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ON OCCURRENCE OF COMPLETE BLOW-UP OF THE SOLUTION FOR A DEGENERATE SEMILINEAR PARABOLIC PROBLEM WITH INSULATED BOUNDARY CONDITIONS

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ABSTRACT. Let a, σ, p, q, r, and m be constants with $a > 0$, $\sigma > 0$, $p \geq 0$, $q \geq 0$, $r > 1$, and $m > 0$. This article studies the following degenerate semilinear parabolic initial-boundary value problem,

$$\xi^q u_\tau - u_{\xi\xi} = \xi^p u^r \text{ for } 0 < \xi < a,\ 0 < \tau < \sigma,$$
$$u(\xi, 0) = u_0(\xi) = m \text{ for } 0 \leq \xi \leq a,$$
$$u_\xi(0, \tau) = 0 = u_\xi(a, \tau) \text{ for } \tau > 0.$$

We derive criteria for u to blow up in finite time, and estimate the blow-up rate. We show that the blow-up is regional if $q > p$; the blow-up is complete if $q = p$; and the blow-up cannot be complete if $p > q$.

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1. Introduction

Let a, σ, p, q, r, and m be constants with $a > 0$, $\sigma > 0$, $p \geq 0$, $q \geq 0$, $r > 1$, and $m > 0$. We consider the following degenerate semilinear parabolic initial-boundary value problem,

$$\xi^q u_\tau - u_{\xi\xi} = \xi^p u^r \text{ for } 0 < \xi < a,\ 0 < \tau < \sigma,$$
$$u(\xi, 0) = u_0(\xi) = m \text{ for } 0 \leq \xi \leq a,$$
$$u_\xi(0, \tau) = 0 = u_\xi(a, \tau) \text{ for } 0 < \tau < \sigma.$$

Let $\xi = ax$, $\tau = a^{q+2}t$, $D = (0, 1)$, $\Omega = D \times (0, T)$, $\bar{D}$ and $\bar{\Omega}$ be the closures of D and Ω respectively, and $Lu = x^q u_t - u_{xx}$. The above problem is transformed into

$$(1.1) \qquad \begin{cases} Lu = a^{p+2} x^p u^r \text{ in } \Omega, \\ u(x, 0) = u_0(x) = m > 0 \text{ on } \bar{D}, \\ u_x(0, t) = 0 = u_x(1, t),\ 0 < t < T, \end{cases}$$

where $T = \sigma / a^{q+2} < \infty$.

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A solution u of the problem (1.1) is said to blow up at the point $(\bar{x}, t_b)$ if there exists a sequence $\{(x_n, t_n)\}$ such that $u(x_n, t_n) \to \infty$ as $(x_n, t_n) \to (\bar{x}, t_b)$. The blow-up of u is complete if u blows up at every point $x \in \bar{D}$ at $t = t_b$. The blow-up of u is regional in the case $q > p$, if u blows up at every point $x \in [0, b_1]$ at $t = t_b$, where $b_1 < 1$.

Chan and Dyakevich [1] investigated the blow-up set of the solution for the degenerate semilinear parabolic equation $Lu = a^2 f(u)$ subject to the mixed boundary conditions $u(0,t) = 0 = u_x(1,t)$. Dyakevich [3] studied quenching of the solution for the problem (1.1) with $m = 0$ and with $a^{p+2}x^p u^r$ replaced by the function $x^p f(u)$ satisfying $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c. It was shown that constants p and q determine whether the solution quenches completely, or at one of the boundary points $x = 0$ or $x = 1$. In this article, we investigate the influence of the constants p and q on the blow-up set of the solution u of the problem (1.1).

In Section 2, we discuss existence of a unique classical solution. In Section 3, we investigate the conditions for u to blow up in a finite time t_b, and give an estimate for the blow-up rate. In Section 4, we show that the blow-up is regional if $q > p$, and complete if $q = p$. In Section 5, we show that the blow-up cannot be complete if $p > q$.

2. **Existence of a Unique Classical Solution**

Let $D_\varepsilon = (\varepsilon, 1)$, $\bar{D}_\varepsilon = [\varepsilon, 1]$, $\Omega_\varepsilon = D_\varepsilon \times (0, T)$, where $0 \le \varepsilon < \frac{1}{2}$. We notice that if $\varepsilon = 0$, then $D_\varepsilon = D$. The proof of the following comparison lemma is similar to the proof of Lemma 2.1 in Dyakevich [3].

Lemma 2.1. *For any fixed $\bar{t} \in (0, T)$, and any bounded and nontrivial function $B(x,t)$ on $\bar{D}_\varepsilon \times [0, \bar{t}]$, if*

$$\left.\begin{array}{c} (L - x^p B)\, u \ge 0 \text{ in } D_\varepsilon \times (0, \bar{t}], \\ u(x,0) \ge 0,\ x \in \bar{D}_\varepsilon, \\ u_x(\varepsilon, t) \le 0,\ u(b,t) \ge 0,\ t \in [0, \bar{t}], \end{array}\right\}$$

then $u \ge 0$ on $\bar{D}_\varepsilon \times [0, \bar{t}]$.

Following the idea in the proof of Lemma 1 in Chan and Kaper [2], we have the following result.

Lemma 2.2. *The problem* (1.1) *has at most one solution u. This solution has the following properties: (i). $u > m$ in $\bar{D} \times (0, T)$; (ii). u is a strictly increasing function of t for all $x \in \bar{D}$.*

Proof. Let u_1 and u_2 be two distinct solutions of the problem (1.1) and let $y = u_1 - u_2$. Uniqueness of u follows directly from Lemma 2.1 of [3].

(i). Let $y = u - m$. Because $f(m) = m^r > 0$ and $x^p m^r > 0$ for any $x \in D$, we have:

$$\left.\begin{array}{c} x^q u_t - u_{xx} - a^{p+2} x^p f(u) + a^{p+2} x^p f(m) = x^q y_t - y_{xx} - a^{p+2} x^p r \eta^{r-1} y > 0 \text{ in } \Omega, \\ y(x,0) = 0 \text{ on } \bar{D}, \\ y_x(0,t) = 0 = y_x(1,t)\,,\ 0 < t < T, \end{array}\right\}$$

for some η between u and m. By Lemma 2.1 of [3], $y \geq 0$. By the strong maximum principle [4, p. 39], if $y = 0$ at some point $(x_2, t_2) \in (0,1) \times (0,T)$, then $y = 0$ in $(0,1) \times (0,t_2]$. This contradicts to

$$0 = x^q y_t - y_{xx} - a^{p+2} x^p r \eta^{r-1} y > 0 \text{ in } (0,1) \times (0,t_2].$$

Therefore, $y > 0$ at any point in $(0,1)$. Suppose y attains its minimum value zero at $x = 0$ or $x = 1$. By the parabolic version of Hopf's Lemma [4, p. 49], $y_x(0,t) > 0$ and $y_x(1,t) < 0$. This contradiction shows that $y > 0$ on $\bar{D}$.

(ii). The proof of this result is identical to the proof of Lemma 2.2 (ii) in Dyakevich [3, p. 894]. □

We modify the proof of Lemma 2.3 in Dyakevich [3, p.895] to prove the following result.

Lemma 2.3. *There exists some positive constant* t_0 $(< T)$ *such that the problem* (1.1) *has an upper solution* $\mu(x,t) \in C^{2,1}([0,1] \times [0,t_0])$.

Proof. We consider the problem,

$$\text{(2.1)} \qquad \left.\begin{array}{l} Lu_\varepsilon = a^{p+2} x^p u_\varepsilon^r \text{ in } D_\varepsilon \times (0,t_0]\,, \\ u_\varepsilon(x,0) = m \text{ on } \bar{D}_\varepsilon, \\ u_{\varepsilon_x}(\varepsilon,t) = 0 = u_{\varepsilon_x}(1,t) \text{ for } 0 < t \leq t_0. \end{array}\right\}$$

Let $\hat{m} > 1$, $0 < \gamma < \frac{1}{2}$, and $K > \hat{m}$ be chosen such that

$$a^{p+2}(\hat{m} - 1) \geq u_0(x) = m,$$
$$\hat{m}^r \left(a^{p+2}\right)^r < K,$$
$$\hat{m} - 1 < -(K/2)\gamma^2 - \gamma + \hat{m} < \hat{m},$$
$$K^r \left(a^{p+2}\right)^r > 1.$$

Let us construct an upper solution $\mu(x,t) \in C^{2,1}(\bar{D} \times [0,t_0])$ for all u_ε, where $\varepsilon < \gamma$. Let

$$\theta(x) = \begin{cases} -\frac{K}{2}x^2 - x + \hat{m}, & 0 \leq x \leq \gamma, \\ \hat{h}(x)\,, & \gamma < x < 1-\gamma, \\ -\frac{K}{2}(1-x)^2 - (1-x) + \hat{m}, & 1-\gamma \leq x \leq 1, \end{cases}$$

where $\hat{h}(x)$ is a positive C^∞ function chosen such that $\theta(x)$ is in $C^2\left(\bar{D}\right)$ and $\hat{m} - 1 < \hat{h}(x) \leq \hat{m}$. We note that $\theta'(x) < 0$ for $0 \leq x \leq \gamma$ and $\theta'(0) = -1 < 0$,

$\theta'(\gamma) = -K\gamma - 1 < 0$ and $\theta'(1) = 1 > 0$. Also, $\max_{0\le x\le\gamma}\theta(x) = \hat{m}$ and $\min_{0\le x\le\gamma}\theta(x) = -(K/2)\gamma^2 - \gamma + \hat{m} > \hat{m} - 1$.

There exists some t_1 such that the initial-value problem,

$$\tau'(t) = \frac{\left(1 + \max_{\gamma\le x\le 1}|\theta''|\right) a^{p+2}K^r\tau^r}{\gamma^q\left(\min_{\gamma\le x\le 1}\theta\right)},\ \tau(0) = a^{p+2},$$

has a unique solution for $0 \le t \le t_1$. Let us choose some constant t_0 in $(0, t_1]$ such that

$$\begin{aligned} \hat{m}^r\tau^r(t_0) &\le K, \\ \tau(t_0) &\le a^{p+2}K^r(\tau(0))^r \le a^{p+2}K^r\tau^r. \end{aligned}$$

Let $\mu(x,t) = \theta(x)\tau(t)$. For any $x \in [0,\gamma]$ and $t \in (0, t_0]$, $x^q\theta\tau' \ge 0$ and $\theta''(x) = -K < 0$. Therefore,

$$\begin{aligned} L\mu - a^{p+2}x^p\mu^r &= x^q\theta\tau' - \tau\theta'' - a^{p+2}x^p\theta^r\tau^r \\ &\ge \tau(0)K - a^{p+2}\theta^r(0)\tau^r(t_0)) \\ &= a^{p+2}\left[K - \hat{m}^r\tau^r(t_0))\right] \\ &\ge 0. \end{aligned}$$

We have for $x \in (\gamma, 1]$,

$$\begin{aligned} L\mu - a^{p+2}x^p\mu^r &\ge \gamma^q\left(\min_{\gamma\le x\le 1}\theta\right)\tau'(t) - \tau(t_0)\left(\max_{\gamma\le x\le 1}|\theta''|\right) - a^{p+2}\theta^r\tau^r \\ &\ge \gamma^q\left(\min_{\gamma\le x\le 1}\theta\right)\tau'(t) - a^{p+2}K^r\tau^r\left(\max_{\gamma\le x\le 1}|\theta''|\right) - a^{p+2}K^r\tau^r \\ &\ge \gamma^q\left(\min_{\gamma\le x\le 1}\theta\right)\left(\tau'(t) - \frac{\left(1 + \max_{\gamma\le x\le 1}|\theta''|\right)a^{p+2}K^r\tau^r}{\gamma^q\left(\min_{\gamma\le x\le 1}\theta\right)}\right) \\ &= 0. \end{aligned}$$

We also have $\mu(x,0) = a^{p+2}\theta(x) \ge a^{p+2}(\hat{m} - 1) \ge u_0(x) = m$, $\mu_x(0,t) = \theta_x(0)\tau(t) < 0$, $\mu_x(1,t) = \theta_x(1)\tau(t) > 0$ and $\mu(x,t) \in C^{2,1}(\bar{D}\times[0,t_0])$. The function $y = \mu - u_\varepsilon$ satisfies

$$\left.\begin{aligned} Ly - x^p r\vartheta^r y \ge 0 \text{ in } D_\varepsilon \times (0, t_0], \\ y(0) > 0,\ x \in \bar{D}_\varepsilon, \\ y_x(\varepsilon, t) < 0,\ y_x(1,t) > 0,\ t \in [0, t_0], \end{aligned}\right\}$$

where ϑ is between μ and u_ε for all $\varepsilon < \gamma$. By Lemma 2.1 in Dyakevich [3, p. 896–898], $y = \mu - u_\varepsilon \ge 0$. □

The proofs of the following two results can be found in Dyakevich [3, p. 896–898].

Lemma 2.4. *Let $0 < \varepsilon_1 < \varepsilon_2 < \gamma$ and suppose that u_{ε_1} and u_{ε_2} are solutions of the problem* (2.1) *on $(0, t_0)$. If $p < q$, then $u_{\varepsilon_x} < 0$ and $u_{\varepsilon_1} > u_{\varepsilon_2}$ in Ω_{ε_2}. If $p > q$, then $u_{\varepsilon_x} > 0$ and $u_{\varepsilon_1} < u_{\varepsilon_2}$ in Ω_{ε_2}.*

Theorem 2.5. *The problem* (1.1) *has a classical solution $C\left(\bar{D}\right) \cap C^{2,1}((0,1] \times [0,t_0])$.*

We modify the proof of Theorem 2.6 in Dyakevich [3, p. 898] to obtain the following continuation theorem.

Theorem 2.6. *Let T be the supremum over t_0 for which the problem* (1.1) *has a unique solution $u(x,t) \in C(\bar{D}) \cap C^{2,1}((0,1] \times [0,t_0])$. Then, there is a unique solution $u(x,t) \in C(\bar{D} \times [0,T)) \cap C^{2,1}((0,1] \times [0,T))$. If $T < \infty$, then u is unbounded in Ω.*

Proof. Let us suppose that u is bounded above by some positive constant $M > 1/\left(2a^{p+2}\right)$ in Ω. We would like to show that u can be continued into a time interval $[0, T + \tilde{t}_0]$ for some positive $\tilde{t}_0$. Let a positive constant K^* be such that $1 < \left(2Ma^{p+2}\right)^r < K^*$ and a positive constant $\tilde{\gamma}$ is such that $-\frac{K^*}{2}\tilde{\gamma}^2 - \tilde{\gamma} + 2M > M$. Let

$$\tilde{\theta}_1(x) = \begin{cases} -\frac{K^*}{2}x^2 - x + 2M, & 0 \le x \le \tilde{\gamma}, \\ \tilde{h}\left(x\right), \tilde{\gamma} < x < 1 - \tilde{\gamma} & \\ -\frac{K^*}{2}\left(1-x\right)^2 - \left(1-x\right) + 2M, & 1 - \tilde{\gamma} \le x \le 1, \end{cases}$$

where $\tilde{h}\left(x\right)$ is a positive C^∞ function chosen such that $\tilde{\theta}_1(x)$ is in $C^2\left(\bar{D}\right)$ and $M < \tilde{h}\left(x\right) \le 2M$. By construction, $\tilde{\theta}_1\left(x\right) > M \ge u\left(x,t\right) \ge u_0\left(x\right) = m$ for any $t \le T$. Also, we notice that $\tilde{\theta}_{1_x}\left(0\right) < 0 = u_x\left(0,t\right)$, and $\tilde{\theta}_1'\left(1\right) > 0 = u_x\left(1,t\right)$ for $t > 0$.

With $\tilde{\theta}_1(x)$ as the initial function at T, we are to construct an upper solution $\tilde{\mu}\left(x,t\right)$ of $u\left(x,t\right)$ on $\bar{D} \times [T, T + \tilde{t}_0]$ for some positive $\tilde{t}_0$. There exists some t_2 such that the initial-value problem,

$$\tilde{\tau}_1'\left(t - T\right) = \frac{a^{p+2}\left(2M\right)^r\left(\max\limits_{\tilde{\gamma} \le x \le 1}\left|\tilde{\theta}_1''\right| + 1\right)\tilde{\tau}_1^r\left(t - T\right)}{\tilde{\gamma}^q \min\limits_{\tilde{\gamma} \le x \le 1}\tilde{\theta}_1}, \; \tilde{\tau}_1(T - T) = a^{p+2},$$

has a unique solution $\tilde{\tau}_1(t - T)$ for $T \le t \le T + \tilde{t}_2$. Let $\tilde{\mu}(x,t) = \tilde{\theta}_1(x)\tilde{\tau}_1(t - T)$, and $\tilde{t}_0$ be chosen such that $0 < \tilde{t}_0 \le \tilde{t}_2$ and

$$\begin{aligned} \left(2M\right)^r \tilde{\tau}_1^r\left(\tilde{t}_0\right) &\le K^*, \\ \tilde{\tau}_1\left(\tilde{t}_0\right) &\le \left(2M\right)^r a^{p+2}\tilde{\tau}_1^r\left(t - T\right). \end{aligned}$$

Since $x^q\tilde{\theta}_1\tilde{\tau}_1'\left(t\right) \ge 0$, and $\tilde{\theta}_1''\left(x\right) = -K^*$, we obtain for any $x \in (0, \tilde{\gamma}]$ and $t \in \left[T, T + \tilde{t}_0\right]$,

$$L\tilde{\mu} - a^{p+2}x^p\tilde{\mu}^r \ge K^*\tilde{\tau}_1 - a^{p+2}\tilde{\theta}_1^r\tilde{\tau}_1^r \ge a^{p+2}\left(K^* - \left(2M\right)^r\tilde{\tau}_1^r\left(\tilde{t}_0\right)\right) \ge 0.$$

It follows from $\tilde{\tau}_1(t-T) \geq a^{p+2}$ for $t \in \left[T, T+\tilde{t}_0\right]$ that for $x \in (\tilde{\gamma}, 1]$ and $t \in \left[T, T+\tilde{t}_0\right]$,

$$\begin{aligned}
L\tilde{\mu} - a^{p+2}x^p\tilde{\mu}^r &\geq \tilde{\gamma}^q \left(\min_{\tilde{\gamma}\leq x\leq 1} \tilde{\theta}_1\right) \tilde{\tau}_1'(t-T) - \tilde{\tau}_1(t-T)\left(\max_{\tilde{\gamma}\leq x\leq 1}\left|\tilde{\theta}_1''\right|\right) \\
&- a^{p+2}\tilde{\theta}_1^r\tilde{\tau}_1^r(t-T) \\
&\geq \tilde{\gamma}^q \left(\min_{\tilde{\gamma}\leq x\leq 1} \tilde{\theta}_1\right) \tilde{\tau}_1'(t-T) - a^{p+2}(2M)^r\tilde{\tau}_1^r(t-T)\left(\max_{\tilde{\gamma}\leq x\leq 1}\left|\tilde{\theta}_1''\right|\right) \\
&- a^{p+2}(2M)^r\tilde{\tau}_1^r(t-T) \\
&\geq \tilde{\gamma}^q \min_{\tilde{\gamma}\leq x\leq 1}\tilde{\theta}_1\left(\tilde{\tau}_1'(t-T) - \frac{a^{p+2}(2M)^r\tilde{\tau}_1^r(t-T)\left(\max_{\tilde{\gamma}\leq x\leq 1}\left|\tilde{\theta}_1''\right|+1\right)}{\tilde{\gamma}^q\min_{\tilde{\gamma}\leq x\leq 1}\tilde{\theta}_1}\right) \\
&= 0.
\end{aligned}$$

By Lemma 2.1 of [3], $\tilde{\mu}(x,t)$ is an upper solution of u on $\bar{D}\times\left[T, T+\tilde{t}_0\right]$. As in Lemma 2.4 and Theorem 2.5, we can show that the problem (1.1) has a unique solution $u(x,t) \in C(\bar{D}\times\left[0, T+\tilde{t}_0\right]) \cap C^{2,1}((0,1]\times\left[0, T+\tilde{t}_0\right])$. This contradicts the definition of T. □

3. Occurrence of Blow-up and Blow-up Rate Estimate

Theorem 3.1. *Let $q \geq p$ and $r > 1$. Then there exists some*

$$t_b \leq 1/\left(m^{r-1}a^{p+2}(r-1)\right) < \infty \tag{3.1}$$

such that

$$\lim_{t\to t_b^-}\max_{x\in\bar{D}} u(x,t) = \infty.$$

Proof. Let $\tau(t)$ satisfy

$$\tau'(t) = a^{p+2}\tau^r(t), \quad \tau(0) = m > 0.$$

Then

$$\tau(t) = \left[\frac{1}{m^{1-r} - a^{p+2}(r-1)t}\right]^{\frac{1}{r-1}} \text{ for } 0 \leq t < \hat{t}_b,$$

where

$$\hat{t}_b = \frac{1}{m^{r-1}a^{p+2}(r-1)}.$$

We have for $x \in (0,1)$ and $t \in \left(0, \hat{t}_b\right)$,

$$x^q\tau' - \tau_{xx} - a^{p+2}x^p\tau^r \leq x^q\left(\tau' - a^{p+2}\tau^r\right) = 0.$$

Since τ does not depend on x, we have $\tau_x(0) = \tau_x(1) = 0$, $\tau_{xx}(t) = 0$, and $\tau(0) = m$. Therefore, $\tau(t)$ is the lower solution that blows up at $\hat{t}_b$. We notice that if

$q = p$, then $\tau(t)$ is the unique solution of the problem (1.1) which blows up at $t_b = 1/\left(m^{r-1}a^{p+2}(r-1)\right)$ and the blow-up set is $\bar{D}$. □

Theorem 3.2. *Let $q < p$ and $r > 1$. If $u_0(x) = m > 0$ is sufficiently large, then there exists some $t_b < \infty$ such that*

$$\lim_{t\to t_b^-}\max_{x\in\bar{D}} u(x,t) = \infty.$$

Proof. Let us choose positive constants α, β, γ and ω as follows:

$$\beta > \max\{p-q,\, p+2\},$$
$$\alpha > 2,\ \omega > 0,$$
$$\gamma > \max\left\{2, \tfrac{p+2}{2}\right\}.$$

Let positive constant $\tilde{K}$ satisfy the following:

$$\tilde{K} > \left[\frac{1+\beta(\beta-1)\omega+\alpha(\alpha-1)\gamma^2}{a^{p+2}(r-1)}\right]^{\frac{1}{r-1}}.$$

Let

$$\phi(x,t) = \frac{\tilde{K}}{D^{1/(r-1)}},$$

where

$$D(x,t) = x^\beta(\omega-t) + (1-x^\gamma)^\alpha.$$

We have:

$$\phi_t(x,t) = \frac{\tilde{K}}{(r-1)}D^{-r/(r-1)}x^\beta,$$

$$\phi_x(x,t) = -\frac{\tilde{K}}{(r-1)}D^{-r/(r-1)}\left[\beta(\omega-t)x^{\beta-1} - \alpha(1-x^\gamma)^{\alpha-1}\gamma x^{\gamma-1}\right]$$

$$\begin{aligned}\phi_{xx}(x,t) = {} & \frac{\tilde{K}r}{(r-1)^2}D^{(-2r+1)/(r-1)}\left[(\omega-t)\beta x^{\beta-1} - \alpha\gamma(1-x^\gamma)^{\alpha-1}x^{\gamma-1}\right]^2 \\ & - \frac{\tilde{K}}{(r-1)}D^{-r/(r-1)}\left\{(\omega-t)\beta(\beta-1)x^{\beta-2}\right. \\ & \left. +\alpha(\alpha-1)\gamma^2(1-x^\gamma)^{\alpha-2}x^{2\gamma-2} - \alpha\gamma(\gamma-1)(1-x^\gamma)^{\alpha-1}x^{\gamma-2}\right\}\end{aligned}$$

Therefore, for $x \in (0,1)$ and $t \in (0,\omega)$,

$$\begin{aligned} & L\phi - a^{p+2}x^p\phi^r \\ & \le x^q\frac{\tilde{K}}{(r-1)}D^{-r/(r-1)}x^\beta \\ & + \frac{\tilde{K}}{(r-1)}D^{-r/(r-1)}\left\{\beta(\omega-t)(\beta-1)x^{\beta-2} + \alpha(\alpha-1)\gamma^2(1-x^\gamma)^{\alpha-2}x^{2\gamma-2}\right\} \\ & - a^{p+2}x^p\tilde{K}^rD^{-r/(r-1)} \\ & \le \frac{\tilde{K}}{(r-1)}D^{-r/(r-1)}[x^{q+\beta} + \beta(\omega-t)(\beta-1)x^{\beta-2} + \alpha(\alpha-1)\gamma^2x^{2\gamma-2}\end{aligned}$$

$$
\begin{aligned}
&- a^{p+2}x^p\tilde{K}^{r-1}(r-1)] \\
&\leq \frac{\tilde{K}}{(r-1)}D^{-r/(r-1)}x^p\left[1+\beta(\beta-1)\omega+\alpha(\alpha-1)\gamma^2-a^{p+2}\tilde{K}^{r-1}(r-1)\right] \\
&\leq 0.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
\phi_x(0,t) &= 0, \\
\phi_x(1,t) &= -\frac{\tilde{K}\beta(\omega-t)}{(r-1)((\omega-t))^{r/(r-1)}} \\
&= -\frac{\tilde{K}\beta}{(r-1)(\omega-t)^{1/(r-1)}} < 0,\ 0 \leq t < \omega, \\
\phi(x,0) &= \frac{\tilde{K}}{(x^\beta\omega+(1-x^\gamma)^\alpha)^{1/(r-1)}} > 0,\ 0 \leq x \leq 1.
\end{aligned}
$$

If $u_0(x) = m \geq \phi(x,0)$, then by Lemma 2.1 in [3, p. 893], $\phi(x,t)$ is a lower solution for the problem (1.1), which blows up at $t=\omega$. We notice that the blow-up set of the function $\phi(x,t)$ consists of only one point $x=1$. □

Below we estimate the blow-up rate using similar method as in the proof of Theorem 2.2 in Wang and Chen [5, p. 317].

Theorem 3.3. *If the solution $u(x,t)$ of the problem* (1.1) *blows up at $t=t_b$, then there exists positive constant $\check{K}$ such that*

$$u(x,t) \leq \check{K}(t_b-t)^{-\frac{1}{r-1}},\ \textit{in } D\times(0,t_b).$$

Proof. Let

$$J(x,t) = u_t(x,t) - \hat{k}a^{p+2}u^r(x,t),$$

where the positive constant $\hat{k}$ will be determined later. We have:

$$
\begin{aligned}
J_t(x,t) &= u_{tt} - \hat{k}a^{p+2}ru^{r-1}u_t, \\
J_x(x,t) &= u_{tx} - \hat{k}a^{p+2}ru^{r-1}u_x, \\
J_{xx}(x,t) &= u_{txx} - \hat{k}a^{p+2}r(r-1)u^{r-2}(u_x)^2 - \hat{k}a^{p+2}ru^{r-1}u_{xx}.
\end{aligned}
$$

If we differentiate both sides of (1.1) with respect to t, then we get:

$$x^q u_{tt} - u_{xxt} = a^{p+2}x^p ru^{r-1}u_t.$$

Therefore,

$$
\begin{aligned}
&x^qJ_t - J_{xx} \\
&= x^qu_{tt} - \hat{k}a^{p+2}x^qru^{r-1}u_t - u_{txx} \\
&+ \hat{k}a^{p+2}r(r-1)u^{r-2}(u_x)^2 + \hat{k}a^{p+2}ru^{r-1}u_{xx}
\end{aligned}
$$

$$= a^{p+2}x^p r u^{r-1}u_t - \hat{k}a^{p+2}x^q r u^{r-1}u_t$$
$$+ \hat{k}a^{p+2}r\,(r-1)\,u^{r-2}\,(u_x)^2 + \hat{k}a^{p+2}r u^{r-1}u_{xx}$$
$$= a^{p+2}x^p r u^{r-1}u_t - \hat{k}a^{p+2}r u^{r-1}\,(x^q u_t - u_{xx})$$
$$+ \hat{k}a^{p+2}r\,(r-1)\,u^{r-2}\,(u_x)^2$$
$$= a^{p+2}x^p r u^{r-1}u_t - \hat{k}a^{p+2}r u^{r-1}a^{p+2}x^p u^r$$
$$+ \hat{k}a^{p+2}r\,(r-1)\,u^{r-2}\,(u_x)^2$$
$$= a^{p+2}x^p r u^{r-1}\left(u_t - \hat{k}a^{p+2}u^r\right) + \hat{k}a^{p+2}r\,(r-1)\,u^{r-2}\,(u_x)^2$$
$$= a^{p+2}x^p r u^{r-1}J\,(x,t) + \hat{k}a^{p+2}r\,(r-1)\,u^{r-2}\,(u_x)^2\,.$$

The function J satisfies the following:

$$x^q J_t - J_{xx} - a^{p+2}x^p r u^{r-1}J = \hat{k}a^{p+2}r\,(r-1)\,u^{r-2}\,(u_x)^2 > 0 \text{ in } D\times(0,t_b)\,,$$
$$J_x(0,t) = u_{tx}(0,t) - \hat{k}a^{p+2}\left(ru^{r-1}\right)u_x(0,t) = 0, \text{ for } 0<t<t_b,$$
$$J_x(1,t) = u_{tx}(1,t) - \hat{k}a^{p+2}\left(ru^{r-1}\right)u_x(1,t) = 0, \text{ for } 0<t<t_b.$$

We know from Lemma 2.2 that $u_t > 0$ on $\bar{D}\times[0,t_b)$. Therefore, there exists a positive constant k_1 such that $u_t(x,0) \geq k_1 > 0$ for $x \in [0,1]$. Let $\hat{k}$ be a positive constant such that $\hat{k} \leq \min\{k_1/\left(a^{p+2}m^r\right),\,1\}$ and

$$J(x,0) = u_t(x,0) - \hat{k}a^{p+2}u^r(x,0) \geq k_1 - \hat{k}a^{p+2}m^r \geq 0.$$

Therefore, by Lemma 2.1 of [3], $J(x,t) = u_t(x,t) - \hat{k}a^{p+2}u^r(x,t) \geq 0$ on $\bar{D}\times[0,t_b)$. Integrating $u^{-r}(x,t)u_t(x,t) \geq \hat{k}a^{p+2}$ from $t\,(\geq 0)$ to t_b, we obtain:

$$u(x,t) \leq \left[\frac{1}{\hat{k}\,(r-1)\,a^{p+2}\,(t_b-t)}\right]^{\frac{1}{r-1}} \leq \check{K}\,(t_b-t)^{-\frac{1}{r-1}}\,, \tag{3.2}$$

where $\check{K} \geq \left[\hat{k}\,(r-1)\,a^{p+2}\right]^{-\frac{1}{r-1}}$. □

4. Regional/Complete Blow-up when $q \geq p$

In this section we assume that the solution u of the problem (1.1) blows up and that the blow-up time t_b is a fixed given number corresponding to the given initial function $u_0\,(x) = m > 0$. We would like to investigate the blow-up set. We proved in Theorem 3.1 that if $q = p$, then the blow-up set is $\bar{D}$. Let $0 < \delta < 1$ be an arbitrary constant. We choose

$$\hat{\varepsilon} > 1 - \frac{(1-\delta)}{\sqrt{2q+3}}$$

and observe that $\delta < \hat{\varepsilon} < 1$. Also, let $\varkappa$ be a positive constants such that

$$\varkappa < \frac{1}{\hat{\varepsilon}^{q-p}} - 1. \tag{4.1}$$

We define

$$f(x) = (4q+6)\,x^2 - 2\,(4q+6)\,x - 2\delta^2 + 4\delta + 4q + 4, \tag{4.2}$$

$$0 < B \le \min\left\{\frac{\varkappa}{(q+2)\,2^q f(\delta)}, \frac{1}{(q+2)\,(\hat{\varepsilon}-\delta)^q\,(1-\delta)^q\,|f(\hat{\varepsilon})|}\right\}, \tag{4.3}$$

and

$$\begin{aligned} 0 &< R \\ &= \max\left\{\delta^{q-p}, (1+\varkappa)\,\hat{\varepsilon}^{q-p}, 1 - B\,(q+2)\,(\hat{\varepsilon}-\delta)^q\,(1-\delta)^q\,|f(\hat{\varepsilon})|\right\} \\ &< 1. \end{aligned} \tag{4.4}$$

Theorem 4.1. *Let $p < q$ and $r > 1$. If*

$$m^{r-1} t_b \ge \frac{R}{a^{p+2}\,(r-1)}, \tag{4.5}$$

then the blow-up set for the solution of (1.1) *is* $[0,\delta]$.

Proof. Let

$$\theta(x) = \begin{cases} B\,(x-\delta)^{q+2}\,(2-\delta-x)^{q+2}, & \text{for } \delta \le x \le 1, \\ 0, & \text{for } 0 \le x \le \delta, \end{cases}$$

where the positive constant B is defined in (4.3). From

$$\begin{aligned} \theta'(x) &= B\,(q+2)\,(x-\delta)^{q+1}\,(2-\delta-x)^{q+2} - B\,(x-\delta)^{q+2}\,(q+2)\,(2-\delta-x)^{q+1} \\ &= 2B\,(q+2)\,(x-\delta)^{q+1}\,(2-\delta-x)^{q+1}\,[1-x], \end{aligned}$$

we conclude that $\theta'(x) > 0$ for $\delta < x < 1$. Also,

$$\begin{aligned} \theta''(x) &= B\,(q+2)\,(q+1)\,(x-\delta)^q\,(2-\delta-x)^{q+2} \\ &\quad - 2B\,(q+2)\,(x-\delta)^{q+1}\,(q+2)\,(2-\delta-x)^{q+1} \\ &\quad + B\,(q+2)\,(q+1)\,(x-\delta)^{q+2}\,(2-\delta-x)^q \\ &= B\,(q+2)\,(x-\delta)^q\,(2-\delta-x)^q\,f(x), \end{aligned}$$

where

$$\begin{aligned} f(x) &= (q+1)\,(2-\delta-x)^2 - 2\,(q+2)\,(x-\delta)\,(2-\delta-x) + (q+1)\,(x-\delta)^2 \\ &= (4q+6)\,x^2 - 2\,(4q+6)\,x - 2\delta^2 + 4\delta + 4q + 4. \end{aligned}$$

This quadratic function $f(x)$ has one zero on the interval $\delta \le x \le 1$ at

$$z = 1 - \frac{(1-\delta)}{\sqrt{2q+3}}.$$

Also, $f(x)$ has its vertex at the point $\left(1, -2\,(1-\delta)^2\right)$. The following is true about $\theta''(x)$ on the interval $0 \le x \le 1$:

$$\theta''(x) = 0 \text{ for } 0 \le x \le \delta, \text{ and at } x = z,$$

$$\theta''(x) > 0 \text{ for } \delta < x < z,$$
$$\theta''(x) < 0 \text{ for } z < x < 1.$$

From (4.1), (4.2) and (4.3) we have on the interval $[\delta, \hat{\varepsilon}]$:

$$\begin{aligned} \frac{\theta''(x)}{x^q} &\le \frac{B\,(q+2)\,(x-\delta)^q\,(2-\delta-x)^q\,f(\delta)}{x^q} \\ &\le B\,(q+2)\,2^q f(\delta) \\ &\le \varkappa. \end{aligned} \tag{4.6}$$

Using (4.5), we choose a positive constant E such that

$$\frac{R}{a^{p+2}\,(r-1)} \le E^{r-1} \le m^{r-1} t_b, \tag{4.7}$$

with R defined in (4.4). Let $\tau(x,t)$ be a $C^{2,1}\left([0,1]\times[0,t_b)\right)$ function as follows:

$$\tau(x,t) = \frac{E}{\left((t_b - t) + \theta(x)\right)^{\frac{1}{r-1}}} = ED^{-\frac{1}{r-1}},$$

where $D\,(x,t) = (t_b - t) + \theta(x)$. From (4.5) and (3.1), the blow-up time satisfies the following:

$$\frac{R}{m^{r-1}a^{p+2}\,(r-1)} \le t_b \le \frac{1}{m^{r-1}a^{p+2}\,(r-1)}. \tag{4.8}$$

We have:

$$\begin{aligned} \tau_t(x,t) &= \frac{ED^{-\frac{r}{r-1}}}{(r-1)}, \\ \tau_x(x,t) &= -\frac{ED^{-\frac{r}{r-1}}\theta'(x)}{(r-1)}, \\ \tau_{xx}(x,t) &= \frac{rE\,(\theta'(x))^2\,D^{\frac{-2r+1}{r-1}}}{(r-1)^2} - \frac{ED^{-\frac{r}{r-1}}\theta''(x)}{(r-1)}. \end{aligned}$$

Therefore, using (4.7) and $\theta''(x) = 0$ for $0 \le x \le \delta$, we have for $0 \le x \le \delta$:

$$\begin{aligned} &x^q\tau_t(x,t) - \tau_{xx}(x,t) - a^{p+2}x^p\tau^r(x,t) \\ &= \frac{x^qED^{-\frac{r}{r-1}}}{(r-1)} - \frac{rE\,(\theta'(x))^2\,D^{\frac{-2r+1}{r-1}}}{(r-1)^2} + \frac{ED^{-\frac{r}{r-1}}\theta''(x)}{(r-1)} - a^{p+2}x^pE^rD^{-\frac{r}{r-1}} \\ &= \frac{x^qED^{-\frac{r}{r-1}}}{(r-1)}\left[1 - \frac{r\,(\theta'(x))^2}{(r-1)\,x^qD} + \frac{\theta''(x)}{x^q} - \frac{a^{p+2}\,(r-1)\,E^{r-1}}{x^{q-p}}\right] \\ &\le \frac{x^qED^{-\frac{r}{r-1}}}{(r-1)}\left[1 - \frac{a^{p+2}\,(r-1)\,E^{r-1}}{\delta^{q-p}}\right] \le 0. \end{aligned}$$

Using (4.6) and (4.7), we have for $\delta \le x \le \hat{\varepsilon}$:

$$x^q\tau_t(x,t) - \tau_{xx}(x,t) - a^{p+2}x^p\tau^r(x,t)$$

$$= \frac{x^q E D^{-\frac{r}{r-1}}}{(r-1)} \left[1 - \frac{r\left(\theta'(x)\right)^2}{(r-1)\, x^q D} + \frac{\theta''(x)}{x^q} - \frac{a^{p+2}(r-1)\, E^{r-1}}{x^{q-p}} \right]$$
$$\leq \frac{x^q E D^{-\frac{r}{r-1}}}{(r-1)} \left[1 + \varkappa - \frac{a^{p+2}(r-1)\, E^{r-1}}{\hat{\varepsilon}^{q-p}} \right] \leq 0.$$

Using (4.3), (4.7) and $\theta''(x) < 0$ for $\hat{\varepsilon} \leq x \leq 1$, we have for $\hat{\varepsilon} \leq x \leq 1$:

$$x^q \tau_t(x,t) - \tau_{xx}(x,t) - a^{p+2} x^p \tau^r(x,t)$$
$$= \frac{x^q E D^{-\frac{r}{r-1}}}{(r-1)} \left[1 - \frac{r\left(\theta'(x)\right)^2}{(r-1)\, x^q D} + \frac{\theta''(x)}{x^q} - \frac{a^{p+2}(r-1)\, E^{r-1}}{x^{q-p}} \right]$$
$$\leq \frac{x^q E D^{-\frac{r}{r-1}}}{(r-1)} \left[1 - \min_{\hat{\varepsilon} \leq x \leq 1} |\theta''(x)| - a^{p+2}(r-1)\, E^{r-1} \right]$$
$$\leq \frac{x^q E D^{-\frac{r}{r-1}}}{(r-1)} \left[1 - B\,(q+2)\,(\hat{\varepsilon}-\delta)^q\,(1-\delta)^q\, |f(\hat{\varepsilon})| - a^{p+2}(r-1)\, E^{r-1} \right] \leq 0.$$

From (4.7) we have:

$$\tau(x,0) = \frac{E}{(t_b + \theta(x))^{\frac{1}{r-1}}} \leq \frac{E}{(t_b)^{\frac{1}{r-1}}} \leq m,$$

and

$$\tau_x(0,t) = -\frac{E D^{-\frac{r}{r-1}} \theta'(0)}{(r-1)} = 0,$$
$$\tau_x(1,t) = -\frac{E D^{-\frac{r}{r-1}} \theta'(1)}{(r-1)} = 0.$$

We conclude that $\tau(x,t)$ is a lower solution that blows up at $t = t_b$ on the interval $[0,\delta]$. Therefore, $u\,(x,t)$ also blows up on $[0,\delta]$ at $t = t_b$. If $R = 1$ in (4.5) and (4.8), that is,

$$t_b = \frac{1}{m^{r-1} a^{p+2}\,(r-1)},$$

then the blow-up is complete. This is exactly what happened in the case $p = q$ in Theorem 3.1. □

5. **No Complete Blow-up when** $q < p$

Below we assume that the solution u of the problem (1.1) blows up under the hypotheses of Theorem 3.2 and that the blow-up time t_b is a fixed given number corresponding to the given initial function $u_0\,(x) = m > 0$. Let

$$k_3 = \left[\frac{1}{a^{p+2}\hat{k}\,(r-1)} + \frac{1}{a^{p+2}\hat{k}\,(r-1)\, t_b} \right]^{\frac{1}{r-1}},$$

where the positive constant $\hat{k}$ is defined in Theorem 3.3. Let us choose positive constants β and $k_4 < 1$ such that the following two conditions are satisfied:

$$\left\{ \begin{array}{c} \beta > q+2, \\ 1 - \left(\frac{4r}{(r-1)} + \frac{2(\beta-1)}{\beta} \right) k_4^{2\beta-q-2} - a^{p+2} k_3^{r-1} (r-1) k_4^{p-q} \geq 0. \end{array} \right. \tag{5.1}$$

We modify the proof of Lemma 4.2 in Chan and Dyakevich [1, p. 614] to prove the following result.

Lemma 5.1. *If $p > q$, then the following estimate holds for the solution of the problem* (1.1)*:*

$$u(x, t_b) \leq \frac{k_3}{\left[\frac{1}{\beta} \left(k_4^\beta - x^\beta \right) \right]^{\frac{2}{r-1}}} < \infty \text{ for } x \in [0, k_4).$$

Proof. Let

$$\Phi(x,t) = \frac{k_3}{D^{1/(r-1)}},$$

where

$$D(x,t) = \frac{1}{\beta^2} \left(k_4^\beta - x^\beta \right)^2 + (t_b - t).$$

We have:

$$\begin{aligned}
\Phi_t(x,t) &= \frac{k_3}{(r-1)} D^{-r/(r-1)}, \\
\Phi_x(x,t) &= \frac{k_3}{(r-1)} D^{-r/(r-1)} \frac{2}{\beta} \left(k_4^\beta - x^\beta \right) x^{\beta-1}, \\
\Phi_{xx}(x,t) &= -\frac{k_3 r}{(r-1)^2} D^{(-2r+1)/(r-1)} \frac{2}{\beta} \left(k_8^\beta - x^\beta \right) x^{\beta-1} \frac{2}{\beta^2} \left(k_8^\beta - x^\beta \right) (-\beta) x^{\beta-1} \\
&\quad + \frac{2k_3}{(r-1)\beta} D^{-r/(r-1)} \left[-\beta x^{\beta-1} x^{\beta-1} \right] \\
&\quad + \frac{2k_3}{(r-1)\beta} D^{-r/(r-1)} \left(k_4^\beta - x^\beta \right) (\beta-1) x^{\beta-2} \\
&= \frac{k_3 r}{(r-1)^2} D^{(-2r+1)/(r-1)} \left[\frac{2}{\beta} \left(k_4^\beta - x^\beta \right) x^{\beta-1} \right]^2 \\
&\quad - \frac{2k_3}{(r-1)} D^{-r/(r-1)} x^{2\beta-2} + \frac{2k_3(\beta-1)}{(r-1)\beta} D^{-r/(r-1)} \left(k_4^\beta - x^\beta \right) x^{\beta-2}.
\end{aligned}$$

Using (5.1), we obtain for any $x \in (0, k_4)$ and $0 < t < t_b$,

$$\begin{aligned}
&L\Phi - a^{p+2} x^p \Phi^r \\
&= \frac{k_3}{(r-1) D^{\frac{r}{r-1}}} \left[x^q - \frac{r}{(r-1) D} \left[\frac{2}{\beta} \left(k_4^\beta - x^\beta \right) x^{\beta-1} \right]^2 \right. \\
&\left. + 2x^{2\beta-2} - \frac{2(\beta-1)}{\beta} \left(k_4^\beta - x^\beta \right) x^{\beta-2} - a^{p+2} x^p k_3^{r-1} (r-1) \right]
\end{aligned}$$

$$\geq \frac{k_3 x^q}{(r-1)\, D^{\frac{r}{r-1}}} \left[1 - \frac{4r}{(r-1)} k_4^{2\beta-q-2} - \frac{2(\beta-1)}{\beta} k_4^{2\beta-q-2} - a^{p+2} k_4^{p-q} k_3^{r-1}(r-1)\right]$$
$$\geq 0.$$

It follows from (3.2), $\beta > 1$ and $0 < k_4 < 1$ that

$$\begin{aligned}\Phi(x,0) &= \frac{k_3}{\left\{t_b + \frac{1}{\beta^2}\left(k_4^\beta - x^\beta\right)^2\right\}^{\frac{1}{r-1}}} \\ &\geq \frac{\left[\frac{1}{a^{p+2}\hat{k}(r-1)} + \frac{1}{a^{p+2}\hat{k}(r-1)t_b}\right]^{\frac{1}{r-1}}}{(t_b+1)^{\frac{1}{r-1}}} \\ &= \left[\frac{1}{\hat{k} a^{p+2}(r-1)\, t_b}\right]^{\frac{1}{r-1}} \\ &\geq u(x,0) \ \text{ on } [0,k_4].\end{aligned}$$

Since

$$\Phi_x(0,t) = 0, \quad \Phi(k_4,t) = \frac{k_3}{(t_b - t)^{\frac{1}{r-1}}},$$

it follows from Lemma 2.1, that $\Phi(x,t)$ is an upper solution of the problem (1.1) for $0 \leq x \leq k_4$. Since $\Phi(x,t)$ is bounded at $t = t_b$ for all $0 \leq x < k_4$, we can conclude that the blow-up cannot be complete in the case $p > q$. □

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