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EIGENVALUE PROBLEM FOR ODES WITH A PERTURBED Q-LAPLACE OPERATOR

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ABSTRACT. We investigate the eigenvalue interval for boundary value problem with a one-dimensional perturbed q-Laplace operator. Our results cover also the case when the right-hand side has singularities. Applying variational methods we prove the existence of positive solutions and establish their continuous dependence on functional parameters.

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1. Introduction

This paper is devoted to the eigenvalue problem associated with a second order ODE containing a perturbed one-dimensional q-Laplace operator with a singularity at 0. Our main goal is to discuss when the equation

$$-\left(\left(a(t)\left|u'(t)\right|^{q-2}u'\right)' + \frac{ka(t)}{t}\left|u'(t)\right|^{q-2}u'\right) = f_1(t,u(t)) + \lambda f_2(t,u(t)) \tag{1.1}$$

a.e. in $(0,T)$, where $q \geq 2$, $k > 1$, $T > 0$, $a \in C^1([0,T])$, possesses at least one positive solution satisfying the boundary conditions

$$u'(0) = 0 \text{ and } u(T) = 0. \tag{1.2}$$

Our paper is motivated by the large number of papers associated with similar problems, see for example [1], [2], [3], [4], [5], [6], [8], [9]. The majority of these papers discuss the case $q = 2$ or when the right-hand side of (1) has a special form. The approach presented here is based on methods in calculus of variations. Thus we treat (1.1)–(1.2) as the Euler-Lagrange equation for the following functional

$$J(u) = \int_0^T t^k \left(-F_\lambda(t,u(t)) + \frac{1}{q}a(t)\left|u'(t)\right|^q\right)dt, \tag{1.3}$$

where $F_\lambda(t,u) := \int\limits_0^u (\overline{f_1}(t,l) + \lambda\overline{f_2}(t,l))dl$ and for $i = 1,2$,

$$\overline{f}_i(t,u) = \begin{cases} f_i(t,u) & \text{if } u \in [0,d_1], t \in [0,T] \\ +\infty & \text{if } u \in R\backslash[0,d_1], t \in [0,T] \end{cases}$$

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with positive $d_1 \in I := (-b, c)$, where b and c are fixed positive numbers. We deal with the case when the following assumptions hold

f1 $f_1, f_2 : [0,T] \times I \to R$ *are Caratheodory functions,* λ *is real number* R *such that for almost all* $t \in [0,T]$ *and all* $u \in I$

$$f_1(t,u) + \lambda f_2(t,u) \geq 0$$

and $t \mapsto f_1(t,0) + \lambda f_2(t,0)$ *is not identically zero in a certain subset of* $[0,T]$ *with positive measure.*

f2 *there exists positive* $d \in I$ *such that for* $i = 1,2$, $u \mapsto f_1(t,u) + \lambda f_2(t,u)$ *is increasing in* I *for a.a.* $t \in [0,T]$, *and*

$$\max_{u \in [0,d]} (f_1(\cdot,u) + \lambda f_2(\cdot,u)) \in L^{q'}(0,T),$$

with $q' = \frac{q}{q-1}$.

f3 $a \in C^1([0,T])$ and $a_{\min} := \min_{t \in [0,T]} a(t) > 0$.

Let

$$\begin{aligned} \widetilde{U} \;=\; & \Big\{ u \in C^1([0,T]) : u(T) = 0 \text{ and } u'(0) = 0 \text{ and } u'(t) < 0 \\ & \text{for } t \in [0,T] \text{ and } t^k a(t) \left|u'\right|^{q-2} u' \in A([0,T]) \Big\}, \end{aligned} \tag{1.4}$$

where $A([0,T])$ denotes the space of absolutely continuous functions v such that $v'/t^k \in L^{q'}(0,T)$.

Let us note that in this case J is not necessarily either bounded or continuously differentiable in its natural domain. Therefore we describe the set denoted by U in which J is bounded below and possesses a positive minimizer $\overline{u} \in U$. The special properties of U and the Fenchel equalities for auxiliary functionals allow us to show that $\overline{u}$ is the solution of (1.1)–(1.2). Also in this paper we discuss the continuous dependence of solutions on functional parameters for our problem. Here we employ the schema presented e.g. in [7], [6]. Roughly we prove that a sequence of solutions $(u_m)_{m \in N}$ of the problem

$$\left\{ \begin{array}{c} -\left(\left(a(t)\left|u'(t)\right|^{q-2} u'\right)' + \frac{ka(t)}{t}\left|u'(t)\right|^{q-2} u' \right) \\ = f_1(t,u(t),w(t)) + \lambda f_2(t,u(t),z(t)) \text{ a.e. in } (0,T), \\ u'(0) = 0 \text{ and } u(T) = 0, \end{array} \right. \tag{1.5}$$

corresponding to the sequence of parameters $((w_m, z_m))_{m \in N} \subset L^{p_1}(0,T) \times L^{p_2}(0,T)$, where $p_1, p_2 > 2$, tends uniformly to $\overline{u}$ in $[0,T]$ (up to a subsequence) provided that the sequence of parameters tends almost everywhere in $(0,T)$ to $(w_0, z_0) \in L^{p_1}(0,T) \times L^{p_2}(0,T)$. Moreover we show that $\overline{u}$ is the solution of (1.5) with parameters (w_0, z_0).

Lemma 1.1. *Assume f1, f2 and f3. If u is a solutions of (1.1)–(1.2) such that $u(t) \in I$ then $u'(t) < 0$ for $t \in (0,T)$.*

Proof. Let $h(t) := t^k a(t)|u'(t)|^{q-2}u'(t)$ for all $t \in [0,T]$. Since $h'(t) < 0$ for all $t \in (0,T)$ we see that h is decreasing. Moreover $h(0) = 0$, so we have $h(t) < h(0) = 0$ for $t \in (0,T)$. Therefore, by f3 and definition of h, we see that $u'(t) < 0$ for $t \in (0,T)$. □

Lemma 1.2. *Suppose that f1, f2, f3 hold and assume additionally that for $d \in I$ defined in f2 the following inequality hold*

f4
$$\int_0^T \left(\frac{1}{a(s)s^k} \int_0^s r^k \left(f_1(r,d) + \lambda f_2(r,d) dr \right) \right)^{\frac{1}{q-1}} ds \le d.$$

Then the set $U := \{u \in \widetilde{U}; u(t) \le d \text{ for all } t \in [0,T]\}$ has the following property: for each $u \in U$ there exists $\widetilde{u} \in U$ such that for a.e. in $(0,T)$

$$-\left(a(t)t^k \left|\widetilde{u}'(t)\right|^{q-2} \widetilde{u}'(t) \right)' = t^k (f_1(t,u(t)) + \lambda f_2(t,u(t))). \tag{1.6}$$

Proof. Fix $u \in U$. We show that

$$\widetilde{u}(t) = \int_t^T \left(\frac{1}{a(s)s^k} \int_0^s r^k \left(f_1(r,u(r)) + \lambda f_2(r,u(r)) dr \right) \right)^{\frac{1}{q-1}} ds$$

also belongs to U and satisfies (1.6). To this end we note

$$\widetilde{u}'(t) = -\left(\frac{1}{a(t)t^k} \int_0^t r^k \left(f_1(r,u(r)) + \lambda f_2(r,u(r)) dr \right) \right)^{\frac{1}{q-1}}$$

and further

$$\begin{aligned} & a(t)t^k \left|\widetilde{u}'(t)\right|^{q-2} \widetilde{u}'(t) \\ &= -a(t)t^k \left[\left(\frac{1}{a(t)t^k} \int_0^t r^k \left(f_1(r,u(r)) + \lambda f_2(r,u(r)) dr \right) \right)^{\frac{1}{q-1}} \right]^{q-1} \\ &= -\int_0^t r^k \left(f_1(r,u(r)) + \lambda f_2(r,u(r)) dr \right) \end{aligned}$$

which gives (1.6). It is clear that $\widetilde{u}(T) = 0$, $\widetilde{u} \in C([0,T]) \cap C^1((0,T])$. Moreover, by Hőlder's inequality, we have

$$\begin{aligned} \left|\widetilde{u}'(t)\right|^{q-1} &= \frac{1}{a(t)t^k} \int_0^t r^k f_1(r,u(r)) + \lambda f_2(r,u(r)) dr \\ &\le \frac{1}{a(t)t^k} \left(\int_0^t l^{qk} dl \right)^{1/q} \left(\int_0^t (\overline{f_1}(l,u(l)) + \lambda \overline{f_2}(t,u(t)))^{q'} dl \right)^{1/q'} \\ &\le \frac{1}{a(t)t^k} \left(\frac{1}{qk+1} \right)^{1/q} t^{k+1/q} \left(\int_0^T (\overline{f_1}(l,d) + \lambda \overline{f_2}(t,d))^{q'} dl \right)^{1/q'} \end{aligned}$$

$$\leq \frac{1}{a_{\min}} \left(\frac{1}{qk+1}\right)^{1/q} \left(\int_0^T (\overline{f_1}(l,d) + \lambda \overline{f_2}(t,d))^{q'} dl\right)^{1/q'} t^{1/q}.$$

Therefore

$$\lim_{t\to 0^+} \widetilde{u}'(t) = 0.$$

Taking into account (1.6) we get

$$\left(a(t)t^k \left|\widetilde{u}'(t)\right|^{q-2} \widetilde{u}'(t)\right)' / t^k = (f_1(t,u(t)) + \lambda f_2(t,u(t)))$$

which means, by f2, that $\left(a(t)t^k \left|\widetilde{u}'(t)\right|^{q-2} \widetilde{u}'(t)\right)' / t^k$ belongs to $\in L^{q'}(0,T)$. Finally, by the definition of $\widetilde{u}$ we get $a(t)t^k \left|\widetilde{u}'(t)\right|^{q-2} \widetilde{u}'(t) \in A([0,T])$. □

Theorem 1.3. *Assume that (f1)–(f4) hold. If $(u_m)_{m\in N} \subset U$ is a minimizing sequence of the functional $J : U \to R$ then there exists a sequence $(v_m)_{m\in N} \subset W^{1,q'}(0,T)$ such that*

$$-v_m'(t) = t^k \left(\overline{f_1}(t,u_m)) + \lambda \overline{f_2}(t,u_m)\right) \ \textit{a.e. in } (0,1) \tag{1.7}$$

and

$$\lim_{m\to\infty} \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} + \frac{1}{q} a(t)t^k |u_m'(t)|^q - u_m'(t)v_m(t))dt = 0. \tag{1.8}$$

Proof. Let us note that J is bounded below on U. Indeed, for each $u \in U$ one can see

$$\begin{aligned} J(u) &= \int_0^T \left[-t^k F_\lambda(t,u) + \frac{a(t)t^k}{q} |u'(t)|^q\right] dt \\ &\geq -\int_0^T t^k F_\lambda(t,u(t))dt \geq -\int_0^T t^k u(t)[\left(\overline{f_1}(t,d)) + \lambda \overline{f_2}(t,d)\right)]dt \\ &\geq -dT^k \int_0^T [\left(\overline{f_1}(t,d)) + \lambda \overline{f_2}(t,d)\right)]dt, \end{aligned} \tag{1.9}$$

and further $-\infty < \min := \inf_{u\in\widetilde{U}} J(u) < +\infty$, which implies that for each $\varepsilon > 0$ there exists $m_0 \in N$ such that $J(u_m) < \varepsilon + \min$ for all $m \geq m_0$. Taking into account Lemma 1.2 we infer that for each $u_m \in U$, there exists $(\overline{u}_m)_{m\in N} \subset U$ such that

$$\begin{gathered} -(t^k a(t)|\overline{u}_m'(t)|^{q-2}\overline{u}_m'(t))' = t^k \left(\overline{f_1}(t,u_m(t)) + \lambda \overline{f_2}(t,u_m(t))\right) \text{ a.e. in } (0,T) \\ \overline{u}_m'(0) = 0 \text{ and } \overline{u}_m(T) = 0. \end{gathered} \tag{1.10}$$

We consider the following sequence $(v_m)_{m\in N} \subset W^{1,q'}(0,T)$

$$v_m(t) := t^k a(t) \left|\overline{u}_m'(t)\right|^{q-2} \overline{u}_m'(t) \text{ for } t \in (0,T) \tag{1.11}$$

and note, by (1.10), that

$$\begin{aligned} -v_m'(t) &\in \partial_u\{t^k F_\lambda(t,u_m(t))\} \\ &= \{t^k \left(\overline{f_1}(t,u_m(t)) + \lambda \overline{f_2}(t,u_m(t))\right)\} \text{ a. e. in } (0,T) \end{aligned} \tag{1.12}$$

which can be rewritten as (1.7).

Moreover, by the Fenchel equality for $L^q(0,T) \ni u \mapsto \int_0^T t^k F_\lambda(t,u(t))dt$, we infer that for each $m \geq m_0$

$$\begin{aligned} (1.13) \qquad \min + \varepsilon &> J(u_m) \\ &= \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right) dt + \int_0^T u_m(t) v_m'(t) dt \\ &\quad + \int_0^T \frac{a(t)t^k}{q} |u_m'(t)|^q dt, \end{aligned}$$

where $F_\lambda^*(t,v) := \sup_{u\in R}(uv - F_\lambda(t,u))$ for all $(t,v,\lambda) \in (0,T)\times R\times R$.

On the other hand, for all $u \in U$, we have the estimate

$$\begin{aligned} \min = \inf_{u\in\widetilde{U}} J(u) &\leq \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt - \int_0^T t^k F_\lambda(t,u(t))dt \\ &\leq \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt + \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right) dt - \int_0^T u'(t) v_m(t) dt. \end{aligned}$$

Therefore one sees

$$\begin{aligned} (1.14) \qquad \min &\leq \inf_{u\in\widetilde{U}} \left[\int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt + \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right) dt \right. \\ &\quad \left. - \int_0^T u'(t)v_m(t)dt\right] = \int_0^T t^k F_\lambda^*\left(t, -\frac{v_m'(t)}{t^k}\right) dt \\ &\quad - \sup_{u\in\widetilde{U}}\left[\int_0^T u'(t)v_m(t)dt - \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt\right] \end{aligned}$$

for all $m \in N$. Now, by formula (1.11) and the properties of U we have

$$\begin{aligned} \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} dt &= \int_0^T \overline{u}_m'(t) v_m(t) dt - \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt \\ &\leq \sup_{u\in\widetilde{U}}\left[\int_0^T u'(t)v_m(t)dt - \int_0^T \frac{a(t)t^k}{q}|u'(t)|^q dt\right] \\ &\leq \sup_{z\in L^2(0,T)}\left[\int_0^T z(t)v_m(t)dt - \int_0^T \frac{a(t)t^k}{q}|z(t)|^q dt\right] \\ &= \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} dt, \end{aligned}$$

which implies

$$(1.15) \qquad \sup_{u\in\widetilde{U}}\left[\int_0^T u'(t)v_m(t)dt - \int_0^T \frac{t^k}{2}|u'(t)|dt\right] = \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} dt,$$

that for all $m \in N$. Consequently, (1.14) yields that

$$(1.16) \quad \min \leq \int_0^T t^k F_\lambda^*(t, -\frac{v_m'(t)}{t^k})dt - \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} dt, \text{ for all } m \in N.$$

Combining (1.13) and (1.16) we obtain the estimate

$$
\begin{aligned}
0 &\leq \left(\int_0^T \frac{a(t)t^k}{q} |u_m'(t)|^q dt + \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} dt - \int_0^T u_m'(t) v_m(t) dt \right) \\
&= \left\{ \int_0^T \frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} dt - \int_0^T t^k F_\lambda^*(t, -\frac{v_m'(t)}{t^k}) dt \right\} \\
&+ \left\{ \int_0^T \frac{a(t)t^k}{q} |u'(t)|^q dt + \int_0^T u_m(t) v_m'(t) dt + \int_0^T t^k F_\lambda^*(t, -\frac{v_m'(t)}{t^k}) dt \right\} \\
&\leq - \min + \min + \varepsilon = \varepsilon,
\end{aligned}
$$

for all $m \geq m_0$. Since $\varepsilon > 0$ was arbitrary, we get (1.8). □

Theorem 1.4. *If (f1)–(f4) hold, then problem* (1.1)–(1.2) *possesses at least one solution* $\overline{u} \in U$ *which is a minimizer of* $J : U \to R$.

Proof. We start our proof with the observation that for $a \in R$ large enough the set $S_a := \{u \in U, J(u) \leq a\}$ is nonempty. Let $(u_m)_{m\in N} \subset S_a$ be a minimizing sequence of $J : U \to R$. Taking into account the estimate (1.9), we see that $(t^{k/q} u_m')_{m\in N}$ is bounded in the $L^q(0,T)$-norm, and further $(\left(t^k u_m\right)')_{m\in N}$ is bounded in the $L^q(0,T)$-norm. Thus, going if necessary to a subsequence, $(t^k u_m)_{m\in N}$ is weakly convergent in $W_0^{1,q}(0,T)$ to a certain $\widetilde{z} \in W_0^{1,q}(0,T)$ and, as a consequence, it is uniformly convergent in $[0,T]$. Moreover $(u_m)_{m\in N}$ is bounded in $L^q(0,T)$ so up to a subsequence, $(u_m)_{m\in N}$ tends weakly to a certain $\overline{u} \in L^q(0,T)$. Therefore $\widetilde{z}(t) = t^k \overline{u}(t)$ and further $\overline{u}$ is continuous in $(0,T]$ and $0 \leq \overline{u} \leq d$ in $(0,T]$. Now we show that $\overline{u}' < 0$ and $\overline{u} \in C^1([0,T])$. To this end we see, by Theorem 1.3, that there exists a sequence $(v_m)_{m\in N} \subset W^{1,q'}(0,T)$ such that

$$
-v_m'(t) = t^k \left(f_1(t, u_m(t)) + \lambda \overline{f_2}(t, u_m(t)) \right), \text{ for a.e. } t \in (0,T), \tag{1.17}
$$

and such that

$$
\lim_{m\to\infty} \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} + \frac{a(t)t^k}{q} |u_m'(t)|^q - u_m'(t) v_m(t) \right) dt = 0. \tag{1.18}
$$

Assertion (1.17) leads to the conclusion that $(v_m'/t^k)_{m\in N}$ and $(v_m')_{m\in N}$ are bounded in the $L^{q'}(0,T)$ norm, which implies the weak convergence (up to subsequences) of $(v_m')_{m\in N}$ and $(v_m'/t^k)_{m\in N}$ in $L^{q'}(0,T)$. By (1.18) we can deduce also the boundedness of $(v_m)_{m\in N}$ in $L^{q'}(0,T)$. Finally, going if necessary to a subsequence, $(v_m)_{m\in N}$ is weakly convergent in $W^{1,q'}(0,T)$ to $\overline{v} \in W^{1,q'}(0,T)$. Therefore $(v_m)_{m\in N}$ tends uniformly to $\overline{v}$ in $[0,T]$. Since for all $m \in N$, v_m is continuous and nonpositive, we obtain the continuity and positivity of $\overline{v}$. Our task is now to prove that

$$
\overline{v}'(t) = -t^k (f_1(t, \overline{u}(t)) + \lambda \overline{f_2}(t, \overline{u}(t))) \text{ a.e. in } (0,T) \tag{1.19}
$$

$$
\overline{v}(t) = t^k a(t) \left| \overline{u}'(t) \right|^{q-2} \overline{u}'(t) \text{ a.e. in } (0,T). \tag{1.20}
$$

To this end one notes, by (1.17) and the properties of $(u_m)_{m\in N}$ and $(v'_m)_{m\in N}$,

$$0 \geq \liminf_{m\to\infty} \int_0^T \left(v'_m(t)u_m(t) + t^k F^*_\lambda \left(t, -\frac{v'_m(t)}{t^k}\right) + t^k F_\lambda(t, u_m(t))\right) dt$$
$$\geq \int_0^T \left(\overline{v}'(t)\overline{u}(t) + t^k F^*_\lambda \left(t, -\frac{\overline{v}'(t)}{t^k}\right) + t^k F_\lambda(t, \overline{u}(t))\right) dt \geq 0,$$

where the last inequality is due to the properties of the Fenchel conjugate. Thus we get

(1.21) $$\overline{v}'(t) = -t^k \left(f_1(t,\overline{u}(t)) + \lambda f_2(t,\overline{u}(t))\right) \text{ a.e. in } (0,T).$$

On the other hand, (1.18) gives

$$0 \geq \liminf_{m\to\infty} \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |v_m(t)|^{q'} + \frac{a(t)t^k}{q} |u'_m(t)|^q - u'_m(t)v_m(t)\right) dt$$
$$\geq \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} |\overline{v}(t)|^{q'} + \frac{a(t)t^k}{q} |\overline{u}'(t)|^q - \overline{u}'(t)\overline{v}(t)\right) dt \geq 0.$$

Consequently, applying again the properties of the Fenchel transform, we get

(1.22) $$\overline{v}(t) = t^k a(t) \left|\overline{u}'(t)\right|^{q-2} \overline{u}'(t) \text{ a.e. in } (0,T).$$

Summarizing, assertions (1.21) and (1.22) give

(1.23) $$(t^k a(t) \left|\overline{u}'(t)\right|^{q-2} \overline{u}'(t))' = -t^k f_1(t,\overline{u}(t)) + \lambda f_2(t,\overline{u}(t)) \text{ for a. a. } t \in (0,T)$$

which can be rewritten as (1.1)–(1.2). Moreover it is clear that Lemmas 1.1 and 1.2 yield $\overline{u} \in C^1([0,T])$, $\overline{u}'(0) = 0$, $\overline{u}(T) = 0$, $\overline{u}' < 0$ a.e. in $[0,T]$, $t^k a(t) \left|\overline{u}'(t)\right|^{q-2} \overline{u}'(t) \in L^{q'}(0,T)$. Finally $\overline{u} \in U$.

Finally, by the uniform convergence of $(u_m)_{m\in N}$ to $\overline{u}$ and the weak convergence of $(t^{k/q}u'_m)_{m\in N}$ in $L^q(0,T)$ to $t^{k/q}\overline{u}'$, one gets

$$\inf_{u\in U} J(u) = \liminf_{m\to\infty} \int_0^T t^k \left(-F_\lambda(t,u_m(t)) + \frac{a(t)}{q} |u'_m(t)|^q\right) dt$$
$$\geq \int_0^T t^k \left(-F_\lambda(t,\overline{u}(t)) + \frac{a(t)}{q} |\overline{u}'(t)|^q\right) dt = J(\overline{u}).$$

□

2. **Stability of solutions**

In this section we shall investigate the dependence on functional parameters. Let us consider the set $W \times Z \subset L^{p_1}(0,T) \times L^{p_2}(0,T)$, where $p_1, p_2 > 2$. We start with assumptions which guarantee that for each pair $(w,z) \in W \times Z$ there exists at least one positive and decreasing solution of (1.5). For this we assume f1p *$f_1 : [0,T] \times I \times R \to R$, $f_2 : [0,T] \times I \times R \to R$ are Caratheodory functions, λ is real number R such that for almost all $t \in [0,T]$ and all $u \in I$, $(x,y) \in R^2$*

$$f_1(t,u,x) + \lambda f_2(t,y) \geq 0;$$

f2p *there exists positive $d \in I$ such that for each $(w,z) \in W \times Z$, $u \mapsto f_1(t,u,w(t)) + \lambda f_2(t,u,z(t))$ is increasing in I for a.a. $t \in [0,T]$, and*

$$\max_{u\in[0,d]} \left(f_1(\cdot,u,w(\cdot)) + \lambda f_2(\cdot,u,z(\cdot))\right) \in L^{q'}(0,T),$$

with $q' = \frac{q}{q-1}$, and *$t \mapsto f_1(t,0,w(t)) + \lambda f_2(t,0,z(t))$ is not identically zero in a certain subset of $[0,T]$ with positive measure.*

f4p *for each $(w,z) \in W \times Z$*

$$\int_0^T \left(\frac{1}{a(s)s^k}\int_0^s r^k \left(f_1(r,d,w(r)) + \lambda f_2(r,d,z(r))dr\right)\right)^{\frac{1}{q-1}} ds \leq d.$$

f5p *there exists $M > 0$ such that for each $(w,z) \in W \times Z$*

$$\int_0^T t^k \max_{u\in[0,d]}[f_1(t,u,w(t)) + \lambda f_2(t,u,z(t))]dt \leq M.$$

Theorem 2.1. *Suppose that (f1p), (f2p), (f4p), (f5p) and (f3) hold. Consider the sequence of parameters $(w_m,z_m)_{m\in N} \in W \times Z$such that for each $m \in N$, we denote by $u_m \in U$ a solution of (1.5). If $(w_m,z_m)_{m\in N}$ tends a.e. in $[0,T]$ to (w_0,z_0), then the sequence of solutions $(u_m)_{m\in N}$ tends uniformly (up to a subsequence) to a certain $u_0 \in U$ being a solution of (1.5) for parameters (w_0,z_0).*

Proof. By the previous theorem for each pair $(w_m,z_m)_{m\in N} \in W \times Z$ there exists a solution $u_m \in U$ for problem (1.5), namely

$$-\left(a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right)' = t^k f_1(t,u_m(t),w_m(t)) + \lambda f_2(t,u_m(t),z_m(t)). \tag{2.1}$$

Thus we have

$$\begin{aligned}
&\int_0^T t^k \left|u_m'(t)\right|^q dt \\
&\quad\leq \frac{1}{a_{\min}}\int_0^T a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)u_m'(t)dt \\
&\quad= \frac{1}{a_{\min}}\int_0^T a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)u_m'(t)dt \\
&\quad= \frac{1}{a_{\min}}\left(\left[u(t)a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right]_0^T \right. \\
&\qquad\left. -\int_0^T \left(a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right)' u_m(t)dt\right) \\
&\quad= \frac{1}{a_{\min}}\int_0^T -\left(a(t)t^k \left|u_m'(t)\right|^{q-2} u_m'(t)\right)' u_m(t)dt
\end{aligned}$$

$$= \frac{1}{a_{\min}} \int_0^T t^k \max_{u\in[0,d]} [f_1(t,u,w_m(t)) + \lambda f_2(t,u,z_m(t))]dt \le \frac{M}{a_{\min}}.$$

Therefore we see that $(t^{k/q}u'_m)_{m\in N}$ is bounded in the $L^q(0,T)$-norm, and further $((t^k u_m)')_{m\in N}$ is bounded in the $L^q(0,T)$-norm. Now, employing a reasoning similar to that in the proof of Theorem 1.4, we infer that $(t^k u_m)_{m\in N}$ tends weakly (up to a subsequence) in $W_0^{1,q}(0,T)$ to a certain $x_0 \in W_0^{1,q}(0,T)$. Consequently, it is uniformly convergent in $[0,T]$. On the other hand $(u_m)_{m\in N}$ is bounded in $L^q(0,T)$ so up to a subsequence, $(u_m)_{m\in N}$ is weakly convergent to a certain $u_0 \in L^q(0,T)$. Therefore $x_0(t) = t^k u_0(t)$ and further u_0 is continuous in $(0,T]$ and $0 \le u_0 \le d$ in $(0,T]$. Now we prove that $u'_0 < 0$ and $u_0 \in C^1([0,T])$. For this we consider the sequence

$$v_m(t) = t^k a(t) \left|u'_m(t)\right|^{q-2} u'_m(t) \text{ a.e. in } (0,T).$$

By (2.1),

$$-v'_m(t) = t^k f_1(t,u_m(t),w_m(t)) + \lambda f_2(t,u_m(t),z_m(t)) \text{ a.e. in } (0,T). \tag{2.2}$$

The above assertions and the properties of the sequence $(u_m)_{m\in N}$ guarantee that $(v_m)_{m\in N}$ is bounded in $W^{q'}(0,T)$ and further, it is weakly convergent (up to a subsequence) to $v_0 \in W^{q'}(0,T)$. Finally $(v_m)_{m\in N}$ is uniformly convergent to v_0 in $[0,T]$. Since each $v_m(t) < 0$ we see that $v_0(t) \le 0$ in $(0,T)$ and $v_0 \in C([0,T])$. Moreover we have

$$\begin{aligned} 0 &= \liminf_{m\to\infty} \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} \left|v_m(t)\right|^{q'} \right. \\ &\quad \left. + \frac{a(t)t^k}{q} \left|u'_m(t)\right|^q - u'_m(t)v_m(t) \right) dt \\ &\ge \int_0^T \left(\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}} \left|v_0(t)\right|^{q'} + \frac{a(t)t^k}{q} \left|u'_0(t)\right|^q - u'_0(t)v_0(t) \right) dt \ge 0. \end{aligned} \tag{2.3}$$

We now show

$$\begin{aligned} 0 &= \liminf_{m\to\infty} \int_0^T \left(v'_m(t)u_m(t) + t^k F_\lambda^* \left(t, -\frac{v'_m(t)}{t^k}, w_m(t), z_m(t)\right) \right. \\ &\quad \left. + t^k F_\lambda(t,u_m(t),w_m(t),z_m(t)) \right) dt \\ &\ge \int_0^T \left(v'_0(t)u_0(t) + t^k F_\lambda^* \left(t, -\frac{v'_0(t)}{t^k}, w_0(t), z_0(t)\right) \right. \\ &\quad \left. + t^k F_\lambda(t,u_0(t),w_0(t),z_0(t)) \right) dt \ge 0, \end{aligned} \tag{2.4}$$

where for almost all $t \in [0,T]$ and all $u \in I$, $(x,y) \in R^2$ and $v^* \in R$,

$$F_\lambda(t,u,x,y) := \int_0^u (\overline{f_1}(t,l,x) + \lambda \overline{f_2}(t,l,y))dl,$$

$$F_\lambda^*(t,v^*,x,y) := \sup_{u\in R}\left(uv^* - F_\lambda(t,u,x,y)\right)$$

with

$$\overline{f}_1(t,u,x) = \begin{cases} f_1(t,u,x) \text{ if } u\in[0,d_1],\ t\in[0,T] \\ +\infty \text{ if } u\in R\backslash[0,d_1],\ t\in[0,T] \end{cases}$$

$$\overline{f}_2(t,u,y) = \begin{cases} f_2(t,u,y) \text{ if } u\in[0,d_1],\ t\in[0,T] \\ +\infty \text{ if } u\in R\backslash[0,d_1],\ t\in[0,T]. \end{cases}$$

For this we note that (2.2), convexity of F_λ with respect to the second variable and definition of F_λ yield

$$-\frac{v_m'(t)}{t^k} \in \partial_u F_\lambda(t,u_m(t),w_m(t),z_m(t)))$$

for a.a. $t\in(0,T)$ and all $m\in N$, where $\partial_u F_\lambda$ is the subdifferential of F_λ with respect to the second variable:

$$\partial_u F_\lambda(t,u,x,y) := \{v^*\in R, F_\lambda(t,v,x,y) \geq F_\lambda(t,u,x,y) + v^*\,(v-u) \quad \text{for all } v\in R\}.$$

Now applying the Fenchel equality for the function $F_\lambda(t,\cdot,x,y)$ we get

$$(v_m'(t)u_m(t) + t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right) + t^k F_\lambda(t,u_m(t),w_m(t),z_m(t))) = 0$$

for a.a. $t\in(0,T)$ and all $m\in N$. Thus

$$\lim_{m\to\infty}\int_0^T \left(v_m'(t)u_m(t) + t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)\right.$$
$$\left. + t^k F_\lambda(t,u_m(t),w_m(t),z_m(t))\right)dt = 0.$$

On the other hand, by the assumptions on F_λ and properties of the sequences, we know that

$$\lim_{m\to\infty}\int_0^T v_m'(t)u_m(t)dt = \int_0^T v_0'(t)u_0(t)dt \tag{2.5}$$

and

$$\lim_{m\to\infty}\int_0^T t^k F_\lambda(t,u_m(t),w_m(t),z_m(t))dt = \int_0^T t^k F_\lambda(t,u_0(t),w_0(t),z_0(t))dt. \tag{2.6}$$

Therefore, we infer the existence of the following limit

$$\lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt.$$

Now we note that for all $u\in L^q(0,T)$, $m\in N$ and a.e. $t\in[0,T]$ one has

$$\begin{aligned} & -v_m'(t))u(t) - t^k F_\lambda(t,u(t),w_m(t),z_m(t)) \\ \leq\ & \sup_{r\in R}\left\{-v_m'(t)r - t^k F_\lambda(t,r,w_m(t),z_m(t))\right\} \\ =\ & t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right) \end{aligned}$$

and further

$$\lim_{m\to\infty}\int_0^T \left(-v_m'(t))u(t) - t^k F_\lambda(t,u(t),w_m(t),z_m(t))\right)dt$$
$$\leq \lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt.$$

Combining (2.5), (2.6) and the previous inequality we derive

$$\int_0^T (-v_0'(t))u(t) - t^k F_\lambda(t,u(t),w_0(t),z_0(t)))dt$$
$$\leq \lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt$$

for all $u\in L^q(0,T)$. Consequently

$$\sup_{u\in L^q(0,T)}\left\{\int_0^T \left(-v_0'(t))u(t) - t^k F_\lambda(t,u(t),w_0(t),z_0(t))\right)dt\right\}$$
$$\leq \lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt.$$

Since

$$\int_0^T t^k F_\lambda^*\left(t,-\frac{v_0'(t)}{t^k},w_0(t),z_0(t)\right)dt$$
$$=\sup_{u\in L^q(0,T)}\left\{\int_0^T \left(-v_0'(t))u(t) - t^k F_\lambda(t,u(t),w_0(t),z_0(t))\right)dt\right\},$$

we have

$$\int_0^T t^k F_\lambda^*\left(t,-\frac{v_0'(t)}{t^k},w_0(t),z_0(t)\right)dt \leq \lim_{m\to\infty}\int_0^T t^k F_\lambda^*\left(t,-\frac{v_m'(t)}{t^k},w_m(t),z_m(t)\right)dt. \tag{2.7}$$

Taking into account (2.5), (2.6) and (2.7) we get (2.4).

Assertions (2.3) and (2.4) give

$$\frac{1}{q'(t^k a(t))^{\frac{q'}{q}}}\left|v_0(t)\right|^{q'} + \frac{a(t)t^k}{q}\left|u_0'(t)\right|^q - u_0'(t)v_0(t)) = 0 \text{ a.e in } (0,T)$$

and

$$\left(v_0'(t)u_0(t) + t^k F_\lambda^*\left(t,-\frac{v'(t)}{t^k}\right) + t^k F_\lambda(t,u_0(t),w_0(t),z_0(t))\right) = 0 \text{ a.e in } (0,T).$$

Consequently, by the properties of the Fenchel transform,

$$v_0(t) = t^k a(t)\left|u_0'(t)\right|^{q-2}u_0'(t) \text{ a.e. in } (0,T). \tag{2.8}$$

By (2.1),

$$-v_0'(t) = t^k\left(f_1(t,u_0(t),w_0(t)) + \lambda f_2(t,u_m(t),z_0(t))\right) \text{ a.e. in } (0,T). \tag{2.9}$$

Thus

$$-\left(t^k a(t)\,|u_0'(t)|^{q-2}\,u_0'(t)\right)' = t^k\left(f_1(t,u_0(t),w_0(t)) + \lambda f_2(t,u_m(t),z_0(t))\right) \tag{2.10}$$

a.e. in $(0,T)$. Note that $u_0(T)=0$, $0 \le u_0 \le d$ in $(0,T]$ and u_0 is continuous in $(0,T]$. By (2.8), the continuity of v_0 in $[0,T]$ implies that $u_0 \in C^1((0,T])$. Further (2.8), (2.9) and assumption (f2p) imply $t^k a(t)\,|u_0'(t)|^{q-2}\,u_0' \in A([0,T])$. Now it suffices to show that $u'(t)<0$ for $t\in[0,T]$ and u_0 is continuous at 0. From (2.10) we have the estimates (as in the proof of Lemma 1.2)

$$\begin{aligned}
|u_0'(t)|^{q-1} &= \frac{1}{a(t)t^k}\int_0^t r^k f_1(r,u_0(r)) + \lambda f_2(r,u_0(r))dr \\
&\le \frac{1}{a(t)t^k}\left(\frac{1}{qk+1}\right)^{1/q} t^{k+1/q}\left(\int_0^T (\overline{f_1}(l,d)+\lambda\overline{f_2}(t,d))^{q'}dl\right)^{1/q'} \\
&\le \frac{1}{a_{\min}}\left(\frac{1}{qk+1}\right)^{1/q}\left(\int_0^T (\overline{f_1}(l,d)+\lambda\overline{f_2}(t,d))^{q'}dl\right)^{1/q'} t^{1/q}.
\end{aligned}$$

Finally

$$\lim_{t\to 0^+} u_0'(t) = 0 = u_0'(0).$$

Now (2.10) and Lemma 1.1 lead to the conclusion that $u_0'(t)<0$ for $t\in(0,T)$. Thus $u_0 \in U$. □

Example 2.2. For $\lambda \in (6.935, 8.366)$ and all $(w,z)\in L^{p_1}(0,T)\times L^{p_2}(0,T)$, with $p_1,p_2>2$, the BVP

(2.11)

$$\begin{cases}
-\left(\left(\frac{1}{1+t^2}\,|u'(t)|^2\,u'(t)\right)' + \frac{k}{(1+t^2)t}\,|u'(t)|^2\,u'(t)\right) \\
= \frac{1}{810\sqrt{t}}\left(-u^4(t)\,(1+arctg^2 w(t)) + \lambda\,(u^2(t)+1)\left(1+\sin^2 z(t)\right)\right) \text{ a.e. in } (0,3), \\
u'(0)=0 \text{ and } u(3)=0.
\end{cases}$$

possesses at least one positive solution in the set $U := \{u \in \widetilde{U}\ ;\ u(t)\le 1 \textit{ for all } t\in[0,3]\}$, with

$$\begin{aligned}
\widetilde{U} &= \Big\{u\in C^1([0,T]) : u(3)=0 \text{ and } u'(0)=0 \text{ and } u'(t)<0 \\
&\qquad \text{for } t\in[0,3] \text{ and } \frac{t^k}{1+t^2}\,|u'|^2\,u' \in A([0,T])\Big\}.
\end{aligned} \tag{2.12}$$

Moreover if for each $m\in N$, $u_m\in U$ denotes the solution of (2.11) for (w_m,z_m) and if $(w_m,z_m)_{m\in N}$ tends a.e. in $[0,T]$ to (w_0,z_0), then the sequence of solutions $(u_m)_{m\in N}$ tends uniformly (up to a subsequence) to a certain $u_0\in U$ being a solution of (2.11) for parameters (w_0,z_0).

Proof. We consider (1.5) with $T = 3$, $q = 4$, $a(t) = \frac{1}{1+t^2}$ and

$$\begin{aligned} f_1(t,u,x) &= -\frac{1}{810}\frac{1+arctg^2 x}{\sqrt{t}}u^4; \\ f_2(t,u,y) &= \frac{1}{810}\frac{1+\sin^2 y}{\sqrt{t}}\left(u^2+1\right). \end{aligned}$$

We show that all the ssumptions of Theorem 2.1 are satisfied in this case. First we look for λ such that

$$\begin{aligned} f(\ t,u,x,y) \ &: = f_1(t,u,x) + \lambda f_2(\ t,u,y) \\ &= \frac{1}{810\sqrt{t}}\left(-u^4\left(1+arctg^2 x\right) + \lambda\left(u^2+1\right)\left(1+\sin^2 y\right)\right) \end{aligned}$$

is increasing in $[0,1]$. Note

$$f'_u(t,u,x,y) = \frac{1}{810\sqrt{t}}\left(-4u^3\left(1+arctg^2 x\right) + 2\lambda u\left(1+\sin^2 y\right)\right)$$

and further

$$f'_u(t,u,x,y) = 0 \Leftrightarrow -4u^3\left(1+arctg^2 x\right) + 2\lambda u\left(1+\sin^2 y\right) = 0$$

which gives

$$u_0 = 0 \text{ or } u_1 = \sqrt{\frac{\lambda}{2}}\sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}} \text{ or } u_2 = -\sqrt{\frac{\lambda}{2}}\sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}}.$$

Thus for a.a. $t \in (0,T)$ and all $x, y \in R$

$$\begin{aligned} f'_u(t,u,x,y) &> 0 \text{ for } u \in (-\infty,u_2) \cup (0,u_1) \\ f'_u(t,u,x,y) &< 0 \text{ for } u \in (u_2,0) \end{aligned}$$

which implies that for a.a. $t \in (0,T)$ and $x, y \in R$ the function $f(t,\cdot,x,y)$ is increasing for $u \in (0,u_1)$. Moreover , since $f(t,0,x,y) = \frac{\lambda}{810\sqrt{t}}\left(1+\sin^2 y\right) > 0$, one sees that $f(t,u,x,y) > 0$ for $u \in (0,u_1)$. Thus we obtain

$$1 \le \sqrt{\frac{\lambda}{2}}\sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}}$$

for a.a. $t \in (0,T)$ and all $x, y \in R$. Note that

$$\sqrt{\frac{1}{1+\pi^2/4}} \le \sqrt{\frac{1+\sin^2 y}{1+arctg^2 x}} \le \sqrt{2}$$

for all $x, y \in R$. We take λ such that

$$1 \le \sqrt{\frac{\lambda}{2}}\sqrt{\frac{1}{1+\pi^2/4}},$$

namely

$$\lambda \ge \frac{4+\pi^2}{2} \approx 6.9348. \qquad (2.13)$$

We also look for λ such that

(2.14)
$$\int_0^3 \left(\frac{1}{a(s)s^k} \int_0^s r^k \frac{1}{810\sqrt{r}} \left(-d^4 \left(1 + arctg^2 x\right) + \lambda \left(d^2+1\right)\left(1+\sin^2 y\right)\right) dr \right)^{\frac{1}{3}} ds \le d$$

with $d = 1$. It is easy to see that

$$\begin{aligned}
&\int_0^3 \left(\frac{1}{a(s)s^k} \int_0^s r^k \frac{1}{810\sqrt{r}} \left(-\left(1 + arctg^2 x\right) + 2\lambda \left(1+\sin^2 y\right)\right) dr \right)^{\frac{1}{3}} ds \\
\le\ &\int_0^3 \left(\frac{1}{810} \frac{1}{1+s^2} \int_0^s \frac{1}{\sqrt{r}} \left(- \left(1 + \frac{\pi^2}{4}\right) + 4\lambda \right) dr \right)^{\frac{1}{3}} ds \\
=\ &\int_0^3 \left(\frac{1}{810} \frac{\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)}{1+s^2} \int_0^s \frac{dr}{\sqrt{r}} \right)^{\frac{1}{3}} ds \\
=\ &\sqrt[3]{\frac{1}{810}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)} \int_0^3 \left(\frac{2\sqrt{s}}{1+s^2} \right)^{\frac{1}{3}} ds \\
\le\ &2.7\sqrt[3]{\frac{1}{810}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)} \le \sqrt[3]{\frac{1}{30}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)}.
\end{aligned}$$

Therefore (2.14) holds if

$$\sqrt[3]{\frac{1}{30}\left(4\lambda - \left(1+\frac{\pi^2}{4}\right)\right)} \le 1$$

which is equivalent to

$$\lambda \le \frac{31}{4} + \frac{\pi^2}{16} \approx 8.3669. \tag{2.15}$$

Summarizing, for λ satisfying (2.13) and (2.15) all the assumtions of Theorem 2.1 hold. □

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