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Ψ-STABILITY OF NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper, we give some sufficient conditions for Ψ-(uniform) stability of the trivial solution of the nonlinear differential systems and of a nonlinear Volterra integro-differential system.

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1. PRELIMINARIES

Akinyele [2] introduced the notion of Ψ-stability of degree k with respect to a function $\Psi \in C(R_+, R_+)$, increasing and differentiable on R_+ and such that $\Psi(t) \geq 1$ for $t \geq 0$ and $\lim_{t\to\infty} \Psi(t) = b$, $b \in [1, \infty)$. The fact that the function Ψ is bounded does not enable a deeper analysis, of the asymptotic properties of the solutions of a differential equations, than the notion of stability in sense Lyapunov.

Constantin [6] introduced the notions of degree of stability and degree of boundedness of solutions of an ordinary differential equation, with respect to a continuous positive and nondecreasing function $\Psi : R_+ \to R_+$. Some criteria for these notions are proved there too.

Morchalo [14] introduced the notions of Ψ-stability, Ψ-uniform stability, and Ψ-asymptotic stability of trivial solution of the nonlinear system $x' = f(t, x)$. Several new and sufficient conditions for mentioned types of stability are proved for the linear system $x' = A(t)x$, in this paper Ψ is a scalar continuous function. Diamandescu [16] give some sufficient conditions for Ψ-(uniform) stability of the nonlinear Volterra

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integro-differential system $x' = A(t)x + \int_0^t F(t,s,x(s))ds$, in this paper Ψ is a matrix function. Furthermore, sufficient conditions are given for the uniform Lipschitz stability of the system $x' = f(t,x) + g(t,x)$. For more results, see [3–5, 7–13, 15, 17, 18] and the references therein.

The purpose of our paper is to prove sufficient conditions for Ψ-(uniform) stability of trivial solution of the nonlinear system

$$y' = f(t,y) + g(t,y) \tag{1.1}$$

and the nonlinear Volterra integro-differential system

$$z' = f(t,z) + \int_0^t F(t,s,z(s))ds, \tag{1.2}$$

which can be seen as perturbed systems of

$$x' = f(t,x) \tag{1.3}$$

or the variational system

$$u' = f_x(t, x(t,t_0,x_0))u \tag{1.4}$$

associated with system (1.3). Where $f, g \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $F \in C(D \times \mathbb{R}^n, \mathbb{R}^n)$, $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t < \infty\}$, and $f(t,0) = g(t,0) = F(t,s,0) = 0$ for $(t,s) \in D$, moreover, $f_x = \partial f/\partial x$ exists and continuous on $\mathbb{R}_+ \times \mathbb{R}^n$, and $x(t,t_0,x_0)$ is the solution of (1.3) with $x(t_0,t_0,x_0) = x_0,\ t_0 \ge 0$. The fundamental matrix solution $\Phi(t,t_0,x_0)$ of (1.4) is given by [7]

$$\Phi(t,t_0,x_0) = \frac{\partial}{\partial x_0}(x(t,t_0,x_0)). \tag{1.5}$$

Using the nonlinear variation of constants formula of Alekseev [1], the solutions of the perturbed systems (1.1) and (1.2) with the same initial values as (1.3) are related by

$$y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^t \Phi(t,s,y(s,t_0,x_o))g(s,y(s,t_0,x_o))ds \tag{1.6}$$

and

$$z(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^t \Phi(t,s,z(s,t_0,x_o)) \int_0^s F(s,u,z(u,t_0,x_o))du\,ds. \tag{1.7}$$

We investigate conditions under which the trivial solutions of systems (1.1), (1.2) or (1.3), (1.4) are Ψ-(uniformly) stable on $\mathbb{R}_+$. Here Ψ is a matrix function whose introduction permits us obtaining a mixed behavior for the components of solutions.

In this paper, the definition of Ψ-(uniform) stability is the same as in [16]. Let $\mathbb{R}^n$ denote the Euclidean n-space. For $x = (x_1, x_2, x_3, \dots, x_n)^T \in \mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ be the norm of x. For an $n \times n$ matrix $A = (a_{ij})$, we define the norm $|A| = \sup_{\|x\| \le 1} \|Ax\|$. Let $\Psi = diag[\Psi_1, \Psi_2, \dots, \Psi_n]$, $\Psi_i \in C(\mathbb{R}_+, (0,\infty))$,

$i = 1, 2, \ldots, n$. For $\Psi_i = 1$, $i = 1, 2, \ldots, n$, we obtain the notions of classical stability and uniform-stability, If we replace Ψ with Ψ^k, $k \in \mathbb{Z}\backslash\{0, 1\}$, we obtain stability and uniform-stability of degree k with respect to Ψ in [6].

2. Ψ-STABILITY OF THE SYSTEMS

Theorem 2.1. *If there exist a continuous function $h(t,s) : D \to (0, \infty)$ and the constants $K > 0,\ M > 0$ such that:*

$$||\Psi(t)f(s,x)|| \leq h(t,s)||\Psi(s)x||, \quad \limsup_{t\to\infty} \int_0^t h(t,s)ds = M$$

and $|\Psi(t)\Psi^{-1}(s)| \leq K$ for $0 \leq s \leq t$, then, the trivial solution of system (1.3) is Ψ-uniformly stable on $\mathbb{R}_+$.

Proof. Since $x(t, t_0, x_0) = x_0 + \int_{t_0}^t f(s, x(s, t_0, x_0))ds$, it follows that:

$$\begin{aligned} ||\Psi(t)x(t,t_0,x_0)|| &\leq ||\Psi(t)x_0|| + \int_{t_0}^t ||\Psi(t)f(s,x(s,t_0,x_0))||ds \\ &\leq ||\Psi(t)\Psi^{-1}(t_0)\Psi(t_0)x_0|| + \int_{t_0}^t h(t,s)||\Psi(s)x(s,t_0,x_0)||ds \\ &\leq K||\Psi(t_0)x_0|| + \int_{t_0}^t h(t,s)||\Psi(s)x(s,t_0,x_0)||ds, \end{aligned}$$

this implies by Lipovan's inequality ([17]) that

$$\|\Psi(t)x(t,t_0,x_0)\| \leq K\|\Psi(t_0)x_0\|e^{\int_{t_0}^t h(t,s)ds} \leq Ke^M\|\Psi(t_0)x_0\|,$$

hence the conclusion of the theorem follows.

Theorem 2.2. *If there exist two continuous functions $h(t,s) : D \to (0,\infty)$, $\omega(u) : \mathbb{R}_+ \to (0, \infty)$ and the constants $K > 0$, $M > 0$ such that:*

$$|\Psi(t)\Psi^{-1}(s)| \leq K \text{ and } ||\Psi(t)f(s,x)|| \leq h(t,s)\omega\big(||\Psi(s)x||\big) \text{ for } 0 \leq s \leq t,$$

where $\psi(u)$ is nondecreasing submultiplicative function and

$$\Omega^{-1}\left[\Omega(K) + \frac{\omega(||\Psi(t_0)x_0||)}{||\Psi(t_0)x_0||}\limsup_{t\to\infty}\int_\theta^t h(t,s)ds\right] = M < \infty \text{ for all } 0 \leq t_0 \leq \theta,$$

where $\Omega(u) = \int_{u_0}^u \frac{1}{\omega(s)}ds$, $u_0 \in (0,\infty)$ and $\Omega(\infty) = \infty$, then, the trivial solution of system (1.3) is Ψ-uniformly stable on $\mathbb{R}_+$.

Proof. Since $x(t, t_0, x_0) = x_0 + \int_{t_0}^t f(s, x(s, t_0, x_0))ds$, it follows that:

$$\begin{aligned} ||\Psi(t)x(t,t_0,x_0)|| &\leq ||\Psi(t)x_0|| + \int_{t_0}^t ||\Psi(t)f(s,x(s,t_0,x_0))||ds \\ &\leq K||\Psi(t_0)x_0|| + \int_{t_0}^t h(t,s)\omega(||\Psi(s)x(s,t_0,x_0)||)ds \end{aligned}$$

or, equivalently,

$$\begin{aligned}\frac{||\Psi(t)x(t,t_0,x_0)||}{||\Psi(t_0)x_0||} &\leq K+\int_{t_0}^{t}\frac{h(t,s)}{||\Psi(t_0)x_0||}\omega\big(||\Psi(t_0)x_0||\frac{||\Psi(s)x(s,t_0,x_0)||}{||\Psi(t_0)x_0||}\big)ds\\ &\leq K+\frac{\omega(||\Psi(t_0)x_0||)}{||\Psi(t_0)x_0||}\int_{t_0}^{t}h(t,s)\omega\big(\frac{||\Psi(s)x(s,t_0,x_0)||}{||\Psi(t_0)x_0||}\big)ds\end{aligned}$$

this implies by Lipovan's inequality that

$$\begin{aligned}\|\Psi(t)x(t,t_0,x_0)\| &\leq \|\Psi(t_0)x_0\|\Omega^{-1}\left[\Omega(K)+\frac{\omega(\|\Psi(t_0)x_0\|)}{\|\Psi(t_0)x_0\|}\int_{t_0}^{t}h(t,s)ds\right]\\ &\leq M\|\Psi(t_0)x_0\|,\end{aligned}$$

hence the conclusion of the Theorem follows.

Remark 2.3. Theorems 2.1, 2.2 are based on the fact $x(t,t_0,x_0)=x_0+\int_{t_0}^{t}f(s,x(s,t_0,x_0))ds$ if $x(t,t_0,x_0)$ is a solution of (1.3) which satisfies $x(t_0,t_0,x_0)=x_0$.

Now we give the conditions for Ψ-(uniform) stability of trivial solution of the linear system (1.4), which can be expressed in terms of the fundamental matrix for (1.4).

Theorem 2.4. *Let $\Phi(t,t_0,x_0)$ be a fundamental matrix of* (1.4). *Then*

(a) *The trivial solution of* (1.4) *is Ψ-stable on $\mathbb{R}_+$ if and only if there exists a function $k:\mathbb{R}_+\to(0,\infty)$ such that $|\Psi(t)\Phi(t,t_0,x_0)|\leq k(t_0)$ for $t\geq t_0$ and for $||\Psi(t_0)x_0||$ sufficiently small.*

(b) *The trivial solution of* (1.4) *is Ψ-uniformly stable on $\mathbb{R}_+$ if and only if there exists a positive constant K such that $|\Psi(t)\Phi(t,s,x_0)\Psi^{-1}(s)|\leq K$ for all $0\leq s\leq t$ and for $||\Psi(s)x_0||$ sufficiently small.*

Proof. Let $u(t,t_0,x_0)=\Phi(t,t_0,x_0)x_0$ is the unique solution of (1.4) satisfying $u_0=u(t_0,t_0,x_0)=x_0$.

Suppose first that the trivial solution of (1.4) is Ψ-stable on R+. Then, for $\varepsilon=1$ and $t_0\in\mathbb{R}_+$. There exists $\delta>0$ such that any solution $u(t,t_0,x_0)$ of (1.4) which satisfies $||\Psi(t_0)u_0||=||\Psi(t_0)x_0||<\delta$, there exists and satisfies

$$||\Psi(t)u(t,t_0,x_0)||=||\Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0)\Psi(t_0)x_0||<1 \text{ for } t\geq t_0.$$

Let $v\in\mathbb{R}^n$ be such that $||v||\leq 1$. If we take $x_0=\frac{\delta}{2}\Psi^{-1}(t_0)v$, then $||\Psi(t_0)x_0||<\delta$. Hence, $||\Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0)\frac{\delta}{2}v||<1$ for $t\geq t_0$. Therefore, $|\Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0)|\leq\frac{2}{\delta}$, it is equivalently that $|\Psi(t)\Phi(t,t_0,x_0)|\leq\frac{2}{\delta}|\Psi(t_0)|:=k(t_0)$ for $t\geq t_0$.

Suppose next that there exists a function $k:\mathbb{R}_+\to(0,\infty)$ such that $|\Psi(t)\Phi(t,t_0,x_0)|\leq k(t_0)$ for $t\geq t_0$. For $\varepsilon>0$ and $t_0\in\mathbb{R}_+$, let $\delta(\varepsilon,t_0)=\varepsilon k^{-1}(t_0)|\Psi^{-1}(t_0)|^{-1}$. For $||\Psi(t_0)u_0||=||\Psi(t_0)x_0||<\delta$ and $t\geq t_0$, we have

$$||\Psi(t)u(t,t_0,x_0)||=||\Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0)\Psi(t_0)x_0||<\varepsilon.$$

Thus, the trivial solution of (1.4) is Ψ-stable on R_+. Part (b) is proved similarly and omit its proof. The proof is complete.

Remark 2.5. We generalize Diamandescu's result [16] from linear case to nonlinear case. In the Ψ-stability of [16], our positive function k has reduced to a positive constant K. In fact, the fundamental matrix solution $\Phi(t, t_0, x_0)$ of a linear system is independent of x_0, moreover, $\Phi(t, t_0) = Y(t)Y^{-1}(t_0)$, then we can give the conditions for Ψ-(uniform) stability of linear case in terms of $Y(t)$.

Theorem 2.6. *Let $\Phi(t, t_0, x_0)$ be a fundamental matrix of* (1.4). *If there exist two constants $p \geq 1$, $M > 0$, and a continuous function $\varphi : \mathbb{R}_+ \to (0, \infty)$, a continuous matrix function $\Phi_1(t)$ defined on $\mathbb{R}_+$, such that:*

$$|\Psi(t)\Phi(t, s, x)\Psi^{-1}(s)| \leq |\Psi(t)\Phi_1(t)\Phi_1^{-1}(s)\Psi^{-1}(s)| \text{ for } 0 \leq s \leq t \text{ and for all } x \in \mathbb{R}^n.$$

Then, the trivial solution of system (1.4) *is Ψ-stable on $\mathbb{R}_+$ if one of the following conditions satisfied:*

(i) $\int_0^t \varphi(s)|\Psi(t)\Phi_1(t)\Phi_1^{-1}(s)\Psi^{-1}(s)|^p ds \leq M$, *for all* $t \geq 0$;
(ii) $\int_0^t \varphi(s)|\Phi_1^{-1}(s)\Psi^{-1}(s)\Psi(t)\Phi_1(t)|^p ds \leq M$, *for all* $t \geq 0$.

Proof. Following the proof of Diamandescu [16] Theorem 3.3, we get $|\Psi(t)\Phi_1(t)| \leq K_1$ for $t \geq 0$, where K_1 is a positive constant. Therefore,

$$\begin{aligned} |\Psi(t)\Phi(t, t_0, x_0)| &= |\Psi(t)\Phi(t, t_0, x_0)\Psi^{-1}(t_0)\Psi(t_0)| \\ &\leq |\Psi(t)\Phi_1(t)\Phi_1^{-1}(t_0)\Psi^{-1}(t_0)| \cdot |\Psi(t_0)| \\ &\leq K_1\left|\big(\Psi(t_0)\Phi_1(t_0)\big)^{-1}\right| \cdot |\Psi(t_0)| := k(t_0) \end{aligned}$$

for $t \geq t_0$. Then, the theorem follows immediately from the Theorem 2.4. The proof of case (ii) is similar to case (i) and we omit it.

In the following we consider the Ψ-(uniform) stabilities of the systems (1.1), (1.2) and (1.3).

Since $f(t, 0) = 0$, there exists a sufficiently small $\delta > 0$ such that:

(H0) $f(t, x) = f_x(t, 0)x + p(t, x)$ for $||x|| < \delta$, where $p(t, x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $||p(t, x)|| = o(||x||)(x \to 0)$.

Theorem 2.7. *Let $\Phi(t, t_0, x_0)$ be a fundamental matrix of* (1.4) *and assume that Hypothesis (H0) is satisfied. If there exist a continuous function $\varphi : \mathbb{R}_+ \to (0, \infty)$ and a constant $M > 0$ such that:*

$$\int_0^t \varphi(s)|\Psi(t)\Phi(t, s, 0)\Psi^{-1}(s)|ds \leq M, \text{ for all } t \geq 0$$

and $||\Psi(t)p(t, x)|| \leq q(t)||\Psi(t)x||$, $\sup_{t \geq 0} \frac{q(t)}{\varphi(t)} < \frac{1}{M}$, where $q(t)$ is a nonnegative continuous function on $\mathbb{R}_+$. Then, the trivial solution of system (1.3) *is Ψ-stable on $\mathbb{R}_+$ provided that the trivial solution of system* (1.4) *is Ψ-stable on $\mathbb{R}_+$.*

Proof. From the last assumption we conclude that there exists a function $k : \mathbb{R}_+ \to (0, \infty)$ such that $|\Psi(t)\Phi(t, t_0, 0)| \leq k(t_0)$ for $t \geq t_0$. The solution of (1.3) with initial condition $x(t_0, t_0, x_0) = x_0$ is unique and defined for all $t \geq 0$, by (H0) and the variation of constants formula, we have

$$x(t, t_0, x_0) = \Phi(t, t_0, 0)x_0 + \int_{t_0}^{t} \Phi(t, s, 0)p(s, x(s, t_0, x_0))ds, \ t \geq t_0. \tag{2.1}$$

Hence,

$$\begin{aligned} ||\Psi(t)x(t, t_0, x_0)|| &\leq ||\Psi(t)\Phi(t, t_0, 0)\Psi^{-1}(t_0)\Psi(t_0)x_0|| \\ &\quad + \int_{t_0}^{t} \varphi(s)|\Psi(t)\Phi(t, s, 0)\Psi^{-1}(s)| \frac{||\Psi(s)p(s, x(s, t_0, x_0))||}{\varphi(s)} ds \\ &\leq k(t_0)|\Psi^{-1}(t_0)| \cdot ||\Psi(t_0)x_0|| \\ &\quad + \int_{t_0}^{t} \varphi(s)|\Psi(t)\Phi(t, s, 0)\Psi^{-1}(s)| \frac{q(s)}{\varphi(s)} ||\Psi(s)x(s, t_0, x_0)||ds \end{aligned}$$

for $t \geq t_0$. If we put $b = \sup_{t \geq 0} \frac{q(t)}{\varphi(t)} < \frac{1}{M}$, then,

$$||\Psi(t)x(t, t_0, x_0)|| \leq k(t_0)|\Psi^{-1}(t_0)| \cdot ||\Psi(t_0)x_0|| + Mb \sup_{t \geq t_0} ||\Psi(t)x(t, t_0, x_0)||,$$

hence,

$$||\Psi(t)x(t, t_0, x_0)|| \leq \frac{k(t_0)}{1 - Mb} |\Psi^{-1}(t_0)| \cdot ||\Psi(t_0)x_0||$$

and the conclusion of the theorem follows.

Corollary 2.8. *Suppose that all the assumptions of Theorem* 2.7 *hold, then the conclusion of the Theorem may be replaced by "the trivial solution of system* (1.1) *is* Ψ*-uniform stable on* $\mathbb{R}_+$ *provided that the trivial solution of system* (1.3) *is* Ψ*-uniform stable on* $\mathbb{R}_+$*."*

Proof. Because the trivial solution of system (1.3) is Ψ-uniform stable on $\mathbb{R}_+$, there exists a positive constant K such that $|\Psi(t)\Phi(t, t_0, 0)\Psi^{-1}(t_0)| \leq K$, it is to say that $k(t_0)|\Psi^{-1}(t_0)|$ can be replaced with K in the proof of Theorem 2.7, this completed the proof.

Theorem 2.9. *Assume that Hypothesis (H0) is satisfied and*

$$||\Psi(t)p(t, x)|| \leq q(t)||\Psi(t)x||, \ L = \int_0^\infty q(t)dt < \infty,$$

where $q(t)$ *is a nonnegative continuous function on* $\mathbb{R}_+$*. Then, the trivial solution of system* (1.3) *is* Ψ*-uniform stable on* $\mathbb{R}_+$ *provided that the trivial solution of system* (1.4) *is* Ψ*-uniform stable on* $\mathbb{R}_+$*.*

Proof. Let $\Phi(t, t_0, x_0)$ be a fundamental matrix for system (1.4), from the last assumption, there exists a constant $K > 0$ such that $|\Psi(t)\Phi(t, s, x_0)\Psi^{-1}(s)| \leq$

K for all $0 \leq s \leq t$. Suppose the solution of (1.3) with initial condition $x(t_0, t_0, x_0) = x_0$ is $x(t, t_0, x_0)$, from (2.1) we get

$$\begin{aligned} ||\Psi(t)x(t,t_0,x_0)|| &\leq ||\Psi(t)\Phi(t,t_0,0)\Psi^{-1}(t_0)\Psi(t_0)x_0|| \\ &\quad + \int_{t_0}^{t} ||\Psi(t)\Phi(t,s,0)\Psi^{-1}(s)\Psi(s)p(s,x(s,t_0,x_0))||ds \\ &\leq K||\Psi(t_0)x_0|| + K\int_{t_0}^{t} q(s)||\Psi(s)x(s,t_0,x_0)||ds \end{aligned}$$

for $t \geq t_0 \geq 0$. By Gronwall's inequality, we have

$$||\Psi(t)x(t,t_0,x_0)|| \leq K||\Psi(t_0)x_0||e^{K\int_{t_0}^{t} q(s)ds} \leq Ke^{KL}||\Psi(t_0)x_0||.$$

This shows that the conclusion of the theorem is true.

Theorem 2.10. *Let $\Phi(t, t_0, x_0)$ be a fundamental matrix of* (1.4) *and assume that the trivial solution of system* (1.3) *is Ψ-stable on $\mathbb{R}_+$, moreover,*

$$||\Psi(t)\Phi(t,s,y)g(s,y)|| \leq h(t,s)||\Psi(s)y|| \text{ and } L = \limsup_{t\to\infty} \int_0^t h(t,s)ds < \infty,$$

where h is a continuous nonnegative function on D. Then, the trivial solution of system (1.1) *is Ψ-stable on $\mathbb{R}_+$.*

Proof. Because the trivial solution of system (1.3) is Ψ-stable on $\mathbb{R}_+$, then for $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that: $||\Psi(t)x(t, t_0, x_0)|| < e^{-L}\varepsilon$ for $t \geq t_0$ and for $||\Psi(t_0)x_0|| < \delta$. The solution of (1.1) with initial condition $y(t_0, t_0, x_0) = x_0$ is unique and defined for all $t \geq 0$, by (1.6) we get

$$\begin{aligned} ||\Psi(t)y(t,t_0,x_0)|| &\leq ||\Psi(t)x(t,t_0,x_0)|| \\ &\quad + \int_{t_0}^{t} ||\Psi(t)\Phi(t,s,y(s,t_0,x_0))g(s,y(s,t_0,x_0))||ds \\ &< e^{-L}\varepsilon + \int_{t_0}^{t} h(t,s)||\Psi(s)y(s,t_0,x_0)||ds \end{aligned}$$

for $t \geq t_0 \geq 0$ and for all x_0 which satisfied $||\Psi(t_0)x_0|| < \delta$. By Gronwall's inequality, we have

$$||\Psi(t)y(t,t_0,x_0)|| < e^{-L}\varepsilon e^{\int_{t_0}^{t} h(t,s)ds} \leq \varepsilon.$$

This shows that the conclusion of the theorem is true.

From the proof of the Theorem 2.10, we have the following corollary.

Corollary 2.11. *Suppose that all the assumptions of Theorem* 2.10 *hold except that "the trivial solution of system* (1.3) *is Ψ-stable on $\mathbb{R}_+$" is replaced with "the trivial solution of system* (1.3) *is Ψ-uniform stable on $\mathbb{R}_+$", then the trivial solution of system* (1.1) *is Ψ-uniform stable on $\mathbb{R}_+$.*

Before we give the Ψ-(uniform) stability of trivial solution of system (1.2), we state a hypothesis which is natural in studying the Ψ-(uniform) system (1.2).

(H1) For all $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$, if $||\Psi(t_0)x_0|| < \rho$, then there exists a unique solution $z(t)$ on $\mathbb{R}_+$ of system (1.2) such that $z(t_0, t_0, x_0) = x_0$ and $||\Psi(t)z(t, t_0, x_0)|| \leq \rho$ for all $t \in [0, t_0]$.

Theorem 2.12. *Assume that Hypothesis (H1) is satisfied. Let* $\Phi(t, t_0, x_0)$ *be a fundamental matrix of* (1.4) *and assume that the trivial solution of system* (1.3) *is* Ψ*-stable on* $\mathbb{R}_+$*, moreover,*

$$||\Psi(t)\Phi(t, s, z)F(s, u, z)|| \leq h(s, u)||\Psi(u)z|| \text{ for } (t, s) \in D \text{ and for all } z \in \mathbb{R}^n,$$

$L = \int_0^\infty \int_0^s h(s, u)du\, ds < \infty$, *where* h *is a continuous nonnegative function on* D*. Then, the trivial solution of system* (1.2) *is* Ψ*-stable on* $\mathbb{R}_+$.

Proof. Suppose $\varepsilon > 0$ is arbitrarily chosen. Because the trivial solution of system (1.3) is Ψ-stable on $\mathbb{R}_+$, then for $\frac{1}{2}e^{-L}\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_1 = \delta_1(\varepsilon, t_0) > 0$ such that: $||\Psi(t)x(t, t_0, x_0)|| < \frac{1}{2}e^{-L}\varepsilon$ for $t \geq t_0$ and for $||\Psi(t_0)x_0|| < \delta_1$. From (H1), for $\delta_2(\varepsilon) = \frac{1}{2L}e^{-L}\varepsilon$, let $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ be such that $z(t_0, t_0, x_0) = x_0$ and $||\Psi(t)z(t, t_0, x_0)|| \leq \delta_2(\varepsilon)$ for all $t \in [0, t_0]$. Choose $\delta = \min\{\delta_1, \delta_2\}$, then the solution of (1.2) with initial condition $z(t_0, t_0, x_0) = x_0$ is unique and defined for all $t \geq 0$, by (1.7), for $t \geq t_0$ we get

$$\begin{aligned}
||\Psi(t)z(t, t_0, x_0)|| &\leq ||\Psi(t)x(t, t_0, x_0)|| \\
&\quad + \int_{t_0}^t \int_0^s ||\Psi(t)\Phi(t, s, z(s, t_0, x_0))F(s, u, z(u, t_0, x_0))||du\, ds \\
&< \frac{1}{2}e^{-L}\varepsilon + \int_{t_0}^t \int_0^s h(s, u)||\Psi(u)z(u, t_0, x_0)||du\, ds \\
&\leq \frac{1}{2}e^{-L}\varepsilon + \int_{t_0}^t \int_0^{t_0} h(s, u)||\Psi(u)z(u, t_0, x_0)||du\, ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s h(s, u)||\Psi(u)z(u, t_0, x_0)||du\, ds \\
&\leq e^{-L}\varepsilon + \int_{t_0}^t \int_{t_0}^s h(s, u)||\Psi(u)z(u, t_0, x_0)||du\, ds
\end{aligned}$$

Define $Q(t) = \int_{t_0}^t \int_{t_0}^s h(s, u)||\Psi(u)z(u, t_0, x_0)||du\, ds$, then $||\Psi(t)z(t, t_0, x_0)|| < e^{-L}\varepsilon + Q(t)$, $Q(t)$ is nonnegative , continuously differentiable and increasing on $[t_0, \infty)$. For $t \geq t_0$, we have

$$\begin{aligned}
Q'(t) &= \int_{t_0}^t h(t, u)||\Psi(u)z(u, t_0, x_0)||du \\
&\leq \int_{t_0}^t h(t, u)\big[e^{-L}\varepsilon + Q(u)\big]du
\end{aligned}$$

$$\leq \; e^{-L}\varepsilon \int_{t_0}^{t} h(t,u)du + Q(t) \int_{t_0}^{t} h(t,u)du$$

or, equivalently,

$$Q'(t) - Q(t) \int_{t_0}^{t} h(t,u)du \leq e^{-L}\varepsilon \int_{t_0}^{t} h(t,u)du$$

Multiplying the above inequality by $e^{-\int_{t_0}^{t}\int_{t_0}^{s} h(s,u)du\, ds}$, we get

$$\frac{d}{dt}\left(Q(t)e^{-\int_{t_0}^{t}\int_{t_0}^{s} h(s,u)du\, ds}\right) \leq e^{-L}\varepsilon \frac{d}{dt}\left(-e^{-\int_{t_0}^{t}\int_{t_0}^{s} h(s,u)du\, ds}\right).$$

Consider now the integral on the interval $[t_0, t]$ to obtain

$$Q(t)e^{-\int_{t_0}^{t}\int_{t_0}^{s} h(s,u)du\, ds} \leq e^{-L}\varepsilon(1 - e^{-\int_{t_0}^{t}\int_{t_0}^{s} h(s,u)du\, ds}).$$

So, $Q(t) \leq e^{-L}\varepsilon(e^{\int_{t_0}^{t}\int_{t_0}^{s} h(s,u)du\, ds} - 1)$, and hence $||\Psi(t)z(t,t_0,x_0)|| \leq e^{-L}\varepsilon + Q(t) \leq e^{-L}\varepsilon e^{\int_{t_0}^{t}\int_{t_0}^{s} h(s,u)du\, ds} \leq \varepsilon$ for $t \geq t_0$. Then the trivial solution of system (1.2) is Ψ-stable on $\mathbb{R}_+$.

From the proof of the Theorem 2.12, we have the following corollary.

Corollary 2.13. *Suppose that all the assumptions of Theorem* 2.12 *hold except that "the trivial solution of system* (1.3) *is* Ψ*-stable on* $\mathbb{R}_+$*" is replaced with "the trivial solution of system* (1.3) *is* Ψ*-uniform stable on* $\mathbb{R}_+$*", then the trivial solution of system* (1.2) *is* Ψ*-uniform stable on* $\mathbb{R}_+$*.*

This is because that the δ_1 in the proof of the Theorem 2.12 will be independent of t_0 if the trivial solution of system (1.3) is Ψ-uniform stable on $\mathbb{R}_+$.

3. EXAMPLES

Example 3.1. Consider the nonlinear differential equation

$$y' = -y + e^{-t}\sin t \min\{y^2, y\}. \tag{3.1}$$

In the equation (3.1), $f(t,y) = -y$, $g(t,y) = e^{-t}\sin t \min\{y^2, y\}$, equation (3.1) can be seen as perturbed equation of

$$x' = -x, \tag{3.2}$$

and the variational system through (t_0, x_0) associated with system (3.2) is (3.2) itself, it's fundamental matrix solution $\phi(t,t_0,x_0) = e^{-(t-t_0)}$, independent of x_0. Obviously the trivial solution of equation (3.2) is uniform stable. If we choose the scalar function $\psi(t) = e^t$, since $|\psi(t)\phi(t,s,x_0)\psi^{-1}(s)| = |e^t e^{-(t-s)}e^{-s}| = 1$ for all $0 \leq s \leq t < \infty$, the trivial solution of equation (3.2) is ψ-uniformly stability on $\mathbb{R}_+$. Moreover,

$$|\psi(t)\phi(t,s,y)g(s,y)| \leq e^{-s}|\psi(s)y| = e^{-s}|e^s y| \text{ and } 1 = \limsup_{t\to\infty} \int_0^t e^{-s}ds < \infty,$$

Then the trivial solution of the nonlinear differential equation (3.1) is ψ-uniformly stability on $\mathbb{R}_+$ from our Theorem 2.10.

Example 3.2. Consider the nonlinear differential system

$$\begin{cases} y_1' = y_1 + \min\{y_1, y_1^2\}\sin t, \\ y_2' = -y_2 + \min\{y_2, y_2^2\}\cos t. \end{cases} \tag{3.3}$$

In the equation (3.3), $f(t,y) = (y_1, -y_2)^T$, $g(t,y) = (\min\{y_1, y_1^2\}\sin t, \min\{y_2, y_2^2\}\cos t)^T$, equation (3.3) can be seen as perturbed system of

$$\begin{cases} x_1' = x_1, \\ x_2' = -x_2, \end{cases} \tag{3.4}$$

and the variational system through (t_0, x_0) associated with system (3.4) is (3.4) itself, it's fundamental matrix solution

$$\Phi(t, t_0, x_0) = \begin{pmatrix} e^{(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix},$$

independent of x_0. Because $|\Phi(t, t_0, x_0)|$ is unbounded, the trivial solution of system (3.4) is unstable. Choose the matrix function
$\Psi(t) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}$, since

$$\begin{aligned} |\Psi(t)\Phi(t, s, x_0)\Psi^{-1}(s)| &= \left| \begin{pmatrix} e^{(t-s)} & 0 \\ 0 & e^{-(t-s)} \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} \begin{pmatrix} e^{2s} & 0 \\ 0 & e^{\frac{-s}{2}} \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & e^{-\frac{1}{2}(t-s)} \end{pmatrix} \right| \le 1 \end{aligned}$$

for all $0 \le s \le t < \infty$, then the trivial solution of equation (3.4) is ψ-uniformly stability on $\mathbb{R}_+$. Moreover,

$$\begin{aligned} ||\Psi(t)\Phi(t, s, y)g(s, y)|| &= ||\left[\Psi(t)\Phi(t, s, y)\Psi^{-1}(s)\right]\left[\Psi(s)g(s, y)\right]|| \\ &\le |\Psi(t)\Phi(t, s, y)\Psi^{-1}(s)| \cdot ||\Psi(s)y|| \le e^{-\frac{1}{2}(t-s)}||\Psi(s)y|| \end{aligned}$$

and $2 = \limsup_{t\to\infty} \int_0^t e^{-\frac{1}{2}(t-s)} ds < \infty$. Then the trivial solution of the nonlinear differential system (3.3) is ψ-uniformly stability on $\mathbb{R}_+$ from our Corollary 2.11.

Example 3.3. Consider the nonlinear Volterra integro-differential system

$$\begin{cases} z_1' = z_1 + \int_0^t \min\{z_1(s), z_1^2(s)\}(\sin t)(\cos s)ds, \\ z_2' = -z_2 + \int_0^t \min\{z_2(s), z_2^2(s)\}(\cos t)(\sin s)ds. \end{cases} \tag{3.5}$$

In the equation (3.5), $f(t,z) = (z_1, -z_2)^T$, $F(t,s,z) = (\min\{z_1, z_1^2\}(\sin t)(\cos s), \min\{z_2, z_2^2\}(\cos t)(\sin s))^T$, equation (3.5) can be seen as perturbed system of equation (3.4), by similar discussion and choose the same matrix function $\Psi(t) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}$, we have $|\Psi(t)\Phi(t,s,x_0)\Psi^{-1}(s)| = \left|\begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & e^{-\frac{1}{2}(t-s)} \end{pmatrix}\right| \leq 1$ for all $0 \leq s \leq t < \infty$, then the trivial solution of equation (3.4) is ψ-uniformly stability on $\mathbb{R}_+$. Moreover,

$$\begin{aligned} ||\Psi(t)\Phi(t,s,z)F(s,u,z)|| &= ||\left[\Psi(t)\Phi(t,s,z)\Psi^{-1}(s)\right]\left[\Psi(s)F(s,u,z)\right]|| \\ &\leq |\Psi(t)\Phi(t,s,y)\Psi^{-1}(s)| \cdot ||\Psi(s)z|| \leq e^{-\frac{1}{2}(t-s)}||\Psi(s)y|| \end{aligned}$$

and $2 = \limsup_{t\to\infty} \int_0^t e^{-\frac{1}{2}(t-s)} ds < \infty$. Then the trivial solution of the nonlinear Volterra integro-differential system (3.5) is ψ-uniformly stability on $\mathbb{R}_+$ from our Corollary 2.13.

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