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EXISTENCE, UNIQUENESS AND QUENCHING FOR A PARABOLIC PROBLEM WITH A MOVING NONLINEAR SOURCE ON A SEMI-INFINITE INTERVAL

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ABSTRACT. Let v and T be positive numbers, $D = (0, \infty)$, $\Omega = D \times (0, T]$, and $\bar{D}$ be the closure of D. This article studies the first initial-boundary value problem,

$$u_t - u_{xx} = \delta(x - vt) f(u(x, t)) \text{ in } \Omega,$$
$$u(x, 0) = 0 \text{ on } \bar{D},$$
$$u(0, t) = 0, u(x, t) \to 0 \text{ as } x \to \infty \text{ for } 0 < t \leq T,$$

where $\delta(x)$ is the Dirac delta function, and f is a given function such that $\lim_{u\to c^-} f(u) = \infty$ for some positive constant c. It is shown that the problem has a unique nonnegative continuous solution u, and $u(vt, t)$ is a strictly increasing function of t; also, if u exists for $t \in [0, t_q)$ with $t_q < \infty$, then $\sup\{u(x, t) : 0 \leq x < \infty\}$ reaches c^- at t_q.

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1. INTRODUCTION

Let v and T be positive numbers, $D = (0, \infty)$, $\bar{D} = [0, \infty)$, $\Omega = D \times (0, T]$, and $Hu = u_t - u_{xx}$. We consider the following semilinear parabolic first initial-boundary value problem,

$$\begin{cases} Hu = \delta(x - vt) f(u(x, t)) \text{ in } \Omega, \\ u(x, 0) = 0 \text{ on } \bar{D}, \\ u(0, t) = 0, u(x, t) \to 0 \text{ as } x \to \infty \text{ for } 0 < t \leq T, \end{cases} \tag{1.1}$$

where $\delta(x)$ is the Dirac delta function, and f is a given function such that $\lim_{u\to c^-} f(u) = \infty$ for some positive constant c. We assume that $f(u)$, $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$.

A solution u of the problem (1.1) is a continuous function satisfying (1.1). A solution u of the problem (1.1) is said to quench if there exists some t_q such that

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$\sup\{u(x,t) : x \in D\} \to c^-$ as $t \to t_q$. If t_q is finite, then u is said to quench in a finite time. On the other hand, if $t_q = \infty$, u is said to quench in infinite time.

In Section 2, we convert the problem (1.1) into a nonlinear integral equation, and prove that there exists some t_q such that the integral equation has a unique continuous solution u for $0 \leq t < t_q$. We show that the solution u is the solution of the problem (1.1), and $u(vt,t)$ is a strictly increasing function of t. We also show that if t_q is finite, then u quenches at t_q.

2. EXISTENCE, UNIQUENESS AND QUENCHING

Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.1) is determined by the following system: for x and ξ in D, and t and τ in $(-\infty,\infty)$,

$$HG(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau);\; G(x,t;\xi,\tau) = 0,\; t < \tau,$$
$$G(0,t;\xi,\tau) = 0, \text{ and } G(x,t;\xi,\tau) \to 0 \text{ as } x \to \infty.$$

For $t > \tau$, it is given by

$$G(x,t;\xi,\tau) = \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}}$$

(cf. Duffy [3, p. 183]). To derive the integral equation from the problem (1.1), let us consider the adjoint operator H^*, which is given by $H^*u = -u_t - u_{xx}$. Using Green's second identity, we obtain

$$u(x,t) = \int_0^t G(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau. \tag{2.1}$$

For ease of reference, we state Lemma 2.1 of Chan, Sawangtong and Treeyaprasert [2] as Lemma 2.1 below.

Lemma 2.1. *If $r \in C([0,T])$, then $\int_0^t G(x,t;v\tau,\tau) r(\tau)\, d\tau$ is continuous on $\bar{\Omega}$, where $\bar{\Omega}$ denotes the closure of Ω.*

We modified the techniques in proving Theorem 1 of Chan and Jiang [1] for a stationary source in a bounded domain to obtain the following result for a moving source in an unbounded domain.

Theorem 2.2. *There exists some $t_q\,(\leq \infty)$ such that for $0 \leq t < t_q$, the integral equation (2.1) has a unique nonnegative continuous solution u, and $u(vt,t)$ is a strictly increasing function of t. If t_q is finite, then u quenches at t_q.*

Proof. Let us construct a sequence $\{u_n\}$ by $u_0(x,t) = 0$, and for $n = 0, 1, 2, \ldots$,

$$u_{n+1}(x,t) = \int_0^t G(x,t;v\tau,\tau) f\left(u_n(v\tau,\tau)\right) d\tau. \tag{2.2}$$

Since $G(x,t;v\tau,\tau)$ and $f(0)$ are positive in Ω, we have from (2.2) that $u_1(x,t) > u_0(x,t) = 0$ in Ω. Let us assume that for some positive integer j,

$$0 < u_1 < u_2 < \cdots < u_{j-1} < u_j \text{ in } \Omega.$$

Since f is a strictly increasing function, and $u_j > u_{j-1}$, we have

$$u_{j+1}(x,t) - u_j(x,t) = \int_0^t G(x,t;v\tau,\tau)\left(f\left(u_j(v\tau,\tau)\right) - f\left(u_{j-1}(v\tau,\tau)\right)\right) d\tau > 0.$$

By the principle of mathematical induction,

$$0 < u_1 < u_2 < \cdots < u_{n-1} < u_n \text{ in } \Omega$$

for any positive integer n. To show that $u_n(vt,t)$ is an increasing function of t, let us construct a sequence $\{w_n\}$ in $D \times (0, T-\varepsilon]$ such that for $n = 0,1,2,\ldots$, $w_n(vt,t) = u_n(v(t+\varepsilon), t+\varepsilon) - u_n(vt,t)$, where ε is any positive number less than T. We have $w_0(vt,t) = 0$. By (2.2), we have

$$\begin{aligned} w_1(vt,t) &= u_1(v(t+\varepsilon), t+\varepsilon) - u_1(vt,t) \\ &= f(0)\left[\int_0^{t+\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau,\tau)d\tau - \int_0^t G(vt,t;v\tau,\tau)d\tau\right]. \end{aligned} \tag{2.3}$$

Let $\sigma = \tau - \varepsilon$. Then,

$$\begin{aligned} &\int_0^{t+\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau,\tau)d\tau \\ &= \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau,\tau)d\tau + \int_0^t G(v(t+\varepsilon), t+\varepsilon; v(\sigma+\varepsilon), \sigma+\varepsilon)d\sigma \\ &= \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau,\tau)d\tau + \int_0^t G(v(t+\varepsilon), t; v(\sigma+\varepsilon), \sigma)d\sigma. \end{aligned} \tag{2.4}$$

For $t > \sigma$,

$$\begin{aligned} G(v(t+\varepsilon), t; v(\sigma+\varepsilon), \sigma) &= \frac{e^{-\frac{[v(t+\varepsilon)-v(\sigma+\varepsilon)]^2}{4(t-\sigma)}} - e^{-\frac{[v(t+\varepsilon)+v(\sigma+\varepsilon)]^2}{4(t-\sigma)}}}{\sqrt{4\pi(t-\sigma)}} \\ &> \frac{e^{-\frac{(vt-v\sigma)^2}{4(t-\sigma)}} - e^{-\frac{(vt+v\sigma)^2}{4(t-\sigma)}}}{\sqrt{4\pi(t-\sigma)}} = G(vt,t;v\sigma,\sigma) > 0. \end{aligned} \tag{2.5}$$

We have from (2.4) and (2.5) that

$$\begin{aligned} \int_0^{t+\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau,\tau)d\tau &> \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau,\tau)d\tau \\ &\quad + \int_0^t G(vt,t;v\sigma,\sigma)\, d\sigma. \end{aligned} \tag{2.6}$$

It follows from (2.3) and (2.6) that for $0 < t \le T - \varepsilon$,

$$w_1(vt,t) > f(0)\int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau,\tau)d\tau > 0.$$

Let us assume that for some positive integer j,

$$w_j\left(vt,t\right) = u_j\left(v\left(t+\varepsilon\right),t+\varepsilon\right) - u_j\left(vt,t\right) > 0 \text{ for } 0 < t \leq T-\varepsilon.$$

Then,

$$\begin{aligned} w_{j+1}\left(vt,t\right) = &\int_0^{t+\varepsilon} G(v\left(t+\varepsilon\right),t+\varepsilon;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau \\ &-\int_0^t G(vt,t;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau. \end{aligned}$$

Let $\sigma = \tau - \varepsilon$. Then,

$$\begin{aligned} &\int_0^{t+\varepsilon} G(v\left(t+\varepsilon\right),t+\varepsilon;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau \\ &= \int_0^{\varepsilon} G(v\left(t+\varepsilon\right),t+\varepsilon;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau \\ &+ \int_0^{t} G(v\left(t+\varepsilon\right),t;v\left(\sigma+\varepsilon\right),\sigma)f\left(u_j(v\left(\sigma+\varepsilon\right),\sigma+\varepsilon)\right)d\sigma \\ &> \int_0^{\varepsilon} G(v\left(t+\varepsilon\right),t+\varepsilon;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau \\ &+ \int_0^{t} G(v\left(t+\varepsilon\right),t;v\left(\sigma+\varepsilon\right),\sigma)f\left(u_j(v\sigma,\sigma)\right)d\sigma \end{aligned} \tag{2.7}$$

since $u_j\left(v\left(\sigma+\varepsilon\right),\sigma+\varepsilon\right) > u_j\left(v\sigma,\sigma\right)$ and f is an increasing function. We have from (2.7) and (2.5) that for $0 < t \leq T-\varepsilon$,

$$\begin{aligned} w_{j+1}\left(vt,t\right) > &\int_0^{\varepsilon} G(v\left(t+\varepsilon\right),t+\varepsilon;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau \\ &+\int_0^{t} G(v\left(t+\varepsilon\right),t;v\left(\sigma+\varepsilon\right),\sigma)f\left(u_j(v\sigma,\sigma\right)d\sigma \\ &-\int_0^{t} G(vt,t;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau \\ &>\int_0^{\varepsilon} G(v\left(t+\varepsilon\right),t+\varepsilon;v\tau,\tau)f\left(u_j(v\tau,\tau)\right)d\tau > 0. \end{aligned}$$

By the principle of mathematical induction, $w_n\left(vt,t\right) > 0$ for $0 < t \leq T-\varepsilon$ for all positive integers n. Thus, each $u_n\left(vt,t\right)$ is a strictly increasing function of t.

For any given positive constant M $(< c)$, it follows from (2.2) and $u_n\left(vt,t\right)$ being a strictly increasing function of t that there exists some t_1 such that $u_n \leq M$ for $0 \leq t \leq t_1$ and $n = 0,1,2,\ldots$. In fact, t_1 satisfies

$$f\left(M\right)\int_0^{t_1} G(x,t_1;v\tau,\tau)d\tau \leq M.$$

Let u denote $\lim_{n\to\infty} u_n$. From (2.2) and the Monotone Convergence Theorem (cf. Stromberg [4, p. 288]), we have (2.1) for $0 \leq t \leq t_1$.

Each u_n is continuous by Lemma 2.1. To show that u is continuous, we note from (2.2) that

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t G(x,t;v\tau,\tau)\left[f(u_n(v\tau,\tau)) - f(u_{n-1}(v\tau,\tau))\right]d\tau.$$

Let $S_n = \sup_{\bar{D}\times[0,t_1]}(u_n - u_{n-1})$. By using the Mean Value Theorem,

$$S_{n+1} \le f'(M)\, S_n \sup_{\bar{D}\times[0,t_1]} \int_0^t G(x,t;v\tau,\tau)d\tau. \tag{2.8}$$

For any given positive number ε, we have

$$\begin{aligned}\int_0^t G(x,t;v\tau,\tau)d\tau &= \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} \frac{e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} - e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}}d\tau \\ &\le \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} \frac{1}{\sqrt{4\pi(t-\tau)}}d\tau = \sqrt{\frac{t}{\pi}}.\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we have

$$S_{n+1} \le f'(M)\sqrt{\frac{t}{\pi}}S_n.$$

Let us choose some positive number σ_1 ($\le t_1$) such that for $t \in [0,\sigma_1]$,

$$f'(M)\sqrt{\frac{t}{\pi}} < 1. \tag{2.10}$$

Then, the sequence $\{u_n\}$ converges uniformly to $\lim_{n\to\infty} u_n(x,t)$ for $0 \le t \le \sigma_1$. Thus, the integral equation (2.1) has a nonnegative continuous solution u for $0 \le t \le \sigma_1$. If $\sigma_1 < t_1$, it follows from (2.1) that

$$u(x,t) = \int_0^{\sigma_1} G(x,t;v\tau,\tau)f(u(v\tau,\tau))\,d\tau + \int_{\sigma_1}^t G(x,t;v\tau,\tau)f(u(v\tau,\tau))\,d\tau. \tag{2.11}$$

The first term on the right hand side of (2.11) is continuous. Let

$$z(x,t) = \int_{\sigma_1}^t G(x,t;v\tau,\tau)f(z(v\tau,\tau))\,d\tau.$$

From (2.11), $z < M$. For $\sigma_1 \le t \le t_1$, let us construct a sequence $\{z_i\}$ by $z_0(x,t) = 0$, and for $n = 0,1,2,\ldots,$

$$z_{i+1}(x,t) = \int_{\sigma_1}^t G(x,t;v\tau,\tau)f(z_i(v\tau,\tau))\,d\tau.$$

A proof similar to that for Lemma 2.1 shows that z_i is continuous for $i = 1,2,3,\ldots$. We have

$$z_{i+1}(x,t) - z_i(x,t) = \int_{\sigma_1}^t G(x,t;v\tau,\tau)\left[f(z_i(v\tau,\tau)) - f(z_{i-1}(v\tau,\tau))\right]d\tau.$$

Let $Z_i = \sup_{\bar{D}\times[\sigma_1,\min\{2\sigma_1,t_1\}]}|z_i - z_{i-1}|$. Using the Mean Value Theorem, we have

$$f(z_i(v\tau,\tau)) - f(z_{i-1}(v\tau,\tau)) \le f'(M)\,Z_i.$$

Thus,

$$z_{i+1}(x,t) - z_i(x,t) \leq f'(M) Z_i \int_{\sigma_1}^{t} G(x,t;v\tau,\tau)d\tau \leq \frac{f'(M)\sqrt{t-\sigma_1}}{\sqrt{\pi}} Z_i.$$

It follows from (2.10) that for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$,

$$\frac{f'(M)\sqrt{t-\sigma_1}}{\sqrt{\pi}} < 1. \tag{2.12}$$

Therefore, $\{z_i\}$ converges uniformly to z, and hence, z is a continuous function for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$. From (2.12), u is continuous for $t \in [\sigma_1, \min\{2\sigma_1, t_1\}]$. If $2\sigma_1 < t_1$, then for $2\sigma_1 \leq t \leq t_1$,

$$u(x,t) = \int_0^{2\sigma_1} G(x,t;v\tau,\tau) f(u(v\tau,\tau))\, d\tau + \int_{2\sigma_1}^{t} G(x,t;v\tau,\tau) f(u(v\tau,\tau))\, d\tau.$$

Since the first term on the right-hand side is continuous, we consider the second term. An argument analogous to the above shows that u is continuous for $0 \leq t \leq \min\{3\sigma_1, t_1\}$. By proceeding in this way, the integral equation (2.1) has a continuous solution u for $0 \leq t \leq t_1$.

To prove that u is unique, let us assume that the integral equation (2.1) has two solutions u and $\tilde{u}$ on the interval $[0, t_1]$. Let $\Theta = \sup_{\bar{D}\times[0,t_1]} |u - \tilde{u}|$. From (2.1), we have

$$u(x,t) - \tilde{u}(x,t) = \int_0^t G(x,t;v\tau,\tau)\left(f(u(v\tau,\tau)) - f(\tilde{u}(v\tau,\tau))\right) d\tau.$$

By using the Mean Value Theorem,

$$\Theta \leq f'(M)\,\Theta \int_0^t G(x,t;v\tau,\tau)d\tau \leq \frac{f'(M)\sqrt{t}}{\sqrt{\pi}}\Theta.$$

By (2.10), we have a contradiction for $0 \leq t \leq \sigma_1$. Thus, u is unique for $0 \leq t \leq \sigma_1$. If $\sigma_1 < t_1$, then it follows from (2.1) that

$$u(x,t) = \int_0^{\sigma_1} G(x,t;v\tau,\tau) f(u(v\tau,\tau))\, d\tau + \int_{\sigma_1}^{t} G(x,t;v\tau,\tau) f(u(v\tau,\tau))\, d\tau.$$

Since $u = \tilde{u}$ for $0 \leq t \leq \sigma_1$, we have for $\sigma_1 \leq t \leq t_1$,

$$|u(x,t) - \tilde{u}(x,t)| = \int_{\sigma_1}^{t} G(x,t;v\tau,\tau)\left|f(u(v\tau,\tau)) - f(\tilde{u}(v\tau,\tau))\right| d\tau,$$

from which we obtain

$$\Theta \leq f'(M)\,\Theta \int_{\sigma_1}^{t} G(x,t;v\tau,\tau)d\tau \leq \frac{f'(M)\sqrt{t-\sigma_1}}{\sqrt{\pi}}\Theta.$$

By (2.12), we have a contradiction for $t \in [0, \min\{2\sigma_1, t_1\}]$. Thus, we have uniqueness of a solution for $t \in [0, \min\{2\sigma_1, t_1\}]$. By proceeding in this way, the integral equation (2.1) has a unique continuous solution u for $0 \leq t \leq t_1$.

Let t_q be the supremum of all t_1, where $[0, t_1]$ is the interval for which the integral equation (2.1) has a unique continuous solution $u\,(< c)$. If t_q is finite, and $\sup_{\bar{D}} u(x,t)$

does not reach c^- at t_q, then for any positive constant greater than $\sup_{\bar{D}} u(x, t_q)$, a proof similar to the above shows that there exists an interval $[t_q, t_2]$ such that the integral equation (2.1) has a unique continuous solution u that is bounded away from c. This contradicts the definition of t_q. Hence, if t_q is finite, $\sup_{\bar{D}} u(x,t)$ reaches c^- at t_q.

It follows from $u_n(vt, t)$ being an increasing function of t that $u(vt, t)$ is a non-decreasing function of t. Let $\sigma = \tau - \varepsilon$. Since f is an increasing function, and $u(v(\sigma+\varepsilon), \sigma+\varepsilon) \geq u(v\sigma, \sigma)$, we have

$$\begin{aligned}
&\int_0^{t+\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau, \tau) f(u(v\tau, \tau))\, d\tau \\
&= \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau, \tau) f(u(v\tau, \tau))\, d\tau \\
&+ \int_0^{t} G(v(t+\varepsilon), t+\varepsilon; v(\sigma+\varepsilon), \sigma+\varepsilon) f(u(v(\sigma+\varepsilon), \sigma+\varepsilon))\, d\sigma \\
&\geq \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau, \tau) f(u(v\tau, \tau))\, d\tau \\
&+ \int_0^{t} G(v(t+\varepsilon), t; v(\sigma+\varepsilon), \sigma) f(u(v\sigma, \sigma))\, d\sigma.
\end{aligned}$$

By (2.5),

$$\begin{aligned}
u(v(t+\varepsilon), t+\varepsilon) - u(vt, t) &\geq \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau, \tau) f(u(v\tau, \tau))\, d\tau \\
&+ \int_0^{t} G(v(t+\varepsilon), t; v(\sigma+\varepsilon), \sigma) f(u(v\sigma, \sigma))\, d\sigma \\
&- \int_0^{t} G(vt, t; v\tau, \tau) f(u(v\tau, \tau))\, d\tau \\
&> \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau, \tau) f(u(v\tau, \tau))\, d\tau \\
&+ \int_0^{t} G(vt, t; v\sigma, \sigma) f(u(v\sigma, \sigma))\, d\sigma \\
&- \int_0^{t} G(vt, t; v\tau, \tau) f(u(v\tau, \tau))\, d\tau \\
&= \int_0^{\varepsilon} G(v(t+\varepsilon), t+\varepsilon; v\tau, \tau) f(u(v\tau, \tau))\, d\tau > 0.
\end{aligned}$$

Hence, $u(vt, t)$ is a strictly increasing function of t. □

An argument similar to the proof of Theorem 2.3 of Chan, Sawangtong and Treeyaprasert [2] gives the following result.

Theorem 2.3. *The solution of the integral equation* (2.1) *is the unique solution of the problem* (1.1).

We remark from the above two theorems that if t_q is finite, u quenches at t_q.

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