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ASYMPTOTIC STABILITY IN DELAYED PERIODIC EQUATIONS BY AVERAGE CONDITIONS

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ABSTRACT. A new criterion is proposed for the global asymptotic stability of the positive periodic solution to the following delay logistic equation

$$u'(t) = u(t)[\,r(t) - a(t)u(t) - b(t)u(t-\tau)]$$

with continuous and periodic coefficients. Such condition, given in average form, incorporates some known pointwise assumptions. The same strategy is applied to study the linear case.

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1. INTRODUCTION

Delay differential equations arise in many applications. In particular, there is a great variety of processes in the biological world that involve significant delays. The logistic equations play an important role in models of population growth and their study provides a basilar contribution in the development of the theory of delay differential equations. This article discusses the asymptotic behaviour of all positive solutions to the following delay logistic equation

$$u'(t) = u(t)[\,r(t) - a(t)u(t) - b(t)u(t-\tau)] \tag{1.1}$$

with continuous and T-periodic coefficients. Moreover $a(t) > 0$, $b(t) \geq 0$ and $r(t)$ has a positive mean value, that is

$$m[r] = \frac{1}{T}\int_0^T r(s)\,ds \;> 0\,.$$

It is known (see [2]) that (1.1) admits a positive periodic solution $\overset{\circ}{u}(t)$. Our aim is to study its asymptotic stability by a suitable mean condition involving coefficients $a(t)$, $b(t)$ and $\overset{\circ}{u}(t)$ itself. Freedman and Wu [3] considered the equation

$$u'(t) = u(t)[\,r(t) - a(t)u(t) + b(t)u(t-\tau)]$$

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with $r(t)$, $a(t)$, $b(t)$ continuously differentiable, T-periodic functions, $r(t) > 0$, $a(t) > 0$ and $b(t) \geq 0$. They proved the existence of a positive periodic solution $\overset{\circ}{u}(t)$ and they employed the Razumikhin theorem to show that it is globally attractive if

$$a(t) > \frac{b(t)\,\overset{\circ}{u}(t-\tau)}{\overset{\circ}{u}(t)} \tag{1.2}$$

for all $t \in [0,T]$. Their stability argument can be employed in our case, too, therefore their attractivity result holds also for equation (1.1). An alternative formulation for a delayed logistic equation is

$$x'(t) = -r(t)(1+x(t))(c\,x(t) + x(t-\tau)) \,. \tag{1.3}$$

In [8], it is proved that, if $c \geq 1$ and $\int_0^{+\infty} r(s)\,ds = +\infty$, then every solution of (1.3), $x(t) > -1$, tends to zero as $t \to \infty$, no matter the length of delay τ. For the case $0 < c < 1$, to obtain the same result, the author introduces the inequality (depending on τ)

$$\int_{t-\tau}^{t} r(s)\,ds \leq \frac{1}{2} - \frac{1}{c}\ln(1-c) - \frac{c}{6}\,, \quad \text{for large } t\,.$$

Going back to the logistic equation in form (1.1), introducing a suitable Lyapunov functional, one can obtain the global attractivity of equation (1.1) under the inequality

$$a(t) > b(t+\tau)$$

(see [1],[7],[9]), which can be rewritten in the equivalent form

$$a(t)\overset{\circ}{u}(t) > b(t+\tau)\overset{\circ}{u}(t) \,. \tag{1.4}$$

A previous contribution to the subject of this paper is given in [6] by means of a comparison technique.

Another recent result can be found in [7], in which the author proposes the following assumption for the global attractivity of a periodic solution $\overset{\circ}{u}(t)$

$$a(t)\overset{\circ}{u}(t) > \frac{b(t)\overset{\circ}{u}(t-\tau) + b(t+\tau)\overset{\circ}{u}(t)}{2} \,. \tag{1.5}$$

Now, a natural question is whether, for equation (1.1), it is possible to introduce a more general assumption, ensuring the cited stability property, which incorporates (1.2), (1.4) and (1.5) but takes in a deeper account the periodicity of coefficients. In Theorem 3.3 we demonstrate that the following average condition

$$m[a(t)\,\overset{\circ}{u}(t)] > m[b(t)\,\overset{\circ}{u}(t-\tau)] \tag{1.6}$$

has the sought requisites, giving an affirmative answer to the conjecture advanced in [7].

The basic strategy of Theorem 3.3 can also be applied to the linear case. Indeed, in Theorem 3.4, we prove that average inequality $m[a(t] > m[b(t)]$ guarantees the global asymptotic stability of the linear delay equation

$$x'(t) = -a(t)x(t) - b(t)x(t-\tau) .$$

In this way we extend the relative result proved in [6] in which the relationship between $a(t)$ and $b(t)$ was the following one

$$a(t) - b(t)e^{\int_{t-\tau}^{t}\gamma(s)ds} \geq \gamma(t) ,$$

where $\gamma(t)$ is a T-periodic function with positive average.

2. AN ALTERNATIVE LOGISTIC EQUATION

Let $\alpha(t), \beta(t)$ be continuous, T-periodic functions, $\alpha(t) > 0$, $\beta(t) \geq 0$. In this section we investigate the asymptotic behaviour of solutions $x(t)$, $x(t) > -1$, to the following delay equation

$$x'(t) = (1 + x(t))(-\alpha(t)\,x(t) - \beta(t)\,x(t-\tau)) , \tag{2.1}$$

starting from the corresponding differential equation without the delay term.

Lemma 2.1. *Let $q(t)$ be a continuous, T-periodic function with*

$$m[q(t)] = \frac{1}{T}\int_0^T q(s)\,ds > 0$$

and let $y(t)$ be a solution of equation

$$y'(t) = -(1 + y(t))q(t)\,y(t) \tag{2.2}$$

such that $y(t) > -1$. Then one has

$$\lim_{t\to+\infty} y(t) = 0 .$$

Proof. Note that, making the substitution

$$z(t) = \frac{y(t)}{1 + y(t)}$$

equation (2.2) turns into the linear differential equation

$$z'(t) = -q(t)\,z(t)$$

whose solutions vanish at infinity, because $m[q(t)] > 0$. As a consequence, our statement easily follows. □

Owing to the presence of delay τ, previous argument cannot extend to related delay equations (2.1). First observe that, at the values of t for which a solution $x(t)$ satisfies $x(t) = -x(t-\tau)$, one gets

$$x'(t) = (1 + x(t))(-\alpha(t)\,x(t) + \beta(t)\,x(t))$$

which has the form of equation (2.2) with

$$q(t) = \alpha(t) - \beta(t) \ .$$

Then Lemma 2.1 may suggest that

$$m[\alpha(t)] > m[\beta(t)] \tag{2.3}$$

could be a possible sufficient condition for the asymptotic stability of equation (2.1).

On the other hand, delay equation (2.1) can be rewritten as

$$x'(t) = (1 + x(t))[-(\alpha(t) + \beta(t))\, x(t) + \beta(t)\, (x(t) - x(t - \tau))] \ .$$

In this way, one regards equation (2.1) as a perturbed equation of a nondelayed equation of type (2.2). If time lag τ is small enough, it is reasonable to suppose that perturbated equation (2.1) has a similar asymptotic behaviour as equation

$$x'(t) = (1 + x(t))[-(\alpha(t) + \beta(t))\, x(t)]$$

since addendum $\beta(t)\,(x(t) - x(t - \tau))$ is small. On the contrary, our strategy is to find a delay independent result, so that we have to require that coefficient $\beta(t)$ is *small*, in some sense, with respect to $\alpha(t)$. Some pointwise conditions are already known ([1],[3],[4],[5],[6],[7],[9]). We are going to prove, in Theorem 2.1, that average inequality (2.3) is sufficient for our object.

Definition 2.1. A nontrivial solution $x(t)$ to a delay differential equation is said to be oscillatory iff there exists a sequence $\{t_n\}$ of its zeroes such that $\lim_{n\to\infty} t_n = +\infty$.

The result described in the next theorem plays a central role in reaching our target.

Theorem 2.1. *Assume that inequality* (2.3)

$$m[\alpha(t)] > m[\beta(t)]$$

holds. Then, for every solution $x(t)$, $x(t) > -1$ *of equation* (2.1)*, we have*

$$\lim_{t\to+\infty} x(t) = 0 \ . \tag{2.4}$$

Proof. Let $x(t)$ be a solution of delay equation (2.1), greater than -1, and assume that $x(t)$ is oscillatory. Otherwise, since $x(t)$ is positive (or negative), for t great enough, property (2.4) easily follows from Lemma 2.1 and comparison results for differential equations.

Consider the Lyapunov function

$$V(t) = x(t) - \ln(1 + x(t)) + \frac{1}{2}\int_{t-\tau}^{t} \beta(s + \tau)\, x^2(s)\, ds \ .$$

Note the dependence of $V(t)$ on delay τ and the values of $x(s)$, $t - \tau \le s \le t$, accordingly to what usually happens for delayed equations.

Easy calculations lead to

$$V'(t) = -\alpha(t)\, x^2(t) - \beta(t)\, x(t-\tau)x(t) + \frac{1}{2}\beta(t+\tau)\, x^2(t) - \frac{1}{2}\beta(t)\, x^2(t-\tau)\ .$$

Adding and subtracting term $\frac{\beta(t)}{2}x^2(t)$, we deduce

$$V'(t) = -\alpha(t)\, x^2(t) + \frac{\beta(t)+\beta(t+\tau)}{2}x^2(t) - \frac{\beta(t)}{2}x^2(t) - \frac{\beta(t)}{2}x^2(t-\tau) - \beta(t)\, x(t-\tau)x(t)\ .$$

Setting

$$\lambda(t) = \alpha(t) - \frac{\beta(t)+\beta(t+\tau)}{2}\ .$$

we obtain the following simple expression for the derivative of $V(t)$

$$V'(t) = -\lambda(t)\, x^2(t) - \frac{\beta(t)}{2}\,(x(t)+x(t-\tau))^2 \tag{2.5}$$

where, by hypothesis (2.3),

$$m[\lambda(t)] > 0\ . \tag{2.6}$$

By contradiction, suppose that $x(t)$ does not vanish at infinity and let $\{t_0, t_1, t_2, \dots\}$ be the sequence of its zeroes.

Integrating both sides of (2.5) between t_0 and t, one gets

$$\int_{t_0}^{t} \left(\lambda(s)\, x^2(s) + \frac{\beta(s)}{2}(x(s)+x(s-\tau))^2\right) ds \;=\; V(t_0) - V(t)\ . \tag{2.7}$$

From the contradiction hypothesis on $x(t)$, it follows

$$\lim_{t\to+\infty} \int_{t_0}^{t} \frac{\beta(s)}{2}(x(s)+x(s-\tau))^2\, ds \;= +\infty\ . \tag{2.8}$$

For simplicity, set $h(t) = x^2(t)$. The next step of our proof consists in showing the following claim:

(C) There exists a positive integer k such that, for each $n \geq k$, we have

$$\int_{t_{n-1}}^{t_n} \lambda(s)h(s)\, ds \;> 0\ .$$

Set

$$\Lambda(t) = \int_0^t \lambda(s)\, ds\ .$$

It easy to check that it is possible to write $\lambda(t)$ and $\Lambda(t)$ in the form

$$\lambda(t) = m[\lambda(t)] + q(t), \quad m[q(t)] = 0,$$

$$\Lambda(t) = m[\lambda(t)]\, t + p(t), \quad p(t) \text{ T-periodic.}$$

Let us denote by $\{\sigma_n\}$ the sequence of the maximum and minimum points of $x(t)$, $t_{n-1} < \sigma_n < t_n$, for every n. Each σ_n is a maximum point for $h(t)$, then

$$h'(s) \geq 0 \quad \text{in } [t_{n-1}, \sigma_n], \qquad h'(s) \leq 0 \quad \text{in } [\sigma_n, t_n]\ .$$

Fix a positive integer n. Applying the mean value theorem, we can find $\xi_n \in [t_{n-1}, t_n]$ such that

$$\int_{t_{n-1}}^{t_n} \lambda(s)\, h(s)\, ds = \lambda(\xi) \int_{t_{n-1}}^{t_n} h(s)\, ds = m[\lambda(t)] \int_{t_{n-1}}^{t_n} h(s)\, ds + q(\xi_n) \int_{t_{n-1}}^{t_n} h(s)\, ds \ .$$

Since

$$\int_{t_{n-1}}^{t_n} h(s)\, ds = \int_{t_{n-1}}^{t_n} (-sh'(s))\, ds$$

it follows

$$\int_{t_{n-1}}^{t_n} \lambda(s)\, h(s)\, ds = \tag{2.9}$$

$$m[\lambda(t)] \int_{t_{n-1}}^{t_n} h(s)\, ds + q(\xi_n) \int_{t_{n-1}}^{\sigma_n} (-sh'(s))\, ds + q(\xi_n) \int_{\sigma_n}^{t_n} (-sh'(s))\, ds \ .$$

On the other hand, integrating by parts

$$\int_{t_{n-1}}^{t_n} \lambda(s)\, h(s)\, ds = \int_{t_{n-1}}^{t_n} \frac{\Lambda(s)}{s} (-sh'(s))\, ds \ .$$

Using again the mean value theorem, there exist $\alpha_n \in [t_{n-1}, \sigma_n]$ and $\beta_n \in [\sigma_n, t_n]$ such that

$$\int_{t_{n-1}}^{t_n} \frac{\Lambda(s)}{s} (-sh'(s))\, ds = \tag{2.10}$$

$$m[\lambda(t)] \int_{t_{n-1}}^{t_n} h(s)\, ds + \frac{p(\alpha_n)}{\alpha_n} \int_{t_{n-1}}^{\sigma_n} (-sh'(s))\, ds + \frac{p(\beta_n)}{\beta_n} \int_{\sigma_n}^{t_n} (-sh'(s))\, ds \ .$$

Comparing (2.9) and (2.10), one deduces

$$\frac{p(\alpha_n)}{\alpha_n} = \frac{p(\beta_n)}{\beta_n} = q(\xi_n)$$

which implies that

$$\lim_{n \to +\infty} q(\xi_n) = 0 = m[q(t)] \ .$$

From (2.9), (2.6) and previous property, we obtain claim (C). As a consequence, for n great enough

$$\int_{t_0}^{t_n} \lambda(s) x^2(s)\, ds > 0 \ .$$

Taking into account (2.8), previous inequality implies

$$\lim_{n \to \infty} \int_{t_0}^{t_n} \left(\lambda(s)\, x^2(s) + \frac{\beta(s)}{2} (x(s) + x(s-\tau))^2 \right) ds \ = +\infty \ .$$

On the contrary, using (2.7)

$$\lim_{n \to \infty} \int_{t_0}^{t_n} \left(\lambda(s)\, x^2(s) + \frac{\beta(s)}{2} (x(s) + x(s-\tau))^2 \right) ds \ \leq V(t_0) \ .$$

We conclude that $x(t)$ has to vanish as t goes to infinity, that is (2.4) is proved. □

3. MAIN RESULTS

It is well known that, for any continuous, initial condition $\phi(t)$, there exists a unique solution $u(t)$ of

$$u'(t) = u(t)[\,r(t) - a(t)u(t) - b(t)u(t-\tau)]\ .$$

If $\phi(t) \geq 0$ and $\phi(0) > 0$ then $u(t) > 0$ for $t > 0$. We will call positive such type of solutions.

Next, let us verify that all positive solutions to (1.1) are bounded from above. We assume that $r(t), a(t), b(t)$ are continuous, T-periodic functions, $m[r] > 0$, $a(t) > 0$, $b(t) \geq 0$.

Theorem 3.1. *Denote by* $M = \max_{t\in[0,T]} \dfrac{r(t)}{a(t)} > 0$, *then, for any positive solution* $u(t)$ *to equation* (1.1), *there exists* $t_0 > 0$ *such that*

$$0 < u(t) \leq M + 1, \quad t \geq t_0\ .$$

Proof. Let $\overset{\circ}{v}(t)$ be the positive periodic solution to the logistic equation

$$v'(t) = v(t)\,(r(t) - a(t)\,v(t))\ . \tag{3.1}$$

whose existence is well known.

Consider $\bar{t} > 0$, a maximum point for $\overset{\circ}{v}(t)$, hence

$$\overset{\circ}{v}(\bar{t}) = \frac{r(\bar{t})}{a(\bar{t})} \leq M\ .$$

Since $\overset{\circ}{v}(t)$ attracts all positive solutions, we have

$$\lim_{t\to+\infty} |v(t) - \overset{\circ}{v}(t)| = 0$$

for any positive solution $v(t)$ to logistic equation (3.1). Now, taking $u(t)$ positive solution to (1.1), one yields

$$u'(t) \leq u(t)\,(r(t) - a(t)\,u(t))$$

so that, by comparison results, there exists $t_0 > 0$ such that

$$u(t) \leq v(t) + 1 \leq M + 1, \quad t \geq t_0\ .$$

as required. □

Delay equation (1.1) admits a positive periodic solution as a consequence of the following theorem, proved in [2], using the method of coincidence degree.

Theorem 3.2. *Let* $0 < p \leq q$ *and suppose coefficients* $r(t)$, $a(t)$ *and* $b(t)$ *are as above. Then the following differential equation with delays* σ *and* τ

$$N'(t) = N(t)[\,r(t) - a(t)N^p(t-\sigma) - b(t)N^q(t-\tau)]$$

has at least one positive periodic solution.

At this point, we are in position to formulate our main result.

Theorem 3.3. *Let* $\overset{\circ}{u}(t)$ *be a positive periodic solution to delay equation* (1.1)

$$u'(t) = u(t)[\,r(t) - a(t)u(t) - b(t)u(t-\tau)]$$

and assume that inequality (1.6)

$$m[a(t)\overset{\circ}{u}(t)] > m[b(t)\overset{\circ}{u}(t-\tau)]$$

holds. Then, for every positive solution $u(t)$ *of previous equation, we have*

$$\lim_{t\to+\infty} |u(t) - \overset{\circ}{u}(t)| = 0\ .$$

Proof. Let $u(t)$ be a positive solution of (1.1) and set

$$x(t) = \frac{u(t)}{\overset{\circ}{u}(t)} - 1\ . \tag{3.2}$$

We find that $x(t)$ is a solution, greater than -1, of delay equation (2.1), where new coefficients $\alpha(t)$ and $\beta(t)$ are related to $a(t)$ and $b(t)$ by

$$\alpha(t) = a(t)\overset{\circ}{u}(t), \quad \beta(t) = b(t)\overset{\circ}{u}(t-\tau)\ .$$

Indeed

$$\begin{aligned} x'(t) &= \frac{u(t)}{\overset{\circ}{u}(t)}\left[\frac{u'(t)}{u(t)} - \frac{\overset{\circ}{u}{}'(t)}{\overset{\circ}{u}(t)}\right] \\ &= \frac{u(t)}{\overset{\circ}{u}(t)}\left[-a(t)\,(u(t) - \overset{\circ}{u}(t)) - b(t)\,(u(t-\tau) - \overset{\circ}{u}(t-\tau))\right] \\ &= (1 + x(t))(-\alpha(t)\,x(t) - \beta(t)\,x(t-\tau))\ . \end{aligned}$$

From (1.6), the condition

$$m[\alpha(t)] > m[\beta(t)]$$

is verified. Therefore, by Theorem 2.1

$$\lim_{t\to+\infty} |x(t)| = 0$$

from which, taking into account substitution (3.2),

$$\lim_{t\to+\infty} |u(t) - \overset{\circ}{u}(t)| = 0$$

according with the statement. □

Example. Consider the delay equation

$$u'(t) = u(t)\left[(2 + \frac{\sin^2 t}{2} + \cos t) - (\frac{3}{2}\sin^2 t + \frac{1}{2})\,u(t) - (\cos^2 t + \cos t + \frac{1}{2})\,u(t-1)\right]\ .$$

Obviously this differential equation has $\overset{\circ}{u}(t) = 1$ as positive 2π-periodic solution. Here

$$a(t) = \frac{3}{2}\sin^2 t + \frac{1}{2}, \quad b(t) = \cos^2 t + \cos t + \frac{1}{2}$$

so that none of conditions (1.2), (1.4), (1.5) is satisfied. On the other hand,

$$\frac{3}{4}+\frac{1}{2}=m[a(t)\cdot 1]>m[b(t)\cdot 1]=1$$

that is assumption (1.6) is verified. We conclude that

$$\lim_{t\to\infty} u(t)=1$$

for each positive solution $u(t)$.

Our last result concerns the linear case.

Theorem 3.4. *Consider the delay differential equation*

$$x'(t)=-a(t)x(t)-b(t)x(t-\tau)\ , \tag{3.3}$$

with coefficients $a(t)$ *and* $b(t)$ *continuous, T-periodic,* $a(t)>0$, $b(t)\geq 0$. *If*

$$m[a(t)]>m[b(t)] \tag{3.4}$$

then any solution of (3.3) *goes to zero as* $t\to\infty$.

Proof. For nonoscillatory solutions the statement easily follows. Now take $x(t)$, oscillatory solution of (3.3) and introduce the Lyapunov function

$$V(t)=\frac{x^2(t)}{2}+\frac{1}{2}\int_{t-\tau}^{t} b(s+\tau)\,x^2(s)\,ds\ .$$

Repeating calculations analogous to those in the proof of Theorem 2.1, one yields

$$V'(t)=-\lambda(t)x^2(t)-\frac{b(t)}{2}(x(t)+x(t-\tau))^2$$

where

$$\lambda(t)=a(t)-\frac{b(t)+b(t+\tau)}{2}\ .$$

Owing to hypothesis (3.4), $\lambda(t)$ has positive mean value.

Let $\{t_0,t_1,t_2,\dots\}$ be the sequence of the zeroes of $x(t)$. If $x(t)$ doesn't vanish at infinity, using again the arguments of Theorem 2.1, for each n, one gets

$$\int_{t_0}^{t_n}\left(\lambda(s)x^2(s)+\frac{b(s)}{2}(x(s)+x(s-\tau)^2)\right)\,ds\ <V(t_0),$$

together with the property

$$\lim_{t\to\infty}\int_{t_0}^{t_n}\left(\lambda(s)x^2(s)+\frac{b(s)}{2}(x(s)+x(s-\tau)^2\right)\,ds=+\infty.$$

We conclude that

$$\lim_{t\to\infty} x(t)=0\ .$$

so that the proof is complete. $\square$

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