Dynamic Systems and Applications 20 (2011) 183-192

A NOTE ON A STOCHASTIC PERTURBED DI SIR EPIDEMIC MODEL

QINGSHAN YANG[1], CHUNYAN JI[1,2],
DAQING JIANG[1], DONAL O'REGAN[3], AND RAVI P. AGARWAL[4]

[1]School of Mathematics and Statistics, Northeast Normal University
Changchun 130024, Jilin, P. R. China
[2]Department of Mathematics, Changshu Institute of Technology
Changshu 215500, Jiangsu, P. R. China
[3]School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, Ireland
[4]Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, FL 32901-6975, USA

ABSTRACT. In this paper, we investigate the asymptotic behavior of a DI SIR epidemic model with a stochastic perturbation. The ergodic property is obtained by stochastic Lyapunov functions. We also make simulations to show how the solution goes around the endemic equilibrium of a deterministic system under conditions, which conform to our analytical result.

Key words: Stochastic DI SIR epidemic model; Itô's formula; Stochastic Lyapunov function

1. INTRODUCTION

At the global level, the number of people killed by HIV/AIDS has been growing-from 35 million in 2001 to 38 million in 2003 and over 20 million have died since the first emergence of AIDS in 1981. Hence the HIV/AIDS pandemic has been the greatest public health disaster of modern times. Unfortunately, the dynamics transmission of HIV is quite complex. Recently, many researchers have constructed mathematical models, which reflect the characteristics of this epidemic to some extent. In particular, Hyman et al. [4] proposed a differential infectivity (DI) model that accounted for differences in infectiousness between individuals during the chronic stages, and the correlation between viral loads and rates of developing AIDS. They assumed that the susceptible population was homogeneous and neglected variations in susceptibility, risk behavior, and many other factors associated with the dynamics to the HIV

The work was supported by NSFC of China (No.10971021), the Ministry of Education of China (No.109051), the Ph.D. Programs Foundation of Ministry of China (No.200918), the Graduate Innovative Research Project of NENU (No. 09SSXT117), Young Teachers of Northeast Normal University (No.20090104) and Key Laboratory for Applied Statistics of MOE (KLAS).

Received February 14, 2011 1056-2176 $15.00

spread. They divided the population as susceptible individuals S, the HIV infection population I, which was subdivided into n subgroups, $I_1, I_2, \ldots, I_n$, and the group of AIDS patients A. They presented the DI model:

$$\begin{cases} \dfrac{dS}{dt} = \mu S^0 - \mu S - \displaystyle\sum_{j=1}^{n} \beta_j I_j S, \\ \dfrac{dI_k}{dt} = p_k \displaystyle\sum_{j=1}^{n} \beta_j I_j S - (\mu + \gamma_k) I_k, \ k = 1, 2, \ldots, n, \\ \dfrac{dA}{dt} = \displaystyle\sum_{k=1}^{n} \gamma_k I_k - \delta A, \end{cases} \tag{1.1}$$

where the rate of infection depends upon the transmission probability per partner β_k of individuals in subgroup k, S^0 presents a constant steady state of the susceptible population S, μ is the rate of inflow and outflow, which maintains the equilibrium S^0, p_k is the probability of an individual entering subgroup k, when he is infected, and $\sum_{k=1}^{n} p_k = 1$, γ_k is the rate of leaving the high-risk population because of behavior changes that are induced by either HIV-related illnesses or a positive HIV test and finally δ is the die rate of A which satisfies $\delta \geq \mu$. Obviously, system (1.1) has only two kinds of equilibria: the infection-free equilibrium $E_0 = (S^0, I_1 = 0, I_2 = 0, \ldots, I_n = 0)$ and the endemic equilibrium $E^* = (S^*, I_1^*, I_1^*, \ldots, I_n^*)$. Hyman et al. [4] and Ma et al. [8] showed if $R_0 \leq 1$, the infection-free equilibrium is globally asymptotically stable in the region $G := \{(S, I_k) | 0 \leq N = S + \sum_{k=1}^{n} I_k \leq S^0\}$, while if $R_0 > 1$, the disease-free equilibrium is unstable, and the endemic equilibrium E^* is globally asymptotically stable in the region G, where $R_0 = S^0 \sum_{k=1}^{n} \frac{\beta_k p_k}{\mu + \gamma_k}$.

Allowing for environmental white noise, Jiang et al. ([6]) proposed a reasonable stochastic analogue of system (1.1) given by

$$\begin{cases} dS = (\mu S^0 - \mu S - \displaystyle\sum_{j=1}^{n} \beta_j I_j S) dt + \sigma_S S dB_S(t), \\ dI_k = [p_k \displaystyle\sum_{j=1}^{n} \beta_j I_j S - (\mu + \gamma_k) I_k] dt + \sigma_{I,k} I_k dB_{I,k}(t), \ k = 1, 2, \ldots, n, \\ dA = (\displaystyle\sum_{j=1}^{n} \gamma_j I_j - \delta A) dt + \sigma_A A dB_A(t), \end{cases} \tag{1.2}$$

where $B_S(t), B_{I,k}(t), B_A(t)$ are independent Brownian motions, and $\sigma_S, \sigma_{I,k}, \sigma_A$ are their intensities. They showed there is a unique nonnegative solution to system (1.2) for any nonnegative initial value and under some conditions there is a stability result like

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t E\|X(t) - E_0\|^2 \text{ or } \limsup_{t \to \infty} \frac{1}{t} \int_0^t E\|X(t) - E^*\|^2$$

is small, provided the diffusion coefficients are sufficiently small, here $X(t)$ denotes the solution of system (1.2), and $E \parallel X(t) - X^* \parallel^2 = E[\sum_{k=1}^{n}(x_k(t) - x_k^*)^2]$.

In this paper, as in [5, 7], we focus on the ergodicity of system (1.2) as $R_0 > 1$, which gives complement results to the results of [6]. The paper is organized as follows. In section 2, we utilize a new stochastic Lyapunov function to show system (1.2) is an ergodic diffusion process if the intensities $\sigma_S, \sigma_{I,k}$ are sufficiently small. In section 3, simulations are made to verify our analytical results.

2. THE ERGODIC PROPERTY OF SYSTEM (1.2)

In this section, we discuss the stochastic dynamics of system (1.2) as $R_0 > 1$. First we introduce some sufficient conditions on the ergodic property of diffusion processes. Let $X(t)$ be a regular temporally homogeneous Markov process in $E_l \subset R^l$ described by the stochastic differential equation

$$dX(t) = b(X)dt + \sum_{r=1}^{k} \sigma_r(X)dB_r(t),$$

and the diffusion matrix is defined as follows

$$A(x) = \big(a_{i,j}(x)\big), \quad a_{i,j}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_r^j(x).$$

Theorem 2.1 ([2]). *Assume there exists a bounded domain $U \subset E_l$ with regular boundary, having the following properties:*

(B.1) *In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.*

(B.2) *If $x \in E_l \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$.*

Then, the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$ with density in E_l such that for any Borel set $B \subset E_l$

$$\lim_{t\to\infty} P(t, x, B) = \mu(B),$$

and

$$P_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f\big(x(t)\big)dt = \int_{E_l} f(x)\mu(dx)\right\} = 1,$$

for all $x \in E_l$ and $f(x)$ being a function integrable with respect to the measure μ.

Remark 2.2. The proof is given in [2]. The existence of a stationary distribution with density can be found in Theorem 4.1, P119 and Lemma 9.4, P138. The ergodicity and the weak convergence can be found in Theorem 5.1, P121 and Theorem 7.1, P130.

To show Assumption (B.1) and (B.2), it suffices to prove that there exists some neighborhood U and a non-negative C^2-function such that $A(x)$ is uniformly elliptical in U and for any $x \in E_l \backslash U$, $LV(x) \le -C$ for some $C > 0$ (we refer the reader to [1], P1163).

Theorem 2.3. *Let* $(S(t), I_k(t), I_2(t), \ldots, I_n(t))$ *be the solution of system* (1.2) *with any initial value* $(S(0), I_1(0), I_2(0), \ldots, I_n(0)) \in R_+^{n+1}$. *If* $R_0 > 1$, σ_S, $\sigma_{I,k}$, $k = 1, \ldots, n$ *are positive, and for* $j = 1, \ldots, n$,

$$2n\mu S^{*2} > S^{*2}\sigma_S^2\left(4C_3 + \frac{C_1\sum_{k=1}^n \beta_k I_k^*}{2} + 4n\right) + \sum_{k=1}^n \frac{I_k^{*2}\sigma_{I,k}^2}{p_k}\left[\frac{(C_1+C_2)\beta_k S_k^*}{2} + \frac{2}{p_k}\right],$$

$$\frac{\mu+\gamma_k}{p_j^2} I_j^{*2} > S^{*2}\sigma_S^2\left(2C_3 + \frac{C_1\sum_{k=1}^n \beta_k I_k^*}{2} + 2n\right) + \sum_{k=1}^n \frac{I_k^{*2}\sigma_{I,k}^2}{p_k}\left[\frac{(C_1+C_2)\beta_k S_k^*}{2} + \frac{2}{p_k}\right] + \frac{2I_j^{*2}\sigma_{I,j}^2}{p_j^2},$$

then the diffusion process (1.2) *is ergodic and converges weakly to the stationary distribution* μ, *where* $(S^*, I_1^*, I_2^*, \ldots, I_n^*)$ *is the endemic equilibrium of system* (1.1) *and*

$$C_1 = \sum_{k=1}^n \frac{(2\mu+\gamma_k)^2}{\mu^2(\mu+\gamma_k)}, \quad C_2 = \frac{1}{\mu\sum_{k=1}^n \beta_k I_k^*}\sum_{k=1}^n \frac{(2\mu+\gamma_k)^2}{\mu+\gamma_k}, \quad C_3 = \frac{1}{2\mu}\sum_{k=1}^n \frac{(2\mu+\gamma_k)^2}{\mu+\gamma_k}, \tag{2.1}$$

Proof. When $R_0 > 1$, there is the endemic equilibrium $E^* = (S^*, I_1^*, \ldots, I_n^*)$. Setting the right-hand sides of system (1.1) to be zero, we get

$$\mu S^0 - \mu S^* - \sum_{j=1}^n \beta_j I_j^* S^* = 0, \ p_k \sum_{j=1}^n \beta_j I_j^* S^* - (\mu+\gamma_k) I_k^* = 0, \ k = 1, 2, \ldots, n,$$

which gives

$$\mu S^0 = \mu S^* + \sum_{j=1}^n \beta_j I_j^* S^*, \ \sum_{j=1}^n \beta_j I_j^* S^* = \frac{\mu+\gamma_k}{p_k} I_k^*, \ k = 1, 2, \ldots, n. \tag{2.2}$$

Define

$$V_1(S, I_1, I_2, \ldots, I_n) = \sum_{k=1}^n a_k\left[(S - S^* - S^*\log\frac{S}{S^*}) + \frac{1}{p_k}(I_k - I_k^* - I_k^*\log\frac{I_k}{I_k^*})\right],$$

where $a_k, k = 1, 2, \ldots, n$ are positive constants to be determined later. Then

$$dV_1 = LV_1 dt + \sum_{k=1}^n a_k\left[\left(1 - \frac{S^*}{S}\right)\sigma_S S dB_S(t) + \frac{1}{p_k}\left(1 - \frac{I_k^*}{I_k}\right)\sigma_{I,k} I_k dB_{I,k}(t)\right].$$

where

$$LV_1 := \sum_{k=1}^{n} a_k[(1-\frac{S^*}{S})(\mu S^0 - \mu S - \sum_{j=1}^{n}\beta_j S I_j) + \frac{S^*\sigma_S^2}{2}] \tag{2.3}$$
$$+\sum_{k=1}^{n} a_k[(1-\frac{I_k^*}{I_k})(\sum_{j=1}^{n}\beta_j S I_j - \frac{\mu+\gamma_k}{p_k}I_k) + \frac{I_k^*\sigma_{I,k}^2}{2p_k}].$$

Note that (2.2) implies

$$\begin{aligned}
LV_1 =& \sum_{k=1}^{n} a_k\Bigg[2\mu S^* + \sum_{j=1}^{n}\beta_j S^* I_j^* - \mu S - \frac{\mu+\gamma_k}{p_k}I_k^*\frac{I_k}{I_k^*} - \frac{\mu S^{*2}}{S} - \sum_{j=1}^{n}\frac{\beta_j S^{*2} I_j^*}{S}\\
&+\sum_{j=1}^{n}\beta_j S^* I_j - \sum_{j=1}^{n}\beta_j S I_j\frac{I_k^*}{I_k} + \frac{\mu+\gamma_k}{p_k}I_k^* + \frac{S^*\sigma_S^2}{2} + \frac{I_k^*\sigma_{I,k}^2}{2p_k}\Bigg]\\
=& -\sum_{k=1}^{n}\mu S^* a_k\left[\frac{S}{S^*}+\frac{S^*}{S}-2\right] - \left(\sum_{k=1}^{n} a_k\frac{I_k}{I_k^*}\right)\left(\sum_{j=1}^{n}\beta_j S^* I_j^*\right)\\
&-\left(\sum_{k=1}^{n}\frac{a_k S^*}{S}\right)\left(\sum_{j=1}^{n}\beta_j S^* I_j^*\right) + \left(\sum_{k=1}^{n} a_k\right)\left[\sum_{j=1}^{n}\beta_j S^* I_j^*\frac{I_j}{I_j^*}\right]\\
&-\sum_{k=1}^{n} a_k\left[\sum_{j=1}^{n}\beta_j S^* I_j^*\frac{S}{S^*}\frac{I_j}{I_j^*}\frac{I_k^*}{I_k}\right]\\
&+2(\sum_{k=1}^{n} a_k)\left[\sum_{j=1}^{n}\beta_j S^* I_j^*\right] + \sum_{k=1}^{n}\frac{a_k S^*\sigma_S^2}{2} + \left(\sum_{k=1}^{n}\frac{a_k I_k^*\sigma_{I,k}^2}{2p_k}\right).
\end{aligned} \tag{2.4}$$

The fact that $x-1-\ln x \geq 0$ for $x>0$, yields

$$\sum_{k=1}^{n}\frac{a_k S^*}{S} \geq \sum_{k=1}^{n} a_k\left(1+\ln\frac{S^*}{S}\right),$$
$$\sum_{j=1}^{n}\beta_j S^* I_j^*\frac{S}{S^*}\frac{I_j}{I_j^*}\frac{I_k^*}{I_k} \geq \sum_{j=1}^{n}\beta_j S^* I_j^*\left(1+\ln\frac{S}{S^*}+\ln\frac{I_j}{I_j^*}+\ln\frac{I_k^*}{I_k}\right).$$

Substituting the inequalities above into (2.4), we get

$$\begin{aligned}
LV_1 \leq& -\sum_{k=1}^{n}\mu S^* a_k\left(\frac{S}{S^*}+\frac{S^*}{S}-2\right) - \left(\sum_{k=1}^{n} a_k\frac{I_k}{I_k^*}\right)\left(\sum_{j=1}^{n}\beta_j S^* I_j^*\right)\\
&-\left(\sum_{k=1}^{n} a_k\right)\sum_{j=1}^{n}\beta_j S^* I_j^*\ln\frac{I_j}{I_j^*} + \left(\sum_{k=1}^{n} a_k\right)\sum_{j=1}^{n}\beta_j S^* I_j^*\frac{I_j}{I_j^*} + \sum_{k=1}^{n}\frac{a_k S^*\sigma_S^2}{2}\\
&+\left(\sum_{k=1}^{n}\frac{a_k I_k^*\sigma_{I,k}^2}{2p_k}\right) - \left(\sum_{j=1}^{n}\frac{\beta_j S^* I_j^*}{1+\alpha_j I_j^*}\right)\sum_{k=1}^{n} a_k\ln\frac{I_k^*}{I_k}.
\end{aligned}$$

Let $a_k = \beta_k S^* I_k^*$ and we have

(2.5)

$$LV_1 \leq -\mu S^* \left(\sum_{k=1}^{n} \beta_k S^* I_k^* \right) \left(\frac{S}{S^*} + \frac{S^*}{S} - 2 \right) + \left(\sum_{k=1}^{n} \beta_k S^* I_k^* \right) \frac{S^* \sigma_S^2}{2} + \sum_{k=1}^{n} \frac{\beta_k S^* I_k^{*2} \sigma_{I,k}^2}{2p_k}.$$

Define the C^2 function $V_2 : R^n \to R^+$ as

$$V_2(I_1, I_2, \ldots, I_n) = \sum_{k=1}^{n} a_k \left(\frac{I_k}{p_k} - \frac{I_k^*}{p_k} - \frac{I_k^*}{p_k} \log \frac{I_k}{I_k^*} \right),$$

where a_k, $k = 1, \ldots, n$ are the positive constants defined above. By computation, we note that

$$\begin{aligned}
LV_2 &= \sum_{k=1}^{n} a_k \left(\frac{1}{p_k} - \frac{I_k^*}{p_k I_k} \right) \left(p_k \sum_{j=1}^{n} \beta_j S I_j - (\mu + \gamma_k) I_k \right) + \sum_{k=1}^{n} \frac{a_k I_k^* \sigma_{I,k}^2}{2p_k} \\
&= \left(\sum_{k=1}^{n} a_k \right) \sum_{j=1}^{n} \beta_j (S - S^*)(I_j - I_j^*) + \left(\sum_{k=1}^{n} a_k \right) \sum_{j=1}^{n} (S - S^*) \beta_j I_j^* \\
&\quad - + \left(\sum_{k=1}^{n} a_k \right) \sum_{j=1}^{n} \beta_j S^* I_j \sum_{k=1}^{n} \frac{a_k(\mu + \gamma_k)}{p_k} I_k - \left(\sum_{k=1}^{n} \frac{a_k I_k^*}{I_k} \right) \left(\sum_{j=1}^{n} \beta_j S I_j \right) \\
&\quad + \sum_{k=1}^{n} \frac{a_k(\mu + \gamma_k)}{p_k} I_k^* + \sum_{k=1}^{n} \frac{a_k I_k^* \sigma_{I,k}^2}{2p_k} \\
&= \left(\sum_{k=1}^{n} a_k \right) \sum_{j=1}^{n} \beta_j (S - S^*)(I_j - I_j^*) + \left(\sum_{k=1}^{n} a_k \right) \sum_{j=1}^{n} \beta_j S^* I_j^* \frac{S}{S^*} \\
&\quad + \left(\sum_{k=1}^{n} a_k \right) \sum_{j=1}^{n} \beta_j S^* I_j^* \frac{I_j}{I_j^*} - \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \beta_j S^* I_j^* \frac{S}{S^*} \frac{I_j}{I_j^*} \frac{I_k^*}{I_k} - \left(\sum_{k=1}^{n} \frac{a_k I_k}{I_k^*} \right) \sum_{j=1}^{n} \beta_j S^* I_j^* \\
&\quad + \sum_{k=1}^{n} \frac{a_k I_k^* \sigma_{I,k}^2}{2p_k}.
\end{aligned}$$

Using the fact that $x - 1 - \ln x \geq 0$ for $x > 0$, we have that

$$\begin{aligned}
LV_2 &\leq \left(\sum_{k=1}^{n} \beta_k S^* I_k^* \right) \sum_{j=1}^{n} \beta_j (S - S^*)(I_j - I_j^*) \\
&\quad + \left(\sum_{k=1}^{n} \beta_j S^* I_k^* \right) \sum_{j=1}^{n} \beta_j S^* I_j^* \left(\frac{S}{S^*} + \ln \frac{S^*}{S} - 1 \right) \\
&\quad + \sum_{k=1}^{n} \frac{\beta_k S^* I_k^{*2} \sigma_{I,k}^2}{2p_k} \qquad (2.6) \\
&\leq \left(\sum_{k=1}^{n} \beta_k S^* I_k^* \right) \sum_{j=1}^{n} \beta_j (S - S^*)(I_j - I_j^*)
\end{aligned}$$

$$+\left(\sum_{k=1}^{n}\beta_k S^* I_k^*\right)^2\left(\frac{S}{S^*}+\frac{S^*}{S}-2\right)+\sum_{k=1}^{n}\frac{\beta_k S^* I_k^{*2}\sigma_{I,k}^2}{2p_k}.$$

Define the C^2 function $V_3 : R_+ \to R^+$ as

$$V_3(S) = (S - S^*)^2.$$

Thus

(2.7)

$$\begin{aligned}LV_3 &= 2(S-S^*)(\mu S^* - \mu S - \sum_{j=1}^{n}\beta_j S I_j + \sum_{j=1}^{n}\beta_j S^* I_j^*) + \sigma_S^2 S^2\\ &= -2\mu(S-S^*)^2 - 2\sum_{j=1}^{n}\beta_j (S-S^*)^2 I_j - 2S^*\sum_{j=1}^{n}\beta_j(S-S^*)(I_j - I_j^*) + \sigma_S^2 S^2\\ &\le -2\mu(S-S^*)^2 - 2S^*\sum_{j=1}^{n}\beta_j(S-S^*)(I_j-I_j^*) + \sigma_S^2 S^2.\end{aligned}$$

Define the C^2 function $V_4 : R_+^{n+1} \to R_+$ as

$$V_4(S, I_1, I_2, \dots, I_n) = \sum_{k=1}^{n}(S - S^* + \frac{I_k}{p_k} - \frac{I_k^*}{p_k})^2.$$

By computation,

$$\begin{aligned}LV_4 &= 2\sum_{k=1}^{n}\left(S - S^* + \frac{I_k}{p_k} - \frac{I_k^*}{p_k}\right)\left(\mu S^* - \mu S - \frac{\mu+\gamma_k}{p_k}I_k + \frac{\mu+\gamma_k}{p_k}I_k^*\right)\\ &\quad + \sum_{k=1}^{n}(\sigma_S^2 S^2 + \frac{\sigma_{I,k}^2 I_k^2}{p_k^2})\\ &= -2n\mu(S-S^*)^2 - 2\sum_{k=1}^{n}\frac{\mu+\gamma_k}{p_k^2}(I_k - I_k^*)^2 - 2\sum_{k=1}^{n}\frac{2\mu+\gamma_k}{p_k}(S-S^*)(I_k-I_k^*)\\ &\quad + \sum_{k=1}^{n}\left(\sigma_S^2 S^2 + \frac{\sigma_{I,k}^2 I_k^2}{p_k^2}\right).\end{aligned}$$

Using the fact that $2ab \le a^2 + b^2$, we note

$$\begin{aligned}LV_4 \le -2n\mu(S-S^*)^2 - \sum_{k=1}^{n}\frac{\mu+\gamma_k}{p_k^2}(I_k-I_k^*)^2 + \sum_{k=1}^{n}\frac{(2\mu+\gamma_k)^2}{\mu+\gamma_k}(S-S^*)^2\\ + \sum_{k=1}^{n}(\sigma_S^2 S^2 + \frac{\sigma_{I,k}^2 I_k^2}{p_k^2}).\end{aligned} \tag{2.8}$$

Finally define the C^2 function $V : R_+^{n+1} \to R_+$ as

$$V = C_1 V_1 + C_2 V_2 + C_3 V_3 + V_4,$$

where C_1, C_2 and C_3 are the positive constants defined in (2.1).

Taking (2.5), (2.6), (2.7) and (2.8) into account, we have

$$LV \le -2n\mu(S-S^*)^2 - \sum_{k=1}^{n} \frac{\mu+\gamma_k}{p_k^2}(I_k - I_k^*)^2 + C_3\sigma_S^2 S^2 + C_2 \sum_{k=1}^{n} \frac{\beta_k S_k^* I_k^{*,2}\sigma_{I,k}^2}{2p_k}$$
$$+ \frac{C_1\sigma_S^2}{2}\sum_{k=1}^{n}\beta_k S^{*2} I_k^* + C_1 \sum_{k=1}^{n} \frac{\beta_k S_k^* I_k^{*,2}\sigma_{I,k}^2}{2p_k} + \sum_{k=1}^{n}(\sigma_S^2 S^2 + \frac{\sigma_{I,k}^2 I_k^2}{p_k^2}).$$

Since $a^2 \le 2(a-b)^2 + 2b^2$, then

(2.9)

$$LV \le -2\left[n\mu - (C_3+n)\sigma_S^2\right](S-S^*)^2 - \sum_{k=1}^{n}(\frac{\mu+\gamma_k}{p_k^2} - \frac{2\sigma_{I,k}^2}{p_k^2})(I_k - I_k^*)^2$$
$$+ S^{*2}\sigma_S^2(2C_3 + \frac{C_1\sum_{k=1}^{n}\beta_k I_k^*}{2} + 2n) + \sum_{k=1}^{n} \frac{I_k^{*2}\sigma_{I,k}^2}{p_k}\left[\frac{(C_1+C_2)\beta_k S_k^*}{2} + \frac{2}{p_k}\right]$$
$$= -2\left[n\mu - (C_3+n)\sigma_S^2\right](S-S^*)^2 - \sum_{k=1}^{n}(\frac{\mu+\gamma_k}{p_k^2} - \frac{2\sigma_{I,k}^2}{p_k^2})(I_k - I_k^*)^2 + A,$$

where $A = S^{*2}\sigma_S^{*2}(2C_3 + \frac{C_1\sum_{k=1}^{n}\beta_k I_k^*}{2} + 2n) + \sum_{k=1}^{n} \frac{I_k^{*2}\sigma_{I,k}^2}{p_k}\left[\frac{(C_1+C_2)\beta_k S_k^*}{2} + \frac{2}{p_k}\right]$. Note that if $\sigma_S^2 < \frac{n\mu}{C_3+n}$, $\sigma_{I,k}^2 < \frac{\mu+\gamma_k}{2}$, and $2\left[n\mu - (C_3+n)\sigma_S^2\right]S^{*2} > A$, $(\frac{\mu+\gamma_k}{p_k^2} - \frac{2\sigma_{I,k}^2}{p_k^2})I_k^{*2} > A$, $k = 1,\dots,n$ (i.e. the conditions in Theorem 2.3 hold), then the ellipsoid

$$-2\left[n\mu - (C_3+n)\sigma_S^2\right](S-S^*)^2 - \sum_{k=1}^{n}(\frac{\mu+\gamma_k}{p_k^2} - \frac{2\sigma_{I,k}^2}{p_k^2})(I_k - I_k^*)^2 + A = 0$$

lies entirely in R_+^{n+1}. We can take U to be some neighborhood of the ellipsoid with $\bar{U} \subseteq E_l = R_+^{n+1}$, so for $x \in U \setminus E_l$, $LV \le -C$ for some $C > 0$, which implies condition (B.2) in Lemma (2.1) is satisfied. Also, there is a $M > 0$ such that

$$\sum_{i,j=1}^{n+1}\left(\sum_{k=1}^{n} a_{ik}(x)a_{jk}(x)\right)\xi_i\xi_j = \sigma_S^2 x_1^2 \xi_1^2 + \sum_{k=1}^{n}\sigma_{I,k}^2 x_{k+1}^2 \xi_{k+1}^2$$
$$\ge M \mid \xi \mid^2 \ \text{ all } x \in \bar{U}, \xi \in R^{n+1}.$$

Applying Rayleigh's principle ([9, p342]), condition (B.1) is satisfied. Therefore, the stochastic system (1.2) has a unique stationary distribution $\mu(\cdot)$ and it is an ergodic diffusion process. □

Remark 2.4. From the results of Theorem 2.5 in [6], we can see, if X^* is the equilibrium of the system (1.1), but not of system (1.2), then under some conditions,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E[\|X(s) - X^*\|^2]ds < O(\sigma^2),$$

where $X(t)$ is the solution of system (1.2), $\|X(s)-X^*\|^2 = \sum_{k=1}^{n}(X_k(s)-X_k^*)^2$ and $\sigma^2 = \sigma_S^2 + \sum_{k=1}^{n}\sigma_{I,k}^2$.

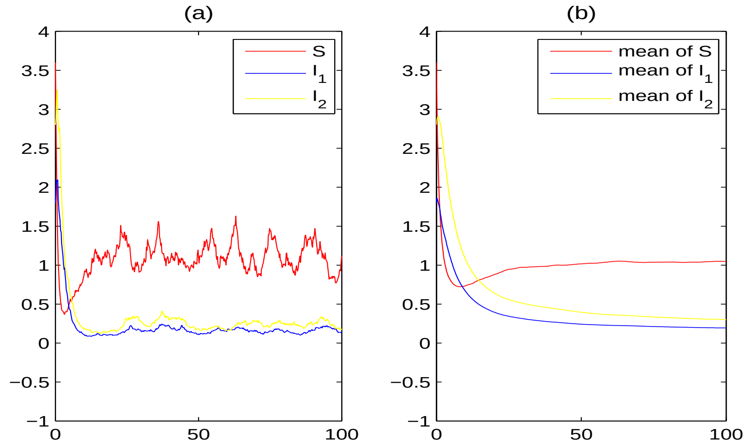
In fact, by the ergodic property of system (1.2), the $\limsup_{t\to\infty}$ can be replaced by $\lim_{t\to\infty}$ and the deviation between the vector $(x_0, x_1, \ldots, x_n)$ and E^* in the $L^2(\mu)$ norm is dominated by the intensities of white noises.

3. SIMULATION

We use the Milstein's higher order method in [3] to find the strong solution of system (1.2) with given initial value and the values of parameters for $n = 2$. The corresponding discretization equation is

$$\begin{cases} S_{k+1} = S_k + \left(\mu S^0 - \mu S_k - \sum_{j=1}^{2}\beta_j S_k I_{j,k}\right)\triangle t + \sigma_S S_k\sqrt{\triangle t}\xi_{1,k} \\ \qquad + \frac{\sigma_S^2}{2}S_k(\triangle t\xi_{1,k}^2 - \triangle t), \\ I_{1,k+1} = I_{1,k} + \left[p_1\sum_{j=1}^{2}\beta_j S_k I_{j,k} - (\mu+\gamma_1)I_{1,k}\right]\triangle t + \sigma_{I,1}I_{1,k}\sqrt{\triangle t}\xi_{2,k} \\ \qquad + \frac{\sigma_{I,1}^2}{2}I_{1,k}(\triangle t\xi_{2,k}^2 - \triangle t), \\ I_{2,k+1} = I_{2,k} + \left[p_2\sum_{j=1}^{2}\beta_j S_k I_{j,k} - (\mu+\gamma_2)I_{2,k}\right]\triangle t + \sigma_{I,2}I_{2,k}\sqrt{\triangle t}\xi_{3,k} \\ \qquad + \frac{\sigma_{I,2}^2}{2}I_{2,k}(\triangle t\xi_{3,k}^2 - \triangle t), \end{cases}$$

where $\xi_{1,k}, \xi_{2,k}$ and $\xi_{3,k}$, $k = 1, 2, \ldots, n$ are independent Gaussian random variables $N(0,1)$, and $\sigma_1, \sigma_2, \sigma_3$ are intensities of white noises. We choose $(S(0), I_1(0), I_2(0)) = (3.6, 1.8, 2.8)$, $\triangle t = 0.2$ and the parameters $S_0 = 2$, $\mu = 0.2$, $\gamma_1 = \gamma_2 = 0.3$, $p_1 = 0.4$, $p_2 = 0.6$, $\beta_1 = \beta_2 = 0.5$, $\sigma_S = \sigma_{I,1} = \sigma_{I,2} = 0.1$ such that the conditions of Theorem 2.3 are satisfied, and the simulations conform the results from visual. Specifically, the left picture (a) in the figure shows the solution of system (1.2) is fluctuating around a fixed point (S^*, I_1^*, I_2^*), where the red, blue and yellow lines represent the population S, I_1, I_2, respectively; In the right picture (b) of the figure, we give the simulation of $\frac{1}{t}\int_0^t S(s)ds$, $\frac{1}{t}\int_0^t I_1(s)ds$ and $\frac{1}{t}\int_0^t I_2(s)ds$ to conform the ergodicity of system (1.2), which are also represented by red, blue and yellow lines, respectively.



REFERENCES

[1] T. C. Gard, Introduction to stochastic differential equations, Marcel Dekker Inc, 1988.

[2] R.Z. Has'minskii, Stochastic stability of differential equations, Sijthoff and Noordhoff, Germantown, MD, 1980.

[3] D. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, Siam Rev. 43(2001) 525–546.

[4] J. Hyman, J. Li, The differential infectivity and staged progression models for the transmission of HIV, Math. Biosci. 155 (1999) 77–109.

[5] C.Y. Ji, D.Q. Jiang, H. Liu, Q.S. Yang, Existence, Uniqueness and ergodicity of positive solution of mutualism system with stochastic perturbation, Math. Probl. Eng. 2010, Article ID 684926, 18 pages.

[6] D.Q. Jiang, C.Y. Ji, N.Z. Shi, J.J. Yu, The long time behavior of DI SIR epidemic model with stochastic perturbation, J. Math. Anal. Appl. 372 (2010) 162–180.

[7] C.Y. Ji, D.Q. Jiang, N.Z. Shi, A note on a predator-prey model with modied Leslie-Gowera Holling-typeII schemes with stochastic perturbation, J. Math. Anal. Appl. (accepted).

[8] Z. Ma, J. Liu, J. Li, Stability analysis for differential infectivity epidemic models, Nonlinear Anal. Real World Appl. 4 (2003) 841–856.

[9] G. Strang, Linear Algebra and Its Applications, Forth Edition, Thomson Leanning, Inc, Latin America, 2006.