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MONOTONE METHOD FOR NONLINEAR CAPUTO FRACTIONAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, by using upper and lower solutions, we develop monotone method for the nonlinear Caputo fractional boundary value problem of order α where $1 < \alpha < 2$. We construct two sequences which converge uniformly and monotonically to the extremal solutions of the nonlinear Caputo fractional boundary value problem.

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1. INTRODUCTION

Qualitative study of fractional differential equations has gained lot of importance recently due to its applications, see [1, 3, 4, 5]. Recently we have developed monotone method for Riemann-Liouville fractional differential equations of order q, $0 < q < 1$, with periodic boundary conditions, see [6].

In this work we develop monotone method for fractional boundary value problem of order α where $1 < \alpha < 2$. For that purpose we develop some basic fractional calculus results which are used as tools to develop comparison results relative to upper and lower solutions of Caputo fractional boundary value problems. Our results include the comparison theorem given in [2] for derivative of order α. We have also computed the Green's function for the linear Caputo fractional boundary value problem. The integral representation of the linear fractional nonhomogeneous problem is useful in developing monotone method for the Caputo fractional boundary value problem.

Finally we prove monotone method for fractional differential equations of order α. Here we combine the method of upper and lower solutions and monotone method to develop two sequences which converge uniformly and monotonically to the minimal and maximal solutions.

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2. COMPARISON THEOREM, GREEN'S FUNCTION AND SOME AUXILIARY RESULTS

In this section we develop some auxiliary results and a comparison theorem relative to boundary value problems with Caputo fractional derivative of order $1 < \alpha < 2$. This will be useful to develop our main result. For that purpose we consider the Boundary Value Problem (BVP):

$$\begin{aligned} -{}^cD^{1+q}u(t) &= f(t, u(t), {}^cD^q u(t)), \\ \alpha_a u(a) - \beta_a {}^cD^q u(a) &= \gamma_a \\ \alpha_b u(b) + \beta_b {}^cD^q u(b) &= \gamma_b, \end{aligned} \tag{2.1}$$

where $\alpha_a, \alpha_b \geq 0$, $\beta_a, \beta_b > 0$, $\gamma_a, \gamma_b \in \mathbb{R}$, and $f \in C\big[[a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big]$.

In (2.1), ${}^cD^\alpha u(t)$ is the Caputo derivative of order $n-1 < \alpha < n$ for $t \in [a,b]$, see [1, 3], where

$${}^cD^\alpha_{a+}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s)ds,$$

and

$${}^cD^\alpha_{b-}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} u^{(n)}(s)ds.$$

In this paper, we study the case $n = 2$ with respect to (2.1). In particular, we choose $0 < q < 1$.

Lemma 2.1. *Let $m(t) \in C^2([a,b], \mathbb{R})$. If there exists $t_1 \in (a,b)$ such that $m(t_1) = 0$ and $m(t) \leq 0$ for $a \leq t \leq b$, then ${}^cD^q m(t_1) = 0$ and ${}^cD^{1+q}m(t_1) \leq 0$.*

Proof. We will compute both $\big({}^cD^q_{a+}m\big)(t_1)$ and $\big({}^cD^q_{b-}m\big)(t_1)$.

By the definition of Caputo derivative and integration by parts,

$$\begin{aligned} \big({}^cD^q_{a+}m\big)(t_1) &= \frac{1}{\Gamma(1-q)} \int_a^{t_1} \frac{m'(s)}{(t_1-s)^q} ds \\ &= \frac{1}{\Gamma(1-q)} \frac{m(s)}{(t_1-s)^q}\bigg|_a^{t_1} - \frac{q}{\Gamma(1-q)} \int_a^{t_1} m(s)(t_1-s)^{-q-1} ds \\ &= -\frac{1}{\Gamma(1-q)} \frac{m(a)}{(t_1-a)^q} - \frac{q}{\Gamma(1-q)} \int_a^{t_1} m(s)(t_1-s)^{-q-1} ds \\ &\geq 0, \end{aligned}$$

Similarly,

$$\begin{aligned} \big({}^cD^q_{b-}m\big)(t_1) &= \frac{1}{\Gamma(1-q)} \int_{t_1}^{b} \frac{m'(s)}{(s-t_1)^q} ds \\ &= \frac{1}{\Gamma(1-q)} \frac{m(s)}{(s-t_1)^q}\bigg|_{t_1}^{b} + \frac{q}{\Gamma(1-q)} \int_{t_1}^{b} m(s)(s-t_1)^{-q-1} ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-q)}\frac{m(b)}{(b-t_1)^q} + \frac{q}{\Gamma(1-q)}\int_{t_1}^{b} m(s)(s-t_1)^{-q-1}ds \\
&\leq 0,
\end{aligned}
$$

Therefore ${}^cD^q m(t_1) = 0$.

In order to show that $\left({}^cD^{1+q}m\right)(t_1) \leq 0$, we will compute $\left({}^cD^{1+q}_{a+}m\right)(t_1)$ and $\left({}^cD^{1+q}_{b-}m\right)(t_1)$. Observe that

$$
\begin{aligned}
\left({}^cD^{1+q}m\right)(t_1) &= {}^cD\left({}^cD^q m(t_1)\right) \\
&= \frac{d}{dt}{}^cD^q m(t_1).
\end{aligned}
$$

Furthermore, ${}^cD^q m(t)$ is continuous and, consequently

$$
\begin{aligned}
\lim_{h\to 0}{}^cD^q_{a+}m(t_1-h) &\geq {}^cD^q_{a+}m(t_1), \text{ and} \\
\lim_{h\to 0}{}^cD^q_{b-}m(t_1+h) &\leq {}^cD^q_{b-}m(t_1).
\end{aligned}
$$

Then, we have that

$$
\begin{aligned}
{}^cD^{1+q}_{a+}m(t_1) &= \frac{d}{dt}{}^cD^q_{a+}m(t_1) \\
&= \lim_{h\to 0}\frac{{}^cD^q_{a+}m(t_1) - {}^cD^q_{a+}m(t_1-h)}{h} \\
&= -\lim_{h\to 0}\frac{{}^cD^q_{a+}m(t_1-h)}{h} \leq 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
{}^cD^{1+q}_{b-}m(t_1) &= \frac{d}{dt}{}^cD^q_{b-}m(t_1) \\
&= \lim_{h\to 0}\frac{{}^cD^q_{b-}m(t_1+h) - {}^cD^q_{b-}m(t_1)}{h} \\
&= \lim_{h\to 0}\frac{{}^cD^q_{b-}m(t_1+h)}{h} \leq 0.
\end{aligned}
$$

Therefore ${}^cD^{1+q}m(t_1) \leq 0$. □

Corollary 2.2. *Let $m \in C^2([a,b],\mathbb{R})$. If m reaches a nonnegative maximum at $t_1 \in (a,b)$, then ${}^cD^q m(t_1) = 0$ and ${}^cD^{1+q}m(t_1) \leq 0$*

Proof. Let K be the maximum of m, it suffices to take $\tilde{m}(t) = m(t) - K$ and apply Lemma 2.1 to $\tilde{m}$. □

Remark 2.3. If $m(t)$ reaches a nonnegative maximum at the endpoint $t = a$ then ${}^cD^q m(a) \leq 0$. Also, if it reaches a nonnegative maximum at $t = b$ then ${}^cD^q m(b) \geq 0$. The proof is similar to the previous cases.

Similarly we can prove,

Lemma 2.4. *Let $m \in C^2([a,b],\mathbb{R})$.*

(a) *If m reaches a nonpositive minimum at $t_1 \in (a,b)$, then ${}^cD^q m(t_1) = 0$ and ${}^cD^{1+q} m(t_1) \geq 0$.*
(b) *If m reaches a nonpositive minimum at $t = a$ then ${}^cD^q m(a) \geq 0$.*
(c) *If m reaches a nonpositive minimum at $t = b$, then ${}^cD^q m(b) \leq 0$*

Now we are ready to prove the following comparison theorem.

Theorem 2.5. *Let $0 < q < 1$ and assume that:*

(i) *$v, w \in C^2([a,b], \mathbb{R})$ are lower and upper solutions, respectively, of the Boundary Value Problem (2.1); i.e.,*

$$\begin{aligned} -{}^cD^{1+q} v(t) &\leq f(t, v(t), {}^cD^q v(t)), \\ \alpha_a v(a) - \beta_a {}^cD^q v(a) &\leq \gamma_a, \\ \alpha_b v(b) + \beta_b {}^cD^q v(b) &\leq \gamma_b, \end{aligned}$$

and

$$\begin{aligned} -{}^cD^{1+q} w(t) &\geq f(t, w(t), {}^cD^q w(t)), \\ \alpha_a w(a) - \beta_a {}^cD^q w(a) &\geq \gamma_a, \\ \alpha_b w(b) + \beta_b {}^cD^q w(b) &\geq \gamma_b. \end{aligned}$$

(ii) *$f_u, f_{{}^cD^q u}$ exist, are continuous on $[a,b]$ with $f_u < 0$ and $f_u \not\equiv 0$ on $\Omega = [t, u, \bar{u})$: $t \in [a,b], w(t) \leq u \leq v(t)$ and $\bar{u} = {}^cD^q v = {}^cD^q w$.*

Then $v(t) \leq w(t)$ on $[a,b]$.

Proof. Assume first that one of the above inequalities is strict and set $m(t) = v(t) - w(t)$. We will show that $m(t) < 0$.

If the conclusion is not true, then there exists a $t_1 \in [a,b]$ such that $m(t_1) = 0$ and $m(t) \leq 0$ on $[a,b]$. If $t_1 \in (a,b)$, then it follows from Lemma 2.1 that ${}^cD^q m(t_1) = 0$, and ${}^cD^{1+q} m(t_1) \leq 0$.

Therefore, $v(t_1) = w(t_1)$, ${}^cD^q v(t_1) = {}^cD^q w(t_1)$, and ${}^cD^{1+q} v(t_1) \leq {}^cD^{1+q} w(t_1)$. Hence,

$$\begin{aligned} & f\big(t_1, v(t_1), {}^cD^q v(t_1)\big) \geq -{}^cD^{1+q} v(t_1) \geq \\ & \geq -{}^cD^{1+q} w(t_1) > f\big(t_1, w(t_1), {}^cD^q w(t_1)\big) = \\ & = f\big(t_1, v(t_1), {}^cD^q v(t_1)\big), \end{aligned}$$

which is a contradiction. Therefore $v(t) < w(t)$ on (a,b).

Now, assume that $t_1 = a$, then $m'(a) \leq 0$ and $m(a) \geq 0$. Thus $v'(a) \leq w'(a)$ and $v(a) \geq w(a)$. Since $m'(a) \leq 0$,

$${}^cD^q m(a) = \lim_{h \to 0} \frac{1}{\Gamma(1-q)} \int_a^{a+h} \frac{m'(s)}{(a+h-s)^q} ds \leq 0.$$

Thus ${}^cD^qm(a) \leq 0$ and ${}^cD^qv(a) \leq {}^cD^qw(a)$. But from the boundary conditions it follows that

$$\alpha_a v(a) - \beta_a {}^cD^q v(a) \leq \alpha_a w(a) - \beta_a {}^cD^q w(a).$$

Hence, $\beta_a({}^cD^qw(a) - {}^cD^qv(a)) \leq 0$ and ${}^cD^qv(a) \geq {}^cD^qw(a)$ because $\beta_a > 0$. Thus ${}^cD^qm(a) = 0$. Therefore, we have from the proof of Lemma 2.1 that

$$\begin{aligned}
{}^cD^{1+q}m(a) &= \frac{d}{dt}{}^cD^qm(a) = \lim_{h\to 0}\frac{{}^cD^qm(a+h) - {}^cD^qm(a)}{h} \\
&= \lim_{h\to 0}\frac{{}^cD^qm(a+h)}{h} \leq 0.
\end{aligned}$$

Proceeding as before, we get the contradiction. Thus $v(a) < w(a)$

Finally, if $t_1 = b$ it follows by a similar argument that $v(b) < w(b)$. Thus $v(t) < w(t)$ on $[a, b]$.

Now assume that the inequalities when v and w are applied to (2.1) are nonstrict. In this case we can show that $v(t) \leq w(t)$ on $[a, b]$. If the conclusion is not true, then $v(t) - w(t)$ has a positive maximum, say M, at some $t_0 \in (a, b)$. We will prove that if such a t_0 exists, then $v(t) - w(t) \equiv M$ for $t \in [a, b]$. If this is not true, then there exists a $\tilde{t} \in [a, b]$ such that $v(\tilde{t}) - w(\tilde{t}) < M$. We show that this leads to a contradiction.

Suppose first that $\tilde{t} > t_0$ and define the function $z(t) = (t-a)^q e^{\alpha(t-a)} - (t_0 - a)^q e^{\alpha(t_0-a)}$, where $\alpha > 0$ is a constant that will be determined later. Note that

$$\begin{aligned}
&z(t) < 0 \text{ for } a \leq t < t_0, \\
&z(t) > 0 \text{ for } t_0 < t \leq b, \\
&z(t_0) = 0.
\end{aligned}$$

Also,

$$\begin{aligned}
{}^cD^qz(t) &= \alpha\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k\Gamma(k+q+2)}{((k+1)!)^2}, \text{ and} \\
{}^cD^{1+q}z(t) &= \alpha^2\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k\Gamma(k+q+3)}{(k+1)!(k+2)!}
\end{aligned}$$

Finally, note that

$$\begin{aligned}
{}^cD^{1+q}z(t) &= \alpha^2\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k\Gamma(k+q+3)}{(k+1)!(k+2)!} \\
&= \alpha^2\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k(k+q+2)\Gamma(k+q+2)}{(k+1)!(k+2)!} \\
&> \alpha^2\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k(k+2)\Gamma(k+q+2)}{(k+1)!(k+2)!} \\
&= \alpha^2\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k\Gamma(k+q+2)}{((k+1)!)^2}
\end{aligned}$$

$$= \alpha^c D^q z(t)$$

Let $m(t) = v(t) - w(t) + \varepsilon z(t)$, where $\varepsilon > 0$. Then

$$\begin{aligned}
{}^cD^{1+q}m(t) &= {}^cD^{1+q}v(t) - {}^cD^{1+q}w(t) + \varepsilon^c D^{1+q}z(t) \\
&= -f(t, v, {}^cD^q v) + f(t, w, {}^cD^q w) + \varepsilon^c D^{1+q}z(t) \\
&= -f(t, v, {}^cD^q v) + f(t, w, {}^cD^q v) \\
&\quad -f(t, w, {}^cD^q v) + f(t, w, {}^cD^q w) + \varepsilon^c D^{1+q}z(t) \\
&= -f_u(t, \xi, {}^cD^q v)(v - w) - f_{{}^cD^q u}(t, w, \eta)({}^cD^q v - {}^cD^q w) \\
&\quad +\varepsilon\alpha^2 \sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k \Gamma(k+q+3)}{(k+1)!(k+2)!}, \text{ for some values } \xi, \eta \in \mathbb{R} \\
&> -f_u(t, \xi, {}^cD^q v)(v - w) - f_{{}^cD^q u}(t, w, \eta)({}^cD^q v - {}^cD^q w) \\
&\quad +\varepsilon\alpha^2 \sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k \Gamma(k+q+2)}{((k+1)!)^2} \\
&= -f_u(t, \xi, {}^cD^q v)m - f_{{}^cD^q u}(t, w, \eta)({}^cD^q m) \\
&\quad +\varepsilon f_u(t, \xi, {}^cD^q v)[(t-a)^q e^{\alpha(t-a)} - (t_0 - a)^q e^{\alpha(t_0 - a)}] \\
&\quad +\varepsilon\alpha\big(f_{{}^cD^q u}(t, w, \eta)\big)\Big[\sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k \Gamma(k+q+2)}{((k+1)!)^2}\Big] \\
&\quad +\varepsilon\alpha^2 \sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k \Gamma(k+q+2)}{((k+1)!)^2} \\
&> -f_u(t, \xi, {}^cD^q v)m - f_{{}^cD^q u}(t, w, \eta)({}^cD^q m) \\
&\quad +\varepsilon f_u(t, \xi, {}^cD^q v)[(t-a)^q e^{\alpha(t-a)}] \\
&\quad +\varepsilon\big(\alpha f_{{}^cD^q u}(t, w, \eta) + \alpha^2\big)\Big[\sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k \Gamma(k+q+2)}{((k+1)!)^2}\Big]
\end{aligned}$$

$$\begin{aligned}
&= -f_u(t, \xi, {}^cD^q v)m - f_{{}^cD^q u}(t, w, \eta)({}^cD^q m) \\
&\quad +\varepsilon f_u(t, \xi, {}^cD^q v)\Big[(t-a)^q \sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k}{k!}\Big] \\
&\quad +\varepsilon\big(\alpha f_{{}^cD^q u}(t, w, \eta) + \alpha^2\big)\Big[\sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k \Gamma(k+q+2)}{((k+1)!)^2}\Big]
\end{aligned}$$

Since $\Gamma(k+2+q) > \Gamma(k+2) = (k+1)!$ for $k \geq 0$, we get from the last expression that if $f_{{}^cD^q u}(t, w, \eta) + \alpha > 0$, then

$$\begin{aligned}
&-f_u(t, \xi, {}^cD^q v)m - f_{{}^cD^q u}(t, w, \eta)({}^cD^q m) \\
&+\varepsilon f_u(t, \xi, {}^cD^q v)\Big[(t-a)^q \sum_{k=0}^{\infty} \frac{\alpha^k (t-a)^k}{k!}\Big]
\end{aligned}$$

$$
\begin{aligned}
& +\varepsilon\big(\alpha f_{{}^cD^qu}(t,w,\eta)+\alpha^2\big)\Big[\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k\Gamma(k+q+2)}{((k+1)!)^2}\Big] \\
> \; & -f_u(t,\xi,{}^cD^qv)m - f_{{}^cD^qu}(t,w,\eta)({}^cD^qm) \\
& +\varepsilon f_u(t,\xi,{}^cD^qv)\Big[(t-a)^q\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k}{k!}\Big] \\
& +\varepsilon\big(\alpha f_{{}^cD^qu}(t,w,\eta)+\alpha^2\big)\Big[\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k}{(k+1)!}\Big] \\
> \; & -f_u(t,\xi,{}^cD^qv)m - f_{{}^cD^qu}(t,w,\eta)({}^cD^qm) \\
& +\varepsilon f_u(t,\xi,{}^cD^qv)\Big[(t-a)^q\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k}{(k+1)!}\Big] \\
& +\varepsilon\big(\alpha f_{{}^cD^qu}(t,w,\eta)+\alpha^2\big)\Big[\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k}{(k+1)!}\Big] \\
= \; & -f_u(t,\xi,{}^cD^qv)m - f_{{}^cD^qu}(t,w,\eta)({}^cD^qm) \\
& +\varepsilon\big((t-a)^qf_u(t,\xi,{}^cD^qv)+\alpha f_{{}^cD^qu}(t,w,\eta)+\alpha^2\big)\Big[\sum_{k=0}^{\infty}\frac{\alpha^k(t-a)^k}{(k+1)!}\Big]
\end{aligned}
$$

Finally, since $(t-a)^qf_u(t,\xi,{}^cD^qv)$ and $f_{{}^cD^qu}(t,w,\eta)$ are bounded on $[a,b]$ we can select α such that $(t-a)^qf_u(t,\xi^cD^qv)+\alpha f_{{}^cD^qu}(t,w,\eta)+\alpha^2>0$ for all $t\in[a,b]$. Therefore,

$$
-{}^cD^{1+q}m(t) < f_u(t,\xi,{}^cD^qv)m + f_{{}^cD^qu}(t,w,\eta)({}^cD^qm) \tag{2.2}
$$

which implies that $m(t)$ can not attain its maximum on (a,b). However, now choosing $0<\varepsilon<\Big[\dfrac{M-v(\tilde t)-w(\tilde t)}{z(\tilde t)}\Big]$ when $\tilde t>t_0$, we get

$$
m(\tilde t) < v(\tilde t)-w(\tilde t)+\frac{M-v(\tilde t)-w(\tilde t)}{z(\tilde t)}z(\tilde t) < M.
$$

Also $m(t_0)=M$, which implies by our choice of $z(t)$ that $m(t)$ has a positive maximum greater than or equal to M at some $t^*\in(t_0,\tilde t)$, which leads to a contradiction because of (2.2).

If $\tilde t<t_0$, we can arrive to a similar conclusion by considering $z(t)=(b-t)^qe^{\alpha(b-t)}-(b-t_0)^qe^{\alpha(b-t_0)}$. Now we have that

$$
\begin{aligned}
& z(t)>0 \text{ for } a\le t<t_0, \\
& z(t)<0 \text{ for } t_0<t\le b, \\
& z(t_0)=0.
\end{aligned}
$$

If $t_0=a$, then clearly $\tilde t>a$. Also, if $t^*>a$ we can arrive as before to a contradiction. If $t^*=a$, then we only have $m(a)=M$ and ${}^cD^qm(a)\le 0$. But from the boundary conditions is follows that ${}^cD^qm(a)\ge 0$ because $\beta_0>0$. Hence it follows that ${}^cD^qm(a)=0$ and we can arrive at a contradiction as before. By a similar argument, we have that if $t_0 = b$, then ${}^cD^q m(b) = 0$. Thus we have that $v(t) - w(t) \equiv M$ on $[a, b]$. As a consequence, it follows that

$$\begin{aligned} 0 &= {}^cD^{1+q}v - {}^cD^{1+q}w \geq f(t, v, {}^cD^q w) - f(t, w, {}^cD^q v) \\ &= f_u(t, \xi, {}^cD^q w)(-M) \geq 0 \end{aligned}$$

for $t \in (a, b)$, where ξ lies between v and w. This implies that $f_u(t, \xi, {}^cD^q w) \equiv 0$ on Ω, which contradicts the assumption (ii). Hence the proof of the theorem is complete. □

The following special case will be useful to prove our main result.

Corollary 2.6. *Let $\rho, r \in C([a, b])$ with $r(t) \geq 0$ on $[a, b]$. Suppose further that $p \in C^2[a, b]$, and*

$$\begin{gathered} -{}^cD^{1+q}p \leq \rho\, {}^cD^q p - rp, \\ \alpha_a p(a) - \beta_a {}^cD^q p(a) \leq 0, \\ \alpha_b p(b) + \beta_b {}^cD^q p(b) \leq 0. \end{gathered}$$

Then $p(t) \leq 0$.

Next we will obtain a representation formula for the solution of the linear BVP of the form

$$\begin{gathered} -{}^cD^{1+q}u + Mu = f(t, u, {}^cD^q u) \\ \alpha_a u(a) - \beta_a {}^cD^q u(a) = \gamma_a, \\ \alpha_b u(b) + \beta_b {}^cD^q u(b) = \gamma_b, \end{gathered} \tag{2.3}$$

where f satisfies the assumptions of Theorem 2.5, by using the Green's function of

$$\begin{gathered} -{}^cD^{1+q}G(t, z) + MG(t, z) = \delta(t - z) \\ \alpha_a G(a, z) - \beta_a {}^cD^q G(a, z) = 0, \\ \alpha_b G(b, z) + \beta_b {}^cD^q G(b, z) = 0, \end{gathered} \tag{2.4}$$

where $\delta(t - z)$ is the Dirac Delta function.

Observe that $A_0 + A_1(t - a)$ and $B_0 + B_1(b - t)$ are two linearly independent solutions of ${}^cD^{1+q}u = 0$. Letting $u_{0,a} = A_0 + A_1(t - a)$, we will compute $u_a(t)$ by the method of succesive approximations

$$\begin{aligned} u_{n,a}(t) &= u_{0,a}(t) + MI^{1+q}u_{n-1}(t) \\ &= u_{0,a}(t) + \frac{M}{\Gamma(1+q)} \int_a^t u_{n-1,a}(s)(t - s)^q ds, \end{aligned}$$

where $I^\alpha u(t)$ is the fractional integral of order α, defined as in [1], and ${}^cD^\alpha\big(I^\alpha u(t)\big)= u(t)$.

Then

$$u_{1,a}(t) = A_0 + A_1(t - a)$$

$$
\begin{aligned}
& +\frac{M}{\Gamma(1+q)}\int_a^t \big[A_0+A_1(s-a)\big](t-s)^q ds \\
= \; & A_0+A_1(t-a) \\
& +M(t-a)^{1+q}\left[\frac{A_0}{\Gamma(2+q)}+\frac{A_1(t-a)}{\Gamma(3+q)}\right]
\end{aligned}
$$

Repeating the process, it follows that

$$
\begin{aligned}
u_{2,a}(t) \; = \; & A_0+A_1(t-a) \\
& +M(t-a)^{1+q}\left[\frac{A_0}{\Gamma(2+q)}+\frac{A_1(t-a)}{\Gamma(3+q)}\right] \\
& +M^2(t-a)^{2+2q}\left[\frac{A_0}{\Gamma(3+2q)}+\frac{A_1(t-a)}{\Gamma(4+2q)}\right]
\end{aligned}
$$

Proceeding inductively, it follows that

$$
u_{n,a}(t)=\sum_{k=0}^{n} M^k(t-a)^{k+kq}\left[\frac{A_0}{\Gamma(k+1+kq)}+\frac{A_1(t-a)}{\Gamma(k+2+kq)}\right]
$$

Therefore,

$$
u_a(t)=\sum_{n=0}^{\infty} M^n(t-a)^{n+nq}\left[\frac{A_0}{\Gamma(n+1+nq)}+\frac{A_1(t-a)}{\Gamma(n+2+nq)}\right]
$$

It follows by direct comparison that this series is convergent, because M is a fixed constant, $(t-a)$ is bounded on $[a,b]$,

$$
\frac{1}{\Gamma(n+1+nq)} \leq \frac{1}{n!},
$$

and

$$
\frac{1}{\Gamma(n+2+nq)} \leq \frac{1}{(n+1)!}
$$

Furthermore, it is uniformly convergent because

$$
\begin{aligned}
u_a(t) \; & = \; \sum_{n=0}^{\infty} M^n(t-a)^{n+nq}\left[\frac{A_0}{\Gamma(n+1+nq)}+\frac{A_1(t-a)}{\Gamma(n+2+nq)}\right] \\
& \leq \; \sum_{n=0}^{\infty}\frac{A_0M^n(b-a)^{n+nq}}{n!}+\sum_{n=0}^{\infty}\frac{A_1M^n(b-a)^{n+nq}(b-a)}{(n+1)!}
\end{aligned}
$$

By the ratio test the last series is convergent. Thus u_a is uniformly convergent by the Weierstrass M-test.

By a similar argument, letting $u_{0,b}=B_0+B_1(b-t)$ we can compute $u_b(t)$ by the method of succesive approximations

$$
u_{n,b}(t)=u_{0,b}(t)+\frac{M}{\Gamma(1+q)}\int_t^b u_{n-1,b}(s)(s-t)^q ds
$$

Then

$$u_{n,b}(t) = \sum_{k=0}^{n} M^k (b-t)^{k+kq} \left[\frac{B_0}{\Gamma(k+1+kq)} + \frac{B_1(b-t)}{\Gamma(k+2+kq)} \right]$$

Therefore,

$$u_b(t) = \sum_{n=0}^{\infty} M^n (b-t)^{n+nq} \left[\frac{B_0}{\Gamma(n+1+nq)} + \frac{B_1(b-t)}{\Gamma(n+2+nq))} \right]$$

Now it is easy to observe that this series is also uniformly convergent. Thus, the Green's function corresponding to (2.4) is given by

$$G(t,z) = \begin{cases} \dfrac{1}{c} u_a(t) u_b(z), \text{ if } a \le z \le t \le b \\ \\ \dfrac{1}{c} u_a(z) u_b(t), \text{ if } a \le t \le z \le b \end{cases}$$

Since $-{}^cD^{1+q}G + MG = \delta(t-z)$, we have that

$$c = u_b(t)\big({}^cD^q u_a(t)\big) - u_a(t)\big({}^cD^q u_b(t)\big).$$

Observe that

$$\frac{d}{dt} u_a(t) = A_1 + \sum_{n=1}^{\infty} \left[\frac{A_0 M^n (n+nq)(t-a)^{n-1+nq}}{\Gamma(n+1+nq)} + \frac{A_1 M^n (n+1+nq)(t-a)^{n+nq}}{\Gamma(n+2+nq)} \right],$$

and

$$\frac{d}{dt} u_b(t) = -B_1 - \sum_{n=1}^{\infty} \left[\frac{B_0 M^n (n+nq)(b-t)^{n-1+nq}}{\Gamma(n+1+nq)} + \frac{B_1 M^n (n+1+nq)(b-t)^{n+nq}}{\Gamma(n+2+nq)} \right],$$

Hence, it can be shown as before by direct comparison and the Weierstrass M-test that $\frac{d}{dt} G(t,z)$ is also uniformly convergent. Therefore, a solution of (2.3) is given by

$$u(t) = \psi(t) + \int_a^b G(t,z) f\big(z, u(z), {}^cD^q u(z)\big) dz, \tag{2.5}$$

where $\psi(t)$ is a solution of the problem

$$\begin{gathered} -{}^cD^{1+q} u + Mu = 0 \\ \alpha_a u(a) - \beta_a {}^cD^q u(a) = \gamma_a, \\ \alpha_b u(b) + \beta_b {}^cD^q u(b) = \gamma_b. \end{gathered}$$

This is indeed a solution of (2.3) because $u(t) \in C^2[a,b]$, it satisfies the boundary conditions and

$$\begin{aligned} &-{}^cD^{1+q} u(t) + Mu(t) = \\ &= -{}^cD^{1+q}\psi(t) - \int_a^b {}^cD^{1+q} G(t,z) f\big(z, u(z), {}^cD^q u(z)\big) dz \\ &\quad + M\psi(t) + \int_a^b MG(t,z) f\big(z, u(z), {}^cD^q u(z)\big) dz \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \left[-{}^cD^{1+q}G(t,z) + MG(t,z)\right] f\big(z, u(z), {}^cD^q u(z)\big)\,dz \\
&\quad - {}^cD^{1+q}\psi(t) + M\psi(t) \\
&= \int_a^b \delta(t-z) f\big(z, u(z), {}^cD^q u(z)\big)\,dz \\
&= f\big(t, u(t), {}^cD^q u(t)\big).
\end{aligned}
$$

We have shown the existence of a solution of (2.3). Now we are ready to show that this solution is unique.

Lemma 2.7. *The Boundary Value Problem* (2.3) *has a unique solution.*

Proof. Let u_1 and u_2 be two arbitrary solutions of (2.3). Then,

$$
\begin{aligned}
-{}^cD^{1+q}u_1 + Mu_1 &= f(t, u_1, {}^cD^q u_1) \\
\alpha_a u_1(a) - \beta_a {}^cD^q u_1(a) &= \gamma_a, \\
\alpha_b u_1(b) + \beta_b {}^cD^q u_1(b) &= \gamma_b,
\end{aligned}
$$

and

$$
\begin{aligned}
-{}^cD^{1+q}u_2 + Mu_2 &= f(t, u_2, {}^cD^q u_2) \\
\alpha_a u_2(a) - \beta_a {}^cD^q u_2(a) &= \gamma_a, \\
\alpha_b u_2(b) + \beta_b {}^cD^q u_2(b) &= \gamma_b,
\end{aligned}
$$

From Theorem 2.5 it follows that $u_1 \le u_2$, and similarly it follows that $u_1 \ge u_2$. Therefore $u_1 = u_2$ on $[a,b]$ and the solution of (2.3) is unique. □

3. MONOTONE ITERATIVE TECHNIQUE

In this section, we develop a monotone method for a BVP of the form (2.1), by using upper and lower solutions. Next we state our main result related to the corresponding nonlinear fractional differential equation with boundary conditions. Consider the following special case of the BVP (2.3),

$$
\begin{aligned}
-{}^cD^{1+q}u(t) &= f\big(t, u(t)\big), \\
\alpha_a u(a) - \beta_a {}^cD^q u(a) &= \gamma_a \\
\alpha_b u(b) + \beta_b {}^cD^q u(b) &= \gamma_b,
\end{aligned}
\tag{3.1}
$$

Theorem 3.1. *Assume that:*

(i) $v_0, w_0 \in C^2[a,b]$ *are such that* $v_0 \le w_0$ *and*

$$
\begin{aligned}
-{}^cD^{1+q}v_0 &\le f\big(t, v_0(t)\big), \\
\alpha_a v_0(a) - \beta_a {}^cD^q v_0(a) &\le \gamma_a \\
\alpha_b v_0(b) + \beta_b {}^cD^q v_0(b) &\le \gamma_b,
\end{aligned}
$$

$$-{}^cD^{1+q}w_0 \geq f\big(t, w_0(t)\big),$$
$$\alpha_a w_0(a) - \beta_a {}^cD^q w_0(a) \geq \gamma_a$$
$$\alpha_b w_0(b) + \beta_b {}^cD^q w_0(b) \geq \gamma_b,$$

(ii) *There exists $M > 0$ such that*

$$f\big(t, u(t)\big) - f\big(t, \xi(t)\big) \geq -M\big(u(t) - \xi(t)\big),$$

for $t \in [a, b]$ and $v_0(t) \leq \xi(t) \leq u(t) \leq w_0(t)$.

Then there exist monotone sequences $\{v_n\}, \{w_n\}$ such that $v_n \to v, w_n \to w$ as $n \to \infty$ uniformly on $[a, b]$ and v, w are extremal solutions of the BVP (3.1). *That is if $u(t)$ is any solution of the periodic boundary value problem* (3.1) *such that $v_0(t) \leq u \leq w_0(t)$, then $v \leq u \leq w$*

Proof. Define the sequences

$$\begin{gathered}
-{}^cD^{1+q}v_n(t) = f\big(t, v_{n-1}(t)\big) - M\big(v_n(t) - v_{n-1}(t)\big), \\
\alpha_a v_n(a) - \beta_a {}^cD^q v_n(a) = \gamma_a, \\
\alpha_b v_n(b) + \beta_b {}^cD^q v_n(b) = \gamma_b,
\end{gathered} \tag{3.2}$$

and

$$\begin{gathered}
-{}^cD^{1+q}w_n(t) = f\big(t, w_{n-1}(t)\big) - M\big(w_n(t) - w_{n-1}(t)\big), \\
\alpha_a w_n(a) - \beta_a {}^cD^q w_n(a) = \gamma_a, \\
\alpha_b w_n(b) + \beta_b {}^cD^q w_n(b) = \gamma_b,
\end{gathered} \tag{3.3}$$

For (3.2), we have from (2.5) the solution in terms of the corresponding Green's function $G_{v,n}(t, z)$

$$v_n(t) = \psi_{v,n}(t) + \int_a^b G_{v,n}(t, z)\big[f\big(z, v_{n-1}(z)\big) + Mv_{n-1}(z)\big]dz,$$

where $\psi_{v,n}(t)$ is computed like $\psi(t)$ in (2.5).

Similarly, we have for (3.3) that

$$w_n(t) = \psi_{w,n}(t) + \int_a^b G_{w,n}(t, z)\big[f\big(z, w_{n-1}(z)\big) + Mw_{n-1}(z)\big]dz.$$

Furthermore, since $v_0, w_0 \in C^2[a, b]$, it follows that $v_1, w_1 \in C^2[a, b]$ and proceeding inductively $v_n, w_n \in C^2[a, b]$ for each n. Also, by Lemma 2.7, v_n and w_n are unique for each n.

Now, define a mapping A by $v_1 = Av_0$, where v_1 is the solution of (3.2) for $n = 1$ and v_0 is the lower solution of (3.1). Also let $p(t) = v_0(t) - v_1(t)$, then from (i) and (3.2)

$$\begin{aligned}
-{}^cD^{1+q}p &= -{}^cD^{1+q}v_0 + {}^cD^{1+q}v_1 \\
&= f\big(z, v_0\big) - f\big(z, v_0\big) + M(v_1 - v_0)
\end{aligned}$$

$$= \quad M(v_1 - v_0) = -M(v_0 - v_1) = -Mp$$

with

$$\alpha_a p(a) - \beta_a {}^c D^q p(a) \le 0,$$
$$\alpha_b p(b) + \beta_b {}^c D^q p(b) \le 0.$$

From Corollary 2.6, we have that $p(t) \le 0$. Thus $v_0(t) \le Av_0(t)$ on $[a,b]$ and $v_0(t) \le v_1(t)$.

By a similar argument, $w_0(t) \ge w_1(t)$.

Now let η and μ be any two solutions such that $v_0 \le \eta \le \mu \le w_0$ and assume that $u_1 = A\eta$ and $u_2 = A\mu$. Letting $p = u_1 - u_2$ and using assumption (ii), we have that

$$\begin{aligned} -{}^c D^{1+q} p &= -{}^c D^{1+q} u_1 + {}^c D^{1+q} u_2 \\ &= f\big(t,\eta\big) - M(u_1 - \eta) - f\big(t,\mu\big) + M(u_2 - \mu) \\ &\le -M(\eta - \mu) - M(u_1 - \eta) + M(u_2 - \mu) \\ &= -M(u_1 - u_2) = -Mp, \end{aligned}$$

with

$$\alpha_a p(a) - \beta_a {}^c D^q p(a) \le 0,$$
$$\alpha_b p(b) + \beta_b {}^c D^q p(b) \le 0.$$

Hence $p(t) \le 0$ on $[a,b]$ and, consequently, $A\eta \le A\mu$. This proves that A is monotone.

Define the sequences $\{v_n\}, \{w_n\}$ such that $v_n = Av_{n-1}, w_n = Aw_{n-1}$. Note that since $v_0 \le Av_0 = v_1$ and $w_0 \ge Aw_0 = w_1$, by monotonicity of A we have that $v_0 \le v_1 \le w_1 \le w_0$. Repeating the process we have that $v_2 \le w_2$, and $v_0 \le v_1 \le v_2 \le w_2 \le w_1 \le w_0$ Proceeding inductively it follows that $v_n \le u \le w_n$, then

$$v_0 \le v_1 \le \cdots \le v_n \le w_n \le \cdots w_1 \le w_0,$$

on $[a,b]$. Now, we are ready to show that $\{v_n\}$ and $\{w_n\}$ are uniformly bounded and equicontinuous.

First we show that they are uniformly bounded.

By hypothesis both $v_0(t)$ and $w_0(t)$ and bounded on $[a,b]$, then there exists a positive constant $\bar{M}$ such that $|v_0(t)| \le \bar{M}$ and $|w_0(t)| \le \bar{M}$ for all $t \in [a,b]$. Since $v_0(t) \le v_n(t) \le w_0(t)$ for all n, it follows that $0 \le v_n(t) - v_0(t) \le w_0(t) - v_0(t)$. Thus $\{v_n(t)\}$ is uniformly bounded.

By a similar argument, $\{w_n(t)\}$ is also uniformly bounded.

To show that $\{v_n\}$ is equicontinuous, fix n and let $a \le t_1 \le t_2 \le b$. Then

$$\big|v_n(t_1) - v_n(t_2)\big| = \Bigg|\psi_{v,n}(t_1) + \int_a^b G_{v,n}(t_1,z)\big[f\big(z, v_{n-1}(z)\big) + Mv_{n-1}(z)\big]dz$$

$$
\begin{aligned}
&\quad - \psi_{v,n}(t_2) - \int_a^b G_{v,n}(t_2,z)\big[f\big(z,v_{n-1}(z)\big)+Mv_{n-1}(z)\big]dz\Bigg| \\
&\le \big|\psi_{v,n}(t_1)-\psi_{v,n}(t_2)\big| \\
&\quad + \Bigg|\int_a^b G_{v,n}(t_1,z)\big[f\big(z,v_{n-1}(z)\big)+Mv_{n-1}(z)\big]dz \\
&\quad - \int_a^b G_{v,n}(t_2,z)\big[f\big(z,v_{n-1}(z)\big)+Mv_{n-1}(z)\big]dz\Bigg| \\
&= \big|\psi_{v,n}(t_1)-\psi_{v,n}(t_2)\big| \\
&\quad + \Bigg|\int_a^b \big[f\big(z,v_{n-1}(z)\big)+Mv_{n-1}(z)\big]\big[G_{v,n}(t_1,z)dz - G_{v,n}(t_2,z)\big]dz\Bigg| \\
&\le \big|\psi_{v,n}(t_1)-\psi_{v,n}(t_2)\big| \\
&\quad + \int_a^b \Bigg|\big[f\big(z,v_{n-1}(z)\big)+Mv_{n-1}(z)\big]\big[G_{v,n}(t_1,z)dz - G_{v,n}(t_2,z)\big]\Bigg|dz.
\end{aligned}
$$

Since f is continuous and $\{v_n\}$ is uniformly bounded, There exists a constant $\hat{K}$ such that $\big|f\big(z,v_{n-1}(z)\big)+Mv_{n-1}(z)\big|\le \bar{K}$. Then

$$
\big|v_n(t_1)-v_n(t_2)\big|\le \big|\psi_{v,n}(t_1)-\psi_{v,n}(t_2)\big|+\bar{K}\int_a^b\Bigg|G_{v,n}(t_1,z)dz-G_{v,n}(t_2,z)\Bigg|dz.
$$

By the Mean Value Theorem, there exist $\xi,\eta\in[t_1,t_2]$ such that,

$$
\begin{aligned}
&\big|\psi_{v,n}(t_1)-\psi_{v,n}(t_2)\big|+\bar{K}\int_a^b\Bigg|G_{v,n}(t_1,z)dz-G_{v,n}(t_2,z)\Bigg|dz \\
= \quad &\big|\frac{d}{dt}\psi_{v,n}(\xi)(t_1-t_2)\big|+\bar{K}\int_a^b\Bigg|\frac{d}{dt}G_{v,n}(\eta,z)(t_1-t_2)\Bigg|dz.
\end{aligned}
$$

Given that $\psi_{v,n}\in C^2[a,b]$, $\dfrac{d}{dt}\psi_{v,n}$ is continuous and bounded on $[a,b]$, and since $\dfrac{d}{dt}G_{v,n}(t,z)$ is uniformly convergent, there exist constants $\bar{K}_1$ and $\bar{K}_2$ such that for all $t_1,t_2\in[a,b]$,

$$
\begin{aligned}
&\big|\frac{d}{dt}\psi_{v,n}(\xi)(t_1-t_2)\big|+\bar{K}\int_a^b\Bigg|\frac{d}{dt}G_{v,n}(\eta,z)(t_1-t_2)\Bigg|dz \\
&\le \bar{K}_1\big|t_1-t_2\big|+\bar{K}_2\int_a^b\big|t_1-t_2\big|dz \\
&= \bar{K}_1\big|t_1-t_2\big|+\bar{K}_2(b-a)\big|t_1-t_2\big|.
\end{aligned}
$$

Thus for this particular n and for all $t_1,t_2\in[a,b]$,

$$
\big|v_n(t_1)-v_n(t_2)\big|\le \big(\bar{K}_1+\bar{K}_2(b-a)\big)\big|t_1-t_2\big|,
$$

or

$$\frac{\left|v_n(t_1) - v_n(t_2)\right|}{\left|t_1 - t_2\right|} \le \left(\bar{K}_1 + \bar{K}_2(b-a)\right),$$

i.e., v_n satisfies a Lipschitz condition on $[a, b]$.

Moreover, since $\{v_n\}$ is uniformly bounded there exists $L > 0$ such that for all n and all $t_1, t_2 \in [a, b]$,

$$\frac{\left|v_n(t_1) - v_n(t_2)\right|}{\left|t_1 - t_2\right|} \le L.$$

Thus $\{v_n\}$ have the same Lipschitz constant and, consequently, $\{v_n\}$ is equicontinuous. By a similar argument $\{w_n\}$ is equicontinuous.

Hence by Arzela-Ascoli's theorem, there exist subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$ which converge to $v(t)$ and $w(t)$, respectively. Since the sequences are monotone, the entire sequences converge.

It remains to show that $v(t)$ and $w(t)$ are extremal solutions of (3.1).

Assume that for some $k > 0$, $v_{k-1} \le u \le w_{k-1}$ on $[a, b]$ where u is a solution of (3.1) such that $v_0 \le u \le w_0$. Then setting $p = v_k - u$ we get that

$$\begin{aligned}
-{}^cD^q p &= -{}^cD^q v_k + {}^cD^q u \\
&= f\left(t, v_{k-1}\right) - M(v_k - v_{k-1}) - f\left(t, u\right) \\
&\le M(u - v_{k-1}) - M(v_k - v_{k-1}) \\
&= -M(v_k - u) \\
&= -Mp
\end{aligned}$$

and

$$\begin{aligned}
\alpha_a p(a) - \beta_a {}^cD^q p(a) \le 0, \\
\alpha_b p(b) + \beta_b {}^cD^q p(b) \le 0.
\end{aligned}$$

By Corollary 2.6, $p(t) \le 0$ on $a \le t \le b$, hence $v_k \le u$. By a similar argument $w_k \ge u$ on $[a, b]$.

Since $v_0 \le u \le w_0$, it follows by induction that $v_n \le u \le w_n$ on $[a, b]$, for all n. Hence $v \le u \le w$ on $[a, b]$, which shows that v and w are minimal and maximal solutions of (3.1), respectively. This completes the proof. □

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