)$ there exists $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Remark 2.6. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi. Then there exists $x \in U$ with $F(x) \cap \Phi(x) \neq \emptyset$ (take $G = \Phi$ in Definition 2.5).

Our next result can be called the "homotopy property" for Φ-epi maps. In our result E will be a topological vector space so automatically a completely regular space. For convenience we state the result if E is a normal space and we remark on the general case after the theorem.

Theorem 2.7. *Let E be a normal topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi and $H : \overline{U} \times [0,1] \to K(E)$ is an upper semicontinuous map with $H(x, 0) = \{0\}$ for $x \in \partial U$. In addition assume the following conditions hold:*

$$\text{(2.1)} \qquad \begin{cases} \text{if } F_1 \in B(\overline{U}, E) \text{ then } F_1(\cdot) + H(\cdot, \mu(\cdot)) \in B(\overline{U}, E) \\ \text{for any continuous map } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \end{cases}$$

and

$$\text{(2.2)} \qquad \begin{cases} \{x \in \overline{U} : F(x) \cap [\Phi(x) + H(x,t)] \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{does not intersect } \partial U. \end{cases}$$

Then $F(\cdot) - H(\cdot, 1) : \overline{U} \to K(E)$ is Φ-epi.

Proof. Let $G \in B_{\Phi}(\overline{U}, E)$. We must show that there exists $x \in U$ with $[F(x) - H(x, 1)] \cap G(x) \neq \emptyset$. Let

$$D = \left\{x \in \overline{U} : F(x) \cap [G(x) + H(x,t)] \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

When $t = 0$ we have $G(\cdot) + H(\cdot, 0) \in B_{\Phi}(\overline{U}, E)$ since from (2.1) we have $G(\cdot) + H(\cdot, 0) \in B(\overline{U}, E)$ and for $x \in \partial U$ we have $G(x) + H(x, 0) = G(x) \subseteq \Phi(x)$ and this together with the fact that F is Φ-epi yields $D \neq \emptyset$. Next we show D is closed. To see this let (x_α) be a net in D (i.e. $F(x_\alpha) \cap [G(x_\alpha) + H(x_\alpha, t_\alpha)] \neq \emptyset$ for some $t_\alpha \in [0,1]$) with $x_\alpha \to x_0 \in \overline{U}$. Without loss of generality assume $t_\alpha \to t_0 \in [0,1]$. Suppose $y_\alpha \in F(x_\alpha)$ with $y_\alpha \in G(x_\alpha) + H(x_\alpha, t_\alpha)$. Since F is upper semicontinuous then [9] implies that there exists $y_0 \in F(x_0)$ and a subnet (y_β) of (y_α) with $y_\beta \to y_0$. The upper semicontinuity of the maps G and H together with $y_\beta \to y_0$ and $y_\beta \in G(x_\beta) + H(x_\beta, t_\beta)$ implies $y_0 \in G(x_0) + H(x_0, t_0)$. Thus $F(x_0) \cap [G(x_0) + H(x_0, t_0)] \neq \emptyset$ i.e. $x_0 \in D$, so D is closed.

Next we note (2.2) guarantees that $D \cap \partial U = \emptyset$ (note if $x \in \partial U$ then $F(x) \cap [G(x) + H(x,t)] \subseteq F(x) \cap [\Phi(x) + H(x,t)]$). Now Urysohn's lemma guarantees that there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$.

Define a map $J : \overline{U} \to K(E)$ by

$$J(x) = G(x) + H(x, \mu(x)).$$

Note $J \in B(\overline{U}, E)$ from (2.1) and for $x \in \partial U$ we have $J(x) = G(x) + H(x, \mu(x)) = G(x) + H(x, 0) = G(x) \subseteq \Phi(x)$. Thus $J \in B_{\Phi}(\overline{U}, E)$. Now since F is Φ-epi there exists $x \in U$ with $F(x) \cap J(x) \neq \emptyset$ i.e. $F(x) \cap [G(x) + H(x, \mu(x))] \neq \emptyset$. Thus $x \in D$ and as a result $\mu(x) = 1$. Consequently $F(x) \cap [G(x) + H(x, 1)] \neq \emptyset$ so $[F(x) - H(x,1)] \cap G(x) \neq \emptyset$. □

Remark 2.8. We can remove the assumption that E is normal in the statement of Theorem 2.7 provided we put conditions on the maps so that D is compact (the existence of the map μ in the proof above is then guaranteed since topological vector spaces are completely regular).

Our next result can be called the "coincidence property" for Φ-epi maps.

Theorem 2.9. *Let E be a normal topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi, $G \in B(\overline{U}, E)$ and assume the following conditions hold:*

$$\text{(2.3)} \qquad \begin{cases} \mu(\cdot)G(\cdot) + (1 - \mu(cdot))\Phi(\cdot) \in B(\overline{U}, E) \textit{ for any} \\ \textit{continuous map } \mu : \overline{U} \to [0,1] \textit{ with } \mu(\partial U) = 0 \end{cases}$$

and

$$\text{(2.4)} \qquad \begin{cases} \{x \in \overline{U} : F(x) \cap [tG(x) + (1-t)\Phi(x)] \neq \emptyset \textit{ for some } t \in [0,1]\} \\ \textit{does not intersect } \partial U. \end{cases}$$

Then there exists $x \in \overline{U}$ with $F(x) \cap G(x) \neq \emptyset$.

Proof. Let

$$D = \left\{x \in \overline{U} : F(x) \cap [tG(x) + (1-t)\Phi(x)] \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

When $t = 0$ note $F(x) \cap \Phi(x) \neq \emptyset$ for some $x \in U$ since $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi, so $D \neq \emptyset$. The same reasoning as in Theorem 2.7 guarantees that D is closed. Also $D \cap \partial U = \emptyset$ from (2.4). Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$.

Define a map $J : \overline{U} \to K(E)$ by

$$J(x) = \mu(x)G(x) + (1 - \mu(x))\Phi(x).$$

Now (2.3) guarantees that $J \in B(\overline{U}, E)$ and for $x \in \partial U$ we have $J(x) = 0 + \Phi(x) = \Phi(x)$, so $J \in B_\Phi(\overline{U}, E)$. Now since F is Φ-epi there exists $x \in U$ with $F(x) \cap J(x) \neq \emptyset$. Thus $x \in D$ and as a result $\mu(x) = 1$. Consequently $F(x) \cap G(x) \neq \emptyset$. □

Remark 2.10. We also have an analogue of Remark 2.8 in this case also.

Finally we restate Theorem 2.9 as a result of Leray-Schauder type.

Theorem 2.11. *Let E be a normal topological vector space and U an open subset of E. Suppose $F \in A_{\partial U}(\overline{U}, E)$ is Φ-epi and $G \in B(\overline{U}, E)$. In addition assume* (2.3) *holds. Then either*

(A1). there exists $x \in \overline{U}$ with $F(x) \cap G(x) \neq \emptyset$,

or

(A2). there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $F(x) \cap [\lambda\, G(x) + (1-\lambda)\Phi(x)] \neq \emptyset$,

holds.

Proof. Suppose $(A2)$ does not hold and $F(x) \cap G(x) = \emptyset$ for $x \in \partial U$ (since otherwise $(A1)$ holds). Also note $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ since $F \in A_{\partial U}(\overline{U}, E)$. Thus

$$\text{there exists } x \in \partial U \text{ and } \lambda \in [0,1] \text{ with } F(x) \cap [\lambda\, G(x) + (1-\lambda)\Phi(x)] \neq \emptyset$$

cannot occur, so (2.4) holds. Now Theorem 2.9 guarantees that there exists $x \in U$ with $F(x) \cap G(x) \neq \emptyset$. □

We now show that the ideas in this section can be applied to other natural situations. Let E be a Hausdorff topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(E, Y)$.

Definition 2.12. We say $F \in A(\overline{U}, Y; L, T)$ if $F : \overline{U} \to 2^Y$ with $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$.

Definition 2.13. We say $F \in B(\overline{U}, Y; L, T)$ if $F : \overline{U} \to 2^Y$ with $(L+T)^{-1}(F+T) \in B(\overline{U}, E)$.

In our next two results we <u>fix</u> a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 2.14. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $(L + T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ for $x \in \partial U$.

Definition 2.15. We say $F \in B_\Phi(\overline{U}, Y; L, T)$ if $F \in B(\overline{U}, Y; L, T)$ and $(L+T)^{-1}(F+T)(x) \subseteq (L+T)^{-1}(\Phi + T)(x)$ for $x \in \partial U$.

Definition 2.16. A map $F \in A_{\partial U}(\overline{U}, Y; L, T)$ is $(L,T)\Phi$-epi if for every map $G \in B_\Phi(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(G+T)(x) \neq \emptyset$.

Remark 2.17. Suppose $F \in A_{\partial U}(\overline{U}, Y; L, T)$ is $(L,T)\Phi$-epi. Then there exists $x \in U$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$ (take $G = \Phi$ in Definition 2.16).

Theorem 2.18. *Let E be a normal topological vector space, Y a topological vector space, U an open subset of E, $L : dom\, L \subseteq E \to Y$ a linear single valued map and $T \in H_L(E, Y)$. Suppose $F \in A_{\partial U}(\overline{U}, Y; L, T)$ is $(L,T)\Phi$-epi and $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}H : \overline{U} \times [0,1] \to K(E)$ an upper semicontinuous map and $(L+T)^{-1}H(x,0) = \{0\}$ for $x \in \partial U$. In addition assume the following conditions hold:*

$$(2.5)\qquad \begin{cases} \text{if } F_1 \in B(\overline{U}, Y; L, T) \text{ then } F_1(\cdot) + H(\\ cdot, \mu(\cdot)) \in B(\overline{U}, Y; L, T) \\ \text{for any continuous map } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \end{cases}$$

and

$$(2.6)\qquad \begin{cases} \{x \in \overline{U} : (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[\Phi(x) + H(x,t) + T(x)] \neq \emptyset \\ \text{for some } t \in [0,1]\} \text{ does not intersect } \partial U. \end{cases}$$

Then $F(\cdot) - H(\cdot, 1)$ is $(L,T)\Phi$-epi.

Proof. Let $G \in B_\Phi(\overline{U}, Y; L, T)$ and

$$D = \{x \in \overline{U} \ : \ (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[G(x) + H(x,t) + T(x)] \neq \emptyset \text{ for some } t \in [0,1]\}.$$

When $t = 0$ we have $G(\cdot) + H(\cdot, 0) \in B(\overline{U}, Y; L, T)$ and for $x \in \partial U$ we have $(L+T)^{-1}[G(x) + H(x,0) + T(x)] = (L+T)^{-1}(G+T)(x) \subseteq (L+T)^{-1}(\Phi+T)(x)$ so $G(\cdot) + H(\cdot, 0) \in B_\Phi(\overline{U}, Y; L, T)$ and this together with the fact that F is $(L,T)\Phi$-epi yields $D \neq \emptyset$. Similar reasoning as in Theorem 2.7 guarantees that D is closed. Also (2.6) guarantees that $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $J : \overline{U} \to 2^Y$ by

$$J(x) = G(x) + H(x, \mu(x)).$$

Now $J \in B(\overline{U}, Y; L, T)$ and for $x \in \partial U$ we have $(L+T)^{-1}(J+T)(x) = (L+T)^{-1}[G(x) + H(x,0) + T(x)] = (L+T)^{-1}(G+T)(x) \subseteq (L+T)^{-1}(\Phi+T)(x)$. Thus $J \in B_\Phi(\overline{U}, Y; L, T)$ so since F is $(L,T)\Phi$-epi there exists $x \in U$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(J+T)(x) \neq \emptyset$. Thus $x \in D$ so $\mu(x) = 1$ and we are finished. $\square$

Remark 2.19. We also have an analogue of Remark 2.8 in this case also.

Remark 2.20. If we change Definition 2.12 (respectively Definition 2.13) to $F \in A(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ with $(L+T)^{-1}F \in A(\overline{U}, E)$ (respectively $(L+T)^{-1}F \in B(\overline{U}, E)$), Definition 2.14 to $F \in A_{\partial U}(\overline{U}, Y; L, T)$

if $F \in A(\overline{U}, Y; L, T)$ with $(L+T)^{-1}F(x) \cap (L+T)^{-1}\Phi(x) = \emptyset$ for $x \in \partial U$, Definition 2.15 to $F \in B_\Phi(\overline{U}, Y; L, T)$ if $F \in B(\overline{U}, Y; L, T)$ and $(L+T)^{-1}F(x) \subseteq (L+T)^{-1}\Phi(x)$ for $x \in \partial U$, Definition 2.16 to $F \in A_{\partial U}(\overline{U}, Y; L, T)$ is $(L, T)\Phi$-epi if for every map $G \in B_\Phi(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}F(x) \cap (L+T)^{-1}G(x) \neq \emptyset$, then we have an analogue of Theorem 2.18 if (2.6) is replaced by

$$\begin{cases} \{x \in \overline{U} : (L+T)^{-1}F(x) \cap (L+T)^{-1}[\Phi(x) + H(x,t)] \neq \emptyset \text{ for some } t \in [0,1]\} \\ \text{does not intersect } \partial U. \end{cases}$$

$(L, T)\Phi$-epi maps of this type when $\Phi = 0$ were discussed in [5].

Theorem 2.21. *Let E be a normal topological vector space, Y a topological vector space, U an open subset of E, $L : dom\, L \subseteq E \to Y$ a linear single valued map and $T \in H_L(E, Y)$. Suppose $F \in A_{\partial U}(\overline{U}, Y; L, T)$ is $(L, T)\Phi$-epi, $G \in B(\overline{U}, Y; L, T)$ and assume the following conditions hold:*

$$\begin{cases} \mu(\cdot)\, G(\cdot) + (1 - \mu(\cdot))\Phi(\cdot) \in B(\overline{U}, Y; L, T) \textit{ for any} \\ \textit{continuous map } \mu : \overline{U} \to [0,1] \textit{ with } \mu(\partial U) = 0 \end{cases} \tag{2.7}$$

and

(2.8)
$$\begin{cases} \{x \in \overline{U} : (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[tG(x) + (1-t)\Phi(x) + T(x)] \neq \emptyset \\ \textit{for some } t \in [0,1]\} \textit{ does not intersect } \partial U. \end{cases}$$

Then there exists $x \in \overline{U}$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(G+T)(x) \neq \emptyset$.

Proof. Let

$$\begin{aligned} D = \{x \in \overline{U} \ \ : \ \ & (L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}[tG(x) + (1-t)\Phi(x) + T(x)] \neq \emptyset \\ & \text{for some } t \in [0,1]\}. \end{aligned}$$

Now $D \neq \emptyset$ is closed and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $J : \overline{U} \to 2^Y$ by

$$J(x) = \mu(x)G(x) + (1 - \mu(x))\Phi(x).$$

Now $J \in B(\overline{U}, Y; L, T)$ and for $x \in \partial U$ we have $(L+T)^{-1}(J+T)(x) = (L+T)^{-1}[0 + (\Phi + T)(x)]$, so $J \in B_\Phi(\overline{U}, Y; L, T)$. Now since F is $(L, T)\Phi$-epi there exists $x \in U$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(J+T)(x) \neq \emptyset$. Thus $x \in D$ and as a result $\mu(x) = 1$, so we are finished. $\square$

Remark 2.22. We also have an analogue of Remark 2.8 in this case also.

3. Φ-ESSENTIAL MAPS

Let E be a completely regular topological space and U an open subset of E.

As in Section 2 we will consider classes $\mathbf{A}$ and $\mathbf{B}$ of maps.

Definition 3.1. We say $F \in A(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F \in \mathbf{A}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$) and $F : \overline{U} \to K(E)$ is an upper semicontinuous map.

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$ in the first two results.

Definition 3.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$.

Definition 3.3. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous map $\Psi : \overline{U} \times [0, 1] \to K(E)$ with $\Psi(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $\Psi_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $\Psi_1 = F$, $\Psi_0 = G$ and $\big\{x \in \overline{U} : \Phi(x) \cap \Psi(x, t) \neq \emptyset$ for some $t \in [0, 1]\big\}$ is relatively compact (here $\Psi_t(x) = \Psi(x, t)$).

Remark 3.4. We note if $H : \overline{U} \times [0, 1] \to K(E)$ is an upper semicontinuous map then (similar reasoning as in Section 2) $M = \big\{x \in \overline{U} : \Phi(x) \cap H(x, t) \neq \emptyset$ for some $t \in [0, 1]\big\}$ is closed so that if M is relatively compact then M is compact.

The following condition will be assumed in our next two results:

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, E). \tag{3.1}$$

Definition 3.5. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to K(E)$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. Otherwise F is Φ-inessential in $A_{\partial U}(\overline{U}, E)$ i.e. there exists a map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ with $J(x) \cap \Phi(x) = \emptyset$ for all $x \in \overline{U}$.

Theorem 3.6. *Let E be a completely regular topological space, U an open subset of E and assume* (3.1) *holds. Suppose $F \in A_{\partial U}(\overline{U}, E)$. Then the following are equivalent:*

(i). F is Φ-inessential in $A_{\partial U}(\overline{U}, E)$;

(ii). there exists a map $G \in A_{\partial U}(\overline{U}, E)$ with $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ and $G(x) \cap \Phi(x) = \emptyset$ for all $x \in \overline{U}$.

Proof. (i) implies (ii) is immediate. Next we prove (ii) implies (i). Suppose there exists a map $G \in A_{\partial U}(\overline{U}, E)$ with $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ and $G(x) \cap \Phi(x) = \emptyset$ for all $x \in \overline{U}$. Let $H : \overline{U} \times [0, 1] \to K(E)$ be a upper semicontinuous map with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$

for any $x \in \partial U$ and $t \in [0,1]$, $H_0 = F$, $H_1 = G$ (here $H_t(x) = H(x,t)$) and $\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset$ for some $t \in [0,1]\}$ is relatively compact. Consider

$$D = \left\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

If $D = \emptyset$ then in particular $\emptyset = \Phi(x) \cap H(x,0) = \Phi(x) \cap F(x)$ for $x \in \overline{U}$ so F is Φ-inessential in $A_{\partial U}(\overline{U}, E)$ (take $J = F$ in Definition 3.5). Next suppose $D \neq \emptyset$. Essentially the same reasoning as in Theorem 2.7 guarantees that D is closed in E so D is compact from Remark 3.4. Also $D \cap \partial U = \emptyset$. Thus (note E is a completely regular topological space) there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J : \overline{U} \to K(E)$ by $J(x) = H(x, \mu(x))$. Note $J \in A(\overline{U}, E)$ and $J|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$. Also note if there exists a $x \in \overline{U}$ with $J(x) \cap \Phi(x) \neq \emptyset$ then $x \in D$ so $\mu(x) = 1$ i.e. $G(x) \cap \Phi(x) \neq \emptyset$, a contradiction. Thus $J \in A_{\partial U}(\overline{U}, E)$ and $J|_{\partial U} = F|_{\partial U}$ and $J(x) \cap \Phi(x) = \emptyset$ for $x \in \overline{U}$. We now claim

$$J \cong F \text{ in } A_{\partial U}(\overline{U}, E). \tag{3.2}$$

If (3.2) is true then F is Φ-inessential in $A_{\partial U}(\overline{U}, E)$.

It remains to show (3.2). Let $Q : \overline{U} \times [0,1] \to K(E)$ be given by $Q(x,t) = H(x, t\mu(x))$. Note $Q : \overline{U} \times [0,1] \to K(E)$ is an upper semicontinuous map, $Q(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ and

$$\left\{x \in \overline{U} : \emptyset \neq \Phi(x) \cap Q(x,t) = \Phi(x) \cap H(x, t\mu(x)) \text{ for some } t \in [0,1]\right\}$$

is closed and compact. Note $Q_0 = F$ and $Q_1 = J$. Finally if there exists a $t \in [0,1]$ and $x \in \partial U$ with $\Phi(x) \cap Q_t(x) \neq \emptyset$ than $\Phi(x) \cap H_{t\mu(x)}(x) \neq \emptyset$ so $x \in D$ and so $\mu(x) = 1$ i.e. $\Phi(x) \cap H_t(x) \neq \emptyset$, a contradiction. Thus (3.2) holds. □

Theorem 3.7. *Let E be a completely regular topological space, U an open subset of E and assume* (3.1) *holds. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Then F is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if and only if G is Φ-essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. F is Φ-inessential in $A_{\partial U}(\overline{U}, E)$ iff there exists a map $\Psi \in A_{\partial U}(\overline{U}, E)$ with $F \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$ and $\Phi(x) \cap \Psi(x) = \emptyset$ for $x \in \overline{U}$ iff (since (3.1) holds) there exists a map $\Psi \in A_{\partial U}(\overline{U}, E)$ with $G \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$ and $\Phi(x) \cap \Psi(x) = \emptyset$ for $x \in \overline{U}$ iff G is Φ-inessential in $A_{\partial U}(\overline{U}, E)$. □

Remark 3.8. If E is a normal topological space then the assumption that

$$\left\{x \in \overline{U} : \Phi(x) \cap \Psi(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is relatively compact can be removed in Definition 3.3 and we still obtain Theorem 3.6 and Theorem 3.7.

Remark 3.9. A result of Theorem 3.7 type was established in [4, Theorem 2.8] but however an assumption was omitted. In [4, Theorem 2.8] the result will work for subclasses of the admissible maps in [4] where $\cong$ is an equivalence relation in that class (for example the Kakutani and acyclic maps [6, 8]).

We next present a result where (3.1) is not needed.

Theorem 3.10. *Let E be a completely regular topological space, U an open subset of E and let $F \in A_{\partial U}(\overline{U}, E)$ be Φ-essential in $A_{\partial U}(\overline{U}, E)$. Suppose there exists an upper semicontinuous map $H : \overline{U} \times [0,1] \to K(E)$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$, $H_0 = F$ and $\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is relatively compact. Then there exists $x \in U$ with $\Phi(x) \cap H_1(x) \neq \emptyset$.*

Proof. Let

$$D = \left\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Note $D \neq \emptyset$ since F is Φ-essential in $A_{\partial U}(\overline{U}, E)$ (note $F \cong F$ in $A_{\partial U}(\overline{U}, E)$). Essentially the same reasoning as in Theorem 2.7 guarantees that D is closed in E so D is compact from Remark 3.4. Also $D \cap \partial U = \emptyset$ (note $H_0 = F$ so for $t = 0$ we have $\Phi(x) \cap H_0(x) = \emptyset$ for $x \in \partial U$ since $F \in A_{\partial U}(\overline{U}, E)$). Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J : \overline{U} \to K(E)$ by $J(x) = H(x, \mu(x))$. Note $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ (note if $x \in \overline{U}$ then $J(x) = H_0(x) = F(x)$ and $J(x) \cap \Phi(x) = F(x) \cap \Phi(x) = \emptyset$). Also as in Theorem 3.6, $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ (take as before $Q : \overline{U} \times [0,1] \to K(E)$ given by $Q(x,t) = H(x, t\mu(x))$). Now since F is Φ-essential in $A_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and as a result $H_1(x) \cap \Phi(x) \neq \emptyset$. □

Remark 3.11. If E is a normal topological space then the assumption that

$$\left\{x \in \overline{U} : \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is relatively compact can be removed in the statement of Theorem 3.10 and we still obtain Theorem 3.10.

Remark 3.12. The result in Theorem 3.10 also holds (proof is easier also) if we change Definition 3.5 as follows: Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to K(E)$ is Φ-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Let E be a Hausdorff topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Finally T :

$E \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(E, Y)$.

Definition 3.13. We say $F \in A(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $(L + T)^{-1}(F + T) \in A(\overline{U}, E)$ (respectively $L + T)^{-1}(F + T) \in B(\overline{U}, E)$).

We now fix a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 3.14. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $(L + T)^{-1}(F + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for $x \in \partial U$.

Definition 3.15. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $\Psi : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(\Psi + T) : \overline{U} \times [0, 1] \to K(E)$ a upper semi continuous map, $(L + T)^{-1}(\Psi(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}(\Psi_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $\Psi_1 = F$, $\Psi_0 = G$ and $\{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(\Psi_t + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\}$ is relatively compact (here $\Psi_t(x) = \Psi(x, t)$).

The following condition will be assumed in our next two results:

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y; L, T). \tag{3.3}$$

Definition 3.16. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. We say F is $(L, T)\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$. Otherwise F is $(L, T)\Phi$-inessential in $A_{\partial U}(\overline{U}, E)$ i.e. there exists a map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for all $x \in \overline{U}$.

Theorem 3.17. *Let E be a completely regular topological vector space, Y a topological vector space, U an open subset of E, $L : dom\, L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, and assume* (3.3) *holds. Suppose $F \in A_{\partial U}(\overline{U}, Y; L, T)$. Then the following are equivalent:*

(i). F is $(L, T)\Phi$-inessential in $A_{\partial U}(\overline{U}, Y; L, T)$;

(ii). there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ and $(L + T)^{-1}(G + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for all $x \in \overline{U}$.

Proof. (i) implies (ii) is immediate. Next we prove (ii) implies (i). Suppose there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ and $(L+T)^{-1}(G+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for all $x \in \overline{U}$. Let $H : \overline{U} \times [0, 1] \to 2^Y$ with $(L+T)^{-1}(H+T) : \overline{U} \times [0, 1] \to K(E)$ an upper semi continuous map, $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}(H_t +$

$T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ (here $H_t(x) = H(x,t)$) and

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is relatively compact. Let

$$D = \left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

If $D = \emptyset$ we are finished. If $D \neq \emptyset$ then note D is compact, $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J : \overline{U} \to 2^Y$ by $J(x) = H(x, \mu(x))$. It is easy to check (a slight modification of the argument in Theorem 3.6) that $J \in A_{\partial U}(\overline{U}, Y; L, T)$, $J|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$, $(L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(J+T)(x) = \emptyset$ for $x \in \overline{U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Thus F is $(L,T)\Phi$-inessential in $A_{\partial U}(\overline{U}, Y; L, T)$. $\square$

Theorem 3.18. *Let E be a completely regular topological vector space, Y a topological vector space, U an open subset of E, $L : dom\, L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E,Y)$, and assume* (3.3) *holds. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, Y; L, T)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Then F is $(L,T)\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if and only if G is $(L,T)\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$.*

Remark 3.19. If E is a normal topological space then the assumption that

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(\Psi_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is relatively compact can be removed in Definition 3.15 and we still obtain Theorem 3.17 and Theorem 3.18.

Theorem 3.20. *Let E be a completely regular topological vector space, Y a topological vector space, U an open subset of E, $L : dom\, L \subseteq E \to Y$ a linear single valued map and $T \in H_L(E,Y)$. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ be $(L,T)\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$. Suppose there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H+T) : \overline{U} \times [0,1] \to K(E)$ an upper semi continuous map, $(L+T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1]$, $H_0 = F$ (here $H_t(x) = H(x,t)$) and*

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \textit{ for some } t \in [0,1]\right\}$$

is relatively compact. Then there exists $x \in U$ with $(L+T)^{-1}(H_1+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$.

Proof. Let

$$D = \left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Note $D \neq \emptyset$ and D is compact, $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J : \overline{U} \to 2^Y$ by $J(x) = H(x, \mu(x))$. Note $J \in A_{\partial U}(\overline{U}, Y; L, T)$, $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Now since F is $(L,T)\Phi$-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$ (i.e. $(L+T)^{-1}(H_{\mu(x)}+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and we are finished. $\square$

Remark 3.21. If E is a normal topological space then the assumption that

$$\left\{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t+T)(x) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is relatively compact can be removed in the statement of Theorem 3.20 and we still obtain Theorem 3.20.

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