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ON NEUTRAL DIFFERENTIAL EQUATIONS AND THE MONOTONE ITERATIVE METHOD

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ABSTRACT. The application of the monotone iterative method to neutral differential equations with deviating arguments is considered in this paper. We formulate existence results giving sufficient conditions which guarantee that such problems have solutions. This approach is new and to the Authors' knowledge, this is the first paper when the monotone iterative method is applied to neutral first–order differential equations with deviating arguments. An example is given to illustrate theoretical results. One may apply a numerical method based on the proposed monotone iterative method to obtain a numerical solution of our problems.

Key words: Neutral differential equations, existence of solutions, monotone iterative method

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1. INTRODUCTION

In this paper, we investigate initial value problems for first order neutral differential equations with delayed arguments of the form:

$$(1.1)\qquad \begin{cases} x'(t) &= f(t, x'(t), x'(\beta(t)), x(t), x(\alpha(t))), \quad t \in J = [0, T], \\ x(0) &= 0, \end{cases}$$

where

$H_1:\ f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ \ \alpha, \beta \in C(J, J)$ and $\alpha(t) \le t,\ \beta(t) \le t$ on J.

If f does not depend on the second and third arguments, then problem (1.1) is not of neutral type which was considered, for example, in paper [1].

Also in this paper, we discuss the following problem:

$$(1.2)\qquad \begin{cases} x'(t) &= g(t, x'(t), x'(\beta(t)), x(t), x(\alpha(t))), \quad t \in J = [0, T], \\ x(T) &= 0, \end{cases}$$

where

$H_1^1:\ g \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ \ \alpha, \beta \in C(J, J)$ and $\alpha(t) \ge t,\ \beta(t) \ge t$ on J.

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If g does not depend on the second and third arguments then problem (1.2) is not of neutral type which was discussed, for example, in paper [2].

An interesting and fruitful technique for proving existence results for nonlinear differential problems is the monotone iterative method (see, for example, [3] for details). We have many applications of this method both to initial and boundary value problems.

We want to apply the monotone iterative method also to the neutral differential problems of types (1.1) and (1.2). To apply it, we first make the substitution $y(t) = x'(t)$. Then problem (1.1) is equivalent to the following equation:

$$y(t) = f\left(t, y(t), y(\beta(t)), \int_0^t y(s)ds, \int_0^{\alpha(t)} y(s)ds\right) \equiv (Fy)(t), \quad t \in J. \tag{1.3}$$

Similarly, problem (1.2) takes now the form:

$$y(t) = g\left(t, y(t), y(\beta(t)), -\int_t^T y(s)ds, -\int_{\alpha(t)}^T y(s)ds\right) \equiv (Gy)(t), \quad t \in J. \tag{1.4}$$

2. MAIN RESULTS

First of all, we investigate problems (1.3) and (1.1).

We say that $u \in C(J, \mathbb{R})$ is called a lower solution of (1.3) if

$$u(t) \leq (Fu)(t), \quad t \in J,$$

and it is an upper solution of (1.3) if the above inequality is reversed.

Now we formulate conditions under which problem (1.3) has extremal solutions in a corresponding sector bounded by lower and upper solutions of problem (1.3).

Theorem 2.1. *Let assumption H_1 hold. Let $y_0, z_0 \in C(J, \mathbb{R})$ be the lower and upper solutions of (1.3), respectively and $y_0(t) \leq z_0(t)$, $t \in J$. In addition, let us assume that the following assumptions hold:*

H_2 : *f is nondecreasing with respect to the last three variables,*
H_3 : *there exists a constant $K > -1$, such that*

$$f(t, u, v_1, v_2, v_3) - f(t, \bar{u}, v_1, v_2, v_3) \leq K(\bar{u} - u)$$

for $y_0 \leq u \leq \bar{u} \leq z_0$.

Then problem (1.3) has, in the sector $[y_0, z_0]_*$, minimum and maximum solutions, where

$$[y_0, z_0]_* = \{v \in C(J, \mathbb{R}) : \ y_0(t) \leq v(t) \leq z_0(t), \quad t \in J\}.$$

Proof. Let

$$\begin{aligned} y_{n+1}(t) &= (Fy_n)(t) - K[y_{n+1}(t) - y_n(t)], \quad t \in J, \\ z_{n+1}(t) &= (Fz_n)(t) - K[z_{n+1}(t) - z_n(t)], \quad t \in J \end{aligned}$$

for $n = 0, 1, \ldots$, with the operator F defined as in formula (1.3).

Observe that functions y_1, z_1 are well defined. First, we prove that

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \quad t \in J. \tag{2.1}$$

Put $p = y_0 - y_1, \ q = z_1 - z_0$. It leads to

$$\begin{aligned} p(t) &\le (Fy_0)(t) - (Fy_0)(t) - Kp(t) = -Kp(t) \\ q(t) &\le (Fz_0)(t) - (Fz_0)(t) - Kq(t) = -Kq(t). \end{aligned}$$

This shows that $y_0(t) \le y_1(t), \ z_1(t) \le z_0(t), \ t \in J$. Now, we put $p = y_1 - z_1$. In view of assumptions H_2, H_3 we have

$$p(t) = (Fy_0)(t) - (Fz_0)(t) - K[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \le -Kp(t).$$

Hence, $y_1(t) \le z_1(t)$ on J. It proves (2.1).

In the next step we show that y_1, z_1 are the lower and upper solutions of problem (1.3). Note that

$$\begin{aligned} y_1(t) &= (Fy_0)(t) - (Fy_1)(t) + (Fy_1)(t) - K[y_1(t) - y_0(t)] \le (Fy_1)(t), \\ z_1(t) &= (Fz_0)(t) - (Fz_1)(t) + (Fz_1)(t) - K[z_1(t) - z_0(t)] \ge (Fz_1)(t), \end{aligned}$$

by assumptions H_2, H_3. This proves that y_1, z_1 are the lower and upper solutions of problem (1.3).

Using the mathematical induction, we can show that

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le y_{n+1}(t) \le z_{n+1}(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t)$$

for $t \in J$ and $n = 0, 1, \ldots$.

It is easy to see that sequences $\{y_n, z_n\}$ converge uniformly and monotonically to the limit functions y and z, respectively; where y and z are solutions of the following problems:

$$\begin{aligned} y(t) &= (Fy)(t), \quad t \in J, \\ z(t) &= (Fz)(t), \quad t \in J \end{aligned}$$

with $y_0 \le y \le z \le z_0$.

Now, we need to show that y and z are extremal solutions of problem (1.3) in the sector $[y_0, z_0]_*$. Let v be any solution of problem (1.3) such that $y_0 \le v \le z_0$. Put $p = y_1 - v, \ q = v - z_1$. Then, in view of assumptions H_2 and H_3, we see that

$$\begin{aligned} p(t) &= (Fy_0)(t) - K[y_1(t) - y_0(t)] - (Fv)(t) \le -Kp(t), \\ q(t) &= (Fv)(t) - (Fz_0)(t) + K[z_1(t) - z_0(t)] \le -Kq(t). \end{aligned}$$

This shows that $y_1 \le v \le z_1$. By induction, we can show that

$$y_n \le v \le z_n.$$

Now, if $n \to \infty$, then we have the assertion of this theorem. $\square$

Our next theorem concerns the case when problem (1.3) has a unique solution.

Theorem 2.2. *Let all assumptions of Theorem* 2.1 *be satisfied. In addition, we assume that Assumption H_4 hold, where*

H_4 : *there exists nonnegative constants $M_1, M_2, M_3, M_4,\ -K \le M_1$ such that*

$$\rho \equiv M_1 + M_2 + T(M_3 + M_4) < 1 \tag{2.2}$$

and

$$\begin{aligned} f(t,u_1,v_1,v_2,v_3) - f(t,\bar{u}_1,\bar{v}_1,\bar{v}_2,\bar{v}_3) \quad &\le \quad M_1(u_1-\bar{u}_1) + M_2(v_1-\bar{v}_1) \\ &\quad\;\; +M_3(v_3-\bar{v}_3) + M_4(v_4-\bar{v}_4) \end{aligned}$$

if $y_0(0) \le \bar{u}_1 \le u_1 \le z_0(0),\ y_0(\beta(t)) \le \bar{v}_1 \le v_1 \le z_0(\beta(t)),\ \int_0^t y_0(s)ds \le \bar{v}_2 \le v_2 \le \int_0^t z_0(s)ds,\ \int_0^{\alpha(t))} y_0(s)ds \le \bar{v}_3 \le v_3 \le \int_0^{\alpha(t))} z_0(s)ds.$

Then problem (1.3) *has, in the sector* $[y_0, z_0]_*$, *a unique solution.*

Proof. Theorem 2.1 guarantees that functions y and z are extremal solutions of problem (1.3) and $y_0(t) \le y(t) \le z(t) \le z_0(t),\ t \in J$. We need to show that $y(t) = z(t),\ t \in J$. Put $p = z - y$. Then $p(t) \ge 0,\ t \in J$. Moreover, in view of Assumption H_4, we get

$$\begin{aligned} 0 \le p(t) \quad &= \quad (Fz)(t) - (Fy)(t) \\ &\le \quad M_1 p(t) + M_2 p(\beta(t)) + M_3 \int_0^t p(s)ds + M_4 \int_0^{\alpha(t)} p(s)ds \\ &\le \quad \rho \max\nolimits_{s\in J} p(s), \end{aligned}$$

so

$$\max_{t\in J} p(t)(1-\rho) \le 0.$$

Hence, $\max\limits_{t\in J} p(t) \le 0$. This shows that $y(t) = z(t),\ t \in J$, so the proof is complete. $\square$

Theorem 2.3. *Let all assumptions of Theorem* 2.2 *hold. Then problem* (1.1) *has, in the sector* $[y_0, z_0]_{**}$, *a unique solution* x, *where*

$$[y_0, z_0]_{**} = \left\{ v \in C(J, I\!R) : \int_0^t y_0(s)ds \le v(t) \le \int_0^t z_0(s)ds,\ \ t \in J \right\}.$$

Proof. From Theorem 2.2, we see that $y_0(t) \le y(t) \le z_0(t),\ t \in J$, where y is the unique solution of problem (1.3). This means that $y_0 \le x' \le z_0$, so

$$\int_0^t y_0(s)ds \le x(t) \le \int_0^t z_0(s)ds,\ \ t \in J,$$

where x is a unique solution of problem (1.1). $\square$

Example 2.4. Let $M_2, M_3, M_4 \geq 0,\ B > 0$ and $\alpha, \beta \in C(J, J),\ \alpha(t) \leq t,\ \beta(t) \leq t$ with $J = [0, T]$. Consider the following linear neutral differential problem:

$$\left\{ \begin{array}{lcl} x'(t) & = & M_2 x(\beta(t)) + M_3 x'(t) + M_4 x'(\alpha(t)) + B, \ \ t \in J, \\ x(0) & = & 0, \end{array} \right. \tag{2.3}$$

where $\rho_1 \equiv M_2 + (M_3 + M_4)T < 1$.

In this case, we have

$$(Fu)(t) = M_2 u(\beta(t)) + M_3 \int_0^t u(s)ds + M_4 \int_0^{\alpha(t))} u(s)ds + B.$$

Take $y_0(t) = 0,\ z_0(t) = a > 0,\ t \in J$ and

$$a[1 - \rho_1] > B. \tag{2.4}$$

Then

$$\begin{array}{lcl} (Fy_0)(t) & = & B > 0 = y_0(t),\ t \in J, \\ (Fz_0)(t) & = & a\left[M_2 + M_3 t + M_4 \alpha(t)\right] + B \leq a\rho_1 + B < a = z_0(t),\ t \in J, \end{array}$$

in view of condition (2.4). This shows that all assumptions of Theorem 2.3 holds, so problem (2.3) has a unique solution in the region $[y_0, z_0]_{**}$.

Theorem 2.2 gives the sufficient conditions under which problem (1.3) has the unique solution. Now, we are going to discuss again this problem giving another sufficient conditions which guarantee that (1.3) has the unique solution. To do it, first, we need some properties connected with delay differential inequalities.

Lemma 2.5. *Let $\alpha \in C(J, J),\ \alpha(t) \leq t$ on J. Suppose that $p \in C^1(J, \mathbb{R})$ and*

$$\left\{ \begin{array}{lcl} q'(t) & \leq & N(t) p(\alpha(t)), \ \ t \in J, \\ q(0) & \leq & 0, \end{array} \right. \tag{2.5}$$

where $N \in C(J, \mathbb{R}_+)$ and $\mathbb{R}_+ = [0, \infty)$.

In addition, assume that

$$\int_0^T N(t)dt < 1.$$

Then $q(t) \leq 0$ on J.

Proof. We need to prove that $q(t) \leq 0,\ t \in J$. Suppose that the above inequality is not true. Then, we can find $t_0 \in (0, T]$ such that $q(t_0) > 0$. Put

$$q(t_1) = \max_{[0,t_0]} q(t) > 0.$$

Integrating the differential inequality in (2.5) from 0 to t_1, we obtain

$$q(t_1) \leq \int_0^{t_1} N(t) q(\alpha(t))dt \leq q(t_1) \int_0^{t_1} N(t)dt \leq q(t_1) \int_0^T N(t)dt < q(t_1).$$

It contradicts assumption that $q(t_0) > 0$. This shows that $q(t) \leq 0$ on J and the proof is complete. $\square$

Lemma 2.6. *Let $\alpha \in C(J, J)$, $\alpha(t) \leq t$ on J. Suppose that $K \in C(J, \mathbb{R})$, $L \in C(J, \mathbb{R}_+)$, $q \in C^1(J, \mathbb{R})$ and*

$$\begin{cases} q'(t) & \leq & K(t)q(t) + L(t)q(\alpha(t)), \quad t \in J, \\ q(0) & \leq & 0. \end{cases} \tag{2.6}$$

In addition assume that

H_5 : $\int_0^T L(t)e^{-\int_{\alpha(t)}^t K(s)ds}dt < 1$.

Then $q(t) \leq 0$ on J.

Proof. Indeed, the assertion holds if $L(t) = 0,\ t \in J$. Let $\int_0^T L(t)dt > 0$. Put

$$p(t) = e^{-\int_0^t K(s)ds}q(t), \quad t \in J.$$

This yields $p(0) = q(0) \leq 0$, and

$$p'(t) = e^{-\int_0^t K(s)ds}\left[-K(t)q(t) + q'(t)\right]$$

so

$$\begin{cases} p'(t) & \leq & L(t)e^{-\int_{\alpha(t)}^t K(s)ds}p(\alpha(t)), \quad t \in J, \\ p(0) & \leq & 0. \end{cases} \tag{2.7}$$

In view of Lemma 2.5, $p(t) \leq 0$ on J, by Assumption H_5. This also proves that $q(t) \leq 0$ on J and the proof is complete. $\square$

Remark 2.7. Note that Assumption H_5 holds if:

$$\int_0^T L(t)dt < 1 \quad \text{provided that} \quad K(t) \geq 0,\ t \in J,$$

and

$$\int_0^T L(t)e^{-\int_0^t K(s)ds}dt < 1 \quad \text{provided that} \quad K(t) \leq 0,\ t \in J.$$

We see that the above two conditions do not depend on α.

Remark 2.8. If we assume that $K(t) = K$, then Assumption H_5 takes the form:

$$\int_0^T L(t)e^{K[\alpha(t)-t]}dt < 1.$$

Remark 2.9. Let all assumptions of Lemma 2.6 hold with $q \in C^1(J, \mathbb{R}_+)$ and $q(0) = 0$ instead of $q \in C^1(J, \mathbb{R})$ and $q(0) \leq 0$, respectively. Then $q(t) = 0$ on J.

Theorem 2.10. *Let all assumptions of Theorem* 2.1 *be satisfied. Assume that f does not depend on the third variable. In addition, we assume that Assumption H_6 hold, where*

H_6 : *there exists nonnegative constants* $M_1, M_3, M_4,\ -K \le M_1 < 1$ *such that*

$$B\int_0^T e^{A[\alpha(t)-t]}dt < 1 \tag{2.8}$$

with

$$A = \frac{M_3}{1-M_1}, \quad B = \frac{M_4}{1-M_1}$$

and

$$\begin{aligned} f(t,u_1,v_1,v_2,v_3) &- f(t,\bar u_1,v_1,\bar v_2,\bar v_3) \\ &\le M_1(u_1-\bar u_1) + M_3(v_3-\bar v_3) + M_4(v_4-\bar v_4) \end{aligned} \tag{2.9}$$

if $y_0(0) \le \bar u_1 \le u_1 \le z_0(0),\ \int_0^t y_0(s)ds \le \bar v_2 \le v_2 \le \int_0^t z_0(s)ds,\ \int_0^{\alpha(t))} y_0(s)ds \le \bar v_3 \le v_3 \le \int_0^{\alpha(t))} z_0(s)ds.$

Then problem (1.3) *has, in the sector* $[y_0, z_0]_*$, *a unique solution.*

Proof. Repeating the proof of Theorem 2.2, we have

$$\begin{aligned} 0 \le p(t) &= (Fz)(t) - (Fy)(t) \\ &\le M_1p(t) + M_3\int_0^t p(s)ds + M_4\int_0^{\alpha(t)} p(s)ds, \end{aligned}$$

so

$$0 \le p(t) \le A\int_0^t p(s)ds + B\int_0^{\alpha(t)} p(s)ds, \tag{2.10}$$

where p is defined as in the proof of Theorem 2.2.

Put $\int_0^t p(s)ds = q(t)$. Then $q(0) = 0,\ q'(t) = p(t)$. Now, inequality (2.10) is replaced by

$$\begin{cases} 0 \le q'(t) &\le Aq(t) + Bq(\alpha(t)),\ t \in J, \\ \quad q(0) &= 0. \end{cases} \tag{2.11}$$

It yields $q(t) = 0,\ t \in J$, by Remark 2.9. This proofs that $\int_0^t p(s)ds = 0$ for all $t \in J$, so $p(t) = 0,\ t \in J$. It means that $y(t) = z(t),\ t \in J$ and the proof is complete. ☐

Now, we are able to formulate the following result.

Theorem 2.11. *Let all assumptions of Theorem* 2.10 *hold. Then problem* (1.1) *has, in the sector* $[y_0, z_0]_{**}$, *a unique solution* x, *where*

$$[y_0, z_0]_{**} = \left\{ v \in C(J, \mathbb{R}) : \int_0^t y_0(s)ds \le v(t) \le \int_0^t z_0(s)ds,\ \ t \in J \right\}.$$

Example 2.12. Consider the following linear problem:

$$\begin{cases} x'(t) &= \lambda x'(t) + \gamma x(t) + \delta,\ \ t \in J = [0,T], \\ x(0) &= 0, \end{cases} \tag{2.12}$$

where $\lambda < 1$, $\gamma, \delta > 0$.

Note that

$$X(t) = \frac{b}{a}\left(e^{at} - 1\right) \quad \text{with} \quad a = \frac{\gamma}{1-\lambda}, \;\; b = \frac{\delta}{1-\lambda}$$

is the unique solution of problem (2.12).

Putting $x' = y$, we translate problem (2.12) into the following one

$$y(t) = a\int_0^t y(s)ds + b \equiv (Fy)(t). \tag{2.13}$$

Let $y_0(t) = 0,\; z_0(t) = be^{at} + c$, where $c \geq 0$ (if $c > 0$, then we need to assume that $aT \leq 1$). Then

$$\begin{aligned} (Fy_0)(t) &= b > 0 = y_0(t), \\ (Fz_0)(t) &= a\textstyle\int_0^t \left[be^{as} + c\right] ds + b \leq a\int_0^t be^{as} ds + b + c = be^{at} + c = z_0(t). \end{aligned}$$

It proves that y_0 and z_0 are lower and upper solutions of (2.13), respectively. This shows that all assumptions of Theorem 2.1 are satisfied with $K = 0$ in Assumption H_3.

Note that condition (2.9) holds with $M_1 = 0,\; M_3 = a,\; M_4 = 0$, so all assumptions of Theorem 2.10 hold with $B = 0$ in condition (2.8). In view of Theorem 2.11, problem (2.12) has a unique solution x and

$$0 \leq x(t) \leq \int_0^t z_0(t)dt = \frac{b}{a}\left(e^{at} - 1\right) + ct,\; t \in J.$$

Now we discuss problems (1.4) and (1.2).

For problem (1.4), we introduce the same definition of the lower and upper solutions as for equation (1.3) with operator G instead of F. Note that in this case, we assume that the upper solution is less than the lower solution of problem (1.4).

Theorem 2.13. *Let Assumption H_1^1 hold. Let $y_0, z_0 \in C(J, I\!R)$ be the lower and upper solutions of (1.4), respectively, and $z_0(t) \leq y_0(t),\; t \in J$. In addition, we assume that the following assumptions hold:*

$H_6:$ *g is nonincreasing with respect to the last three variables,*
$H_7:$ *there exists a constant $K > 1$, such that*

$$g(t, u, v_1, v_2, v_3) - g(t, \bar{u}, v_1, v_2, v_3) \geq -K(\bar{u} - u)$$

for $z_0 \leq u \leq \bar{u} \leq y_0$.

Then problem (1.4) has, in the sector $[z_0, y_0]_1$, minimum and maximum solutions, where

$$[z_0, y_0]_1 = \{v \in C(J, I\!R) : \; z_0(t) \leq v(t) \leq y_0(t), \;\; t \in J\}.$$

Proof. Now, we define only the sequences $\{y_n, z_n\}$ by formulas:

$$\begin{aligned} y_{n+1}(t) &= (Gy_n)(t) + K[y_{n+1}(t) - y_n(t)], \quad t \in J, \\ z_{n+1}(t) &= (Gz_n)(t) + K[z_{n+1}(t) - z_n(t)], \quad t \in J \end{aligned}$$

for $n = 0, 1, \ldots$ with the operator G defined as in formula (1.4). We omit the proof, since it is similar to the proof of Theorem 2.1. $\square$

Theorem 2.14. *Let all assumptions of Theorem* 2.13 *hold. Then problem* (1.2) *has, in the sector* $[z_0, y_0]_2$*, extremal solutions, where*

$$[z_0, y_0]_2 = \left\{ v \in C(J, I\!R) : \int_T^t y_0(s)ds \le v(t) \le \int_T^t z_0(s)ds, \quad t \in J \right\}.$$

Proof. From Theorem 2.13, we see that $z_0(t) \le z(t) \le y(t) \le y_0(t), \ t \in J$, where z and y are minimal and maximal solutions of problem (1.4), respectively, in the sector $[z_0, y_0]_1$. Hence:

$$z_0(t) \le x'(t) \le X'(t) \le y_0(t), \quad t \in J,$$

where x and X are solutions of problem (1.2). Integrating it from t to T, we have the assertion. $\square$

Example 2.15. Consider the following linear problem:

$$\text{(2.14)} \qquad \begin{cases} x'(t) &= 2x'(t) + x(t) + 1, \quad t \in J = [0, T], \ T \le \frac{2}{3}, \\ x(T) &= 0. \end{cases}$$

Note that

$$\text{(2.15)} \qquad X(t) = e^{T-t} - 1, \quad t \in J$$

is the unique solution of problem (2.14).

Putting $x' = y$, we translate problem (2.14) into the following one

$$\text{(2.16)} \qquad y(t) = 2y(t) - \int_t^T y(s)ds + 1 \equiv (Gy)(t), \quad t \in J$$

(see the definition of operator G in Section 1).

Let $y_0(t) = 0, \ z_0(t) = -3$, so $z_0(t) \le y_0(t), \ t \in J$. Then

$$\begin{aligned} (Gy_0)(t) &= 1 > 0 = y_0(t), \\ (Gz_0)(t) &= -6 + 3\int_t^T ds + 1 \le -3 = z_0(t). \end{aligned}$$

This proves that y_0, z_0 are lower and upper solutions of problem (2.16), respectively. Moreover, Assumptions H_5, H_6 hold with $K = 2$. Problem (2.16) has the minimal and maximal solutions z, y and $z_0(t) \le z(t) \le y(t) \le y_0(t), \ t \in J$, by Theorem 2.13.

Now, we need to show that problem (2.16) has the unique solution in the sector $[z_0, y_0]_1$. To do it we put $p = y - z$, so $p(t) \ge 0$ on J. Then

$$p(t) = y(t) - z(t) = (Gy)(t) - (Gz)(t) = 2p(t) - 2\int_t^T p(s)ds,$$

so

$$p(t) = 2\int_t^T p(s)ds. \tag{2.17}$$

Put $Q(t) = \int_t^T p(s)ds$. Then, equation (2.17) is replaced by

$$\begin{cases} Q'(t) &= -2Q(t), \quad t \in J, \\ Q(T) &= 0. \end{cases}$$

Indeed, $Q(t) = 0,\ t \in J$ is the unique solution, so $\int_t^T p(s)ds = 0,\ t \in J$. Because $p(t) \geq 0,\ t \in J$, it proves that $p(t) = 0$ on J. Hence, problem (2.16) has the unique solution in the sector $[z_0, y_0]_1$. Problem (2.14) has the unique solution in the sector $Q = [\int_0^T y_0(s)ds, \int_t^T z_0(s)ds]$, so $Q = [0,\ 3(T-t)]$. Note that the solution X of problem (2.14) belongs to the sector Q, where X is defined by formula (2.15).

Example 2.16. Consider the following linear problem:

$$\begin{cases} x'(t) &= bx'(t) - be^{-x(\alpha(t))} + d,\ t \in J = [0,T], \quad T \leq \\ x(T) &= 0, \end{cases} \tag{2.18}$$

where $\alpha \in C(J,J),\ \alpha(t) \geq t$ and $b > 1,\ d \geq b$. By the substitution $x' = y$, in the place of problem (2.18) we have

$$y(t) = by(t) - be^{\int_{\alpha(t)}^T y(s)ds} + d \equiv (Gy)(t). \tag{2.19}$$

Put $y_0(t) = 0,\ z_0(t) = -a,\ t \in J$ with $a = \frac{d}{b-1}$. Note that $z_0(t) < y_0(t),\ t \in J$, and

$$\begin{aligned} (Gy_0)(t) &= -b + d \geq 0 = y_0(t),\ t \in J, \\ (Gz_0)(t) &= -ba - be^{-a(T-\alpha(t))} + d \leq -ba + d = -a = z_0(t),\ t \in J. \end{aligned}$$

In view of Theorem 2.13, problem (2.18) has the extremal solutions in the region $[z_0, y_0]_1$. By Theorem 2.14, problem (2.18) has a solution in the sector $[0,\ a[T-t]]$.

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