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BOUNDARY VALUE PROBLEMS WITH ADVANCED ARGUMENTS INVOLVING UPPER AND LOWER SOLUTIONS IN REVERSE ORDER

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ABSTRACT. In this paper, we discuss boundary value problems for first order differential-integral equations with advanced arguments. We formulate sufficient conditions, under which such problems have a minimal and a maximal solution in a corresponding region bounded by upper-lower solutions. To get our results we apply a new approach based on Heikkila and V.Lakshmikantham theorem [1]. An example illustrates the results obtained.

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1. INTRODUCTION

In this paper we investigate boundary value problems for first order differential-integral equations with advanced arguments of the form:

$$(1.1)\qquad \begin{cases} x'(t) = f(t, x(t), x(\alpha(t)), \int_t^T k(t,s)x(s)ds) \equiv (\mathcal{F}x)(t), \quad t \in J, \\ \quad 0 \ = g(x(0), x(T)), \end{cases}$$

where

H_1 : $f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k \in C(J \times J, \mathbb{R}_+)$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J, J)$ and $t \leq \alpha(t)$ on J with $\mathbb{R}_+ = [0, \infty)$.

An interesting and fruitful technique for proving existence results for nonlinear differential problems is the monotone iterative method, for details, see, for example [3]. This technique can be used both initial and boundary value problems. We have many applications of this technique to nonlinear boundary value problems for differential equations, we cite only [2]–[4]. In this paper we also applied this method but we use a new approach based on Heikkila and V. Lakshmikantham theorem from [1]. To our knowledge it is a first application of this theorem to problems of type (1.1).

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The organization of this paper is as follows. In Section 2, we present some necessary results which are useful in the next investigations of this paper. First, we discuss the existence of solutions to problem (1.1) with an initial condition given at the end point T. Next, we discuss a differential-integral inequality with the advanced argument α. We apply both results in the next section to obtain Theorem (3.2) which is the main result of this paper. Also, in Section 3, an example is added to verify assumptions and theoretical results. In the last Section 4, we give some generalizations when problem (1.1) has more advanced arguments α_i.

2. PRELIMINARIES

First, we formulate a theorem which is useful in our investigations.

Consider the initial value problem of the form

$$x'(t) = (\mathcal{F}x)(t), \quad t \in J, \quad x(T) = k_0 \in \mathbb{R}, \tag{2.1}$$

where operator $\mathcal{F}$ is defined as in problem (1.1).

Theorem 2.1. *Suppose that*

(i) : $f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k \in C(J \times J, \mathbb{R}_+)$, $\alpha \in C(J, J)$, $t \leq \alpha(t)$ *on* J,

(ii) : *there exist nonnegative constants* L_1, L_2, L_3 *such that*

$$|f(t, x_1, x_2, x_3) - f(t, \bar{x}_1, \bar{x}_2, \bar{x}_3)| \leq L_1|x_1 - \bar{x}_1| + L_2|x_2 - \bar{x}_2| + L_3|x_3 - \bar{x}_3|$$

for $t \in J$, $x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathbb{R}$.

Then problem (2.1) *has a unique solution* $x \in C^1(J, \mathbb{R})$.

Proof. Integrating (2.1), we have

$$x(t) = k_0 - \int_t^T (\mathcal{F}x)(s)ds \equiv (Ax)(t), \quad t \in J.$$

Put

$$\|x\|_* = \max_{t \in J} e^{\lambda(t-T)}|x(t)| \quad \text{for } \lambda \geq L_1 + L_2 + L_3K, \quad \lambda \geq 1$$

and

$$q \equiv \left(1 - e^{-\lambda T}\right) < 1, \quad K = \max\{k(t, s) : t, s \in J\}.$$

We show that operator A is a contraction. Let $u, v \in C(J, \mathbb{R})$. Then, in view of assumptions (i) and (ii), we obtain

$$\|Au - Av\|_* \leq \max_{t \in J} e^{\lambda(t-T)} \int_t^T |(\mathcal{F}u)(s) - (\mathcal{F}v)(s)|ds$$

$$\leq \max_{t \in J} e^{\lambda(t-T)} \int_t^T \Big[L_1|u(s) - v(s)| + L_2|u(\alpha(s)) - v(\alpha(s))|$$

$$+ L_3 \int_s^T k(s, \tau)|u(\tau) - v(\tau)|d\tau\Big]ds$$

$$\leq \|u-v\|_* \max_{t\in J} e^{\lambda(t-T)} \int_t^T \left[L_1 e^{-\lambda(s-T)} + L_2 e^{-\lambda[\alpha(s)-T]} + L_3 K \int_s^T e^{-\lambda(\tau-T)} d\tau\right] ds$$
$$\leq \|u-v\|_* (L_1+L_2+L_3K) \max_{t\in J} e^{\lambda t} \int_t^T e^{-\lambda s} ds$$
$$= \|u-v\|_* \frac{L_1+L_2+L_3K}{\lambda} q \leq q\|u-v\|_*.$$

Then, problem (2.1) has a unique solution, by the Banach fixed point theorem. This ends the proof. □

Remark 2.2. Let $\alpha \in C(J,J)$, $t \leq \alpha(t)$ on J. Suppose that $K, h \in C(J,\mathbb{R})$, L, $M \in C(J,\mathbb{R}_+)$ and let $k \in C(J\times J, \mathbb{R}_+)$. We consider a linear problem of the form:

$$\begin{cases} x'(t) = (\mathcal{L}x)(t) - h(t), & t \in J, \\ x(T) = \bar{h} \in \mathbb{R}, \end{cases} \tag{2.2}$$

with

$$(\mathcal{L}x)(t) = K(t)p(t) + L(t)p(\alpha(t)) + M(t)\int_t^T k(t,s)p(s)ds.$$

By the proof of Theorem 2.1, we see that solving (2.2) is equivalent to solving a fixed point problem with operator $\mathcal{A}_h$ defined by

$$(\mathcal{A}_h x)(t) = \bar{h} - \int_t^T [(\mathcal{L}x)(s) - h(s)]ds.$$

Problem (2.2) has a unique solution, by Theorem 2.1.

Now, we concentrate our attention to differential-integral inequalities with advanced arguments α.

Lemma 2.3. *Let $\alpha \in C(J,J)$, $t \leq \alpha(t)$ on J. Suppose that $K \in C(J,\mathbb{R})$, L, $M \in C(J,\mathbb{R}_+)$ $p \in C^1(J,\mathbb{R})$ and*

$$\begin{cases} p'(t) \geq (\mathcal{L}p(t), & t \in J, \\ p(T) \leq 0, \end{cases} \tag{2.3}$$

where operator $\mathcal{L}$ is defined as in Remark 2.2.

In addition, we assume that

$H_2:\ \rho \leq 1$ *with*

$$\rho = \int_0^T \left[L(t)e^{\int_t^{\alpha(t)} K(s)ds} + M(t)e^{\int_t^T K(\tau)d\tau} \int_t^T k(t,s)e^{-\int_s^T K(\tau)d\tau} ds\right] dt.$$

Then $p(t) \leq 0$ on J.

Proof. Indeed, the assertion holds if $L(t) = M(t) = 0$, $t \in J$. Assume that the above condition is not true. Put

$$q(t) = e^{\int_t^T K(s)ds} p(t), \quad t \in J.$$

This and (2.3) give $q(T) = p(T) \le 0$, and

$$\begin{aligned} q'(t) &= e^{\int_t^T K(s)ds}\left[-K(t)p(t) + p'(t)\right] \\ &\ge e^{\int_t^T K(s)ds}\left[L(t)p(\alpha(t)) + M(t)\int_t^T k(t,s)p(s)ds\right]; \end{aligned}$$

so

$$\begin{cases} q'(t) \ge L(t)e^{\int_t^{\alpha(t)} K(s)ds}q(\alpha(t)) \\ \qquad\qquad +M(t)e^{\int_t^T K(\tau)d\tau}\int_t^T k(t,s)e^{-\int_s^T K(\tau)d\tau}q(s)ds, \\ q(T) \le 0. \end{cases} \tag{2.4}$$

We need to prove that $q(t) \le 0$, $t \in J$. Suppose that the inequality $q(t) \le 0$, $t \in J$ is not true. Then, we can find $t_0 \in [0,T)$ such that $q(t_0) > 0$. Put

$$q(t_1) = \min_{[t_0,T]} q(t) \le 0.$$

Integrating the differential inequality in (2.4) from t_0 to t_1, we obtain

$$\begin{aligned} q(t_1) - q(t_0) \ge \int_{t_0}^{t_1} \Big[& L(t)e^{\int_t^{\alpha(t)} K(s)ds}q(\alpha(t)) \\ & + M(t)e^{\int_t^T K(\tau)d\tau}\int_t^T k(t,s)e^{-\int_s^T K(\tau)d\tau}q(s)ds\Big]dt \\ \ge\ & q(t_1)\rho \ge q(t_1). \end{aligned}$$

It contradicts the assumption that $q(t_0) > 0$. This proves that $q(t) \le 0$ on J. This also proves that $p(t) \le 0$ on J and the proof is complete. □

Remark 2.4. Let $K(t) \ge 0$ on J. Then assumption H_2 holds if we assume that $\rho_1 \le 1$ with

$$\rho_1 = \int_0^T e^{\int_t^T K(\tau)d\tau}\left[L(t) + M(t)\int_t^T k(t,s)ds\right]dt.$$

Note that constant ρ_1 does not depend on α.

3. EXISTENCE OF SOLUTIONS OF PROBLEM (1.1)

Now, we derive a fixed point result for nondecreasing mappings in ordered spaces which play a central role in our investigations. We say that $Q : [a,b] \to [a,b]$ is nondecreasing if $Qx \le Qy$ for $x, y \in [a,b]$ and $x \le y$. We say that $x \in [a,b]$ is the least fixed point of Q in $[a,b]$ if $x = Qx$ and if $x \le y$ whenever $y \in [a,b]$ and $y = Qy$. The greatest fixed point of Q in $[a,b]$ is defined similarly, by reversing the inequality. If both least and greatest fixed point of Q in $[a,b]$ exist, we call them extremal fixed points of Q in $[a.b]$.

Theorem 3.1 (see [1])**.** *Let $[a, b]$ be an ordered interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to [a, b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subset Q([a, b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the sequence of Q-iteration of a converges to the least fixed point x_* of Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover,*

$$x_* = \min\{y \in [a, b] : \ y \geq Qy\}, \ \textit{and } x^* = \max\{y \in [a, b] : y \leq Qy\}.$$

Let us introduce the following definition.

We say that $u \in C^1(J, \mathbb{R})$ is a lower solution of (1.1) if

$$u'(t) \leq (\mathcal{F}u)(t), \quad t \in J, \quad g(u(0), u(T)) \leq 0,$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

Now we formulate the main result of this paper.

Theorem 3.2. *Let assumption H_1 hold. Let $y_0, z_0 \in C^1(J, \mathbb{R})$ be lower and upper solutions of* (1.1)*, respectively and $z_0(t) \leq y_0(t)$, $t \in J$. In addition, we assume that*

H_3 : *there exist functions $K \in C(J, \mathbb{R}),\ L, M \in C(J, \mathbb{R}_+)$ such that assumption H_2 is satisfied and*

$$f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3) \geq -K(t)[v_1 - u_1] - L(t)[v_2 - u_2] - M(t)[v_3 - u_3]$$

if $z_0(t) \leq u_1 \leq v_1 \leq y_0(t)$, $z_0(\alpha(t)) \leq u_2 \leq v_2 \leq y_0(\alpha(t))$, $\int_t^T k(t, s)z_0(s)ds \leq u_3 \leq v_3 \leq \int_t^T k(t, s)y_0(s)ds$,

H_4 : *g is nondecreasing in the first variable and there exists a constant $m > 0$ such that*

$$g(v, u) - g(v, \bar{u}) \leq m(\bar{u} - u) \quad \textit{if } z_0(T) \leq u \leq \bar{u} \leq y_0(T).$$

Then problem (1.1) *has, in the sector $[z_0, y_0]$, extremal solutions, where*

$$[z_0, y_0] = \{w \in C^1(J, \mathbb{R}) : z_0(t) \leq w(t) \leq y_0(t), t \in J\}.$$

Proof. Let G_h be nondecreasing with respect to h. Choose $h_1, h_2 \in C(J, \mathbb{R})$ such that $h_1(t) \leq h_2(t)$ on J. Let x_1, x_2 denote the solutions of problem (2.2) with h_1, h_2 instead of h, and with G_{h_1}, G_{h_2} instead of $\bar{h}$, respectively. Since problem (2.2) has a unique solution for each $h \in C(J, \mathbb{R}), \bar{h} \in \mathbb{R}$, then x_1, x_2 are well defined. Put $x = x_1 - x_2$. Then,

$$x'(t) = (\mathcal{L}x)(t) - h_1(t) + h_2(t) \geq (\mathcal{L}x)(t), \quad t \in J,$$
$$p(T) = G_{h_1} - G_{h_2} \leq 0.$$

In view of Lemma 2.3, we see that $x_1(t) \leq x_2(t)$ on J; so the operator $\mathcal{A}_h$ is nondecreasing. It is also continuous.

For $u \in [z_0, y_0]$, we put

$$Fu = \mathcal{F}u - \mathcal{L}u, \quad G_u = \frac{1}{m} g(u(0), u(T)) + u(T),$$

where the operator $\mathcal{F}$ is defined as in problem (1.1). Indeed, G_u is nondecreasing with respect to u, by assumption H_4. We define the operator $A = \mathcal{A}_F$. Let $x_1 = Az_0,\ x_2 = Ay_0$, so

$$\begin{cases} x_1'(t) = (\mathcal{L}x_1)(t) + (Fz_0)(t), \\ x_1(T) = G_{z_0}, \end{cases}$$

and

$$\begin{cases} x_2'(t) = (\mathcal{L}x_2)(t) + (Fy_0)(t), \\ x_2(T) = G_{y_0}. \end{cases}$$

Now, apply Lemma 2.3 with $x(t) = x_2(t) - y_0(t)$; so it is easy to show, using the definition of the lower solution y_0, that $y_0(t) \geq x_2(t) = (A\, y_0)(t)$. Similarly we can show $(A\, z_0)(t) = x_1(t) \geq z_0(t)$ on J. Put $x(t) = x_1(t) - x_2(t)$. Then

$$\begin{aligned} x'(t) &= (\mathcal{L}x_1)(t) + (Fz_0)(t) - (\mathcal{L}x_2)(t) - (Fy_0)(t) \geq (\mathcal{L}x)(t), \\ x(T) &= G_{z_0} - G_{y_0} \leq 0 \end{aligned}$$

. Using again Lemma 2.3, we see that $x_1(t) \leq x_2(t)$ on J; so the operator A is nondecreasing. It means that $z_0 \leq Au \leq y_0$ for $u \in [z_0, y_0]$. Hence $A : [z_0, y_0] \to [z_0, y_0]$ and operator A is bounded because $\|Au\| \leq \max(\|y_0\|, \|z_0\|) = B$.

Let $\{y_n\}$ be a monotone sequence in $[z_0, y_0]$; so $z_0 \leq Ay_n \leq y_0$. Hence $\|Ay_n\| \leq B$. It is easy to show that $\{Ay_n\}$ is equicontinuous. By Arzeli-Ascoli theorem, $\{Ay_n\}$ is compact. It proves that $\{Ay_n\}$ converges in $A([z_0, y_0])$. Finally, operator A has a least and a greatest fixed point in $[z_0, y_0]$, by Theorem 3.1. It results that problem (1.1) has minimal and maximal solutions in $[z_0, y_0]$. This ends the proof. □

Example 3.3. For $t \in J = [0, 1]$, we consider the problem

$$\begin{cases} x'(t) = 2e^{x(t)} + (\sin t)e^{-2e(\sqrt{t}-t)}x(\sqrt{t}) - C\int_t^1 x(s)ds - A \equiv (\mathcal{F}x)(t), \\ 0 = x(0) + x^2(0) - x(1). \end{cases} \tag{3.1}$$

with $A = \frac{2}{3}(1+e^{-1})$, $0 \leq C \leq \frac{2}{3}(1-2e^{-1})$. Note that $\alpha(t) = \sqrt{t}$, and $t \leq \alpha(t) \leq T = 1$. Put $y_0(t) = t$, $z_0(t) = -1$, $t \in J$. It yields

$$\begin{aligned} (\mathcal{F}y_0)(t) &= 2e^t + (\sin t)e^{-2e(\sqrt{t}-t)}\sqrt{t} - \frac{C}{2}(1-t^2) - A \geq 1 = y_0'(t), \\ (\mathcal{F}z_0)(t) &= 2e^{-1} - (\sin t)e^{-2e(\sqrt{t}-t)} + C(1-t) - A < 0 = z_0'(t), \end{aligned}$$

and

$$g(y_0(0), y_0(1)) = g(0, 1) = -1 < 0, \;\; g(z_0(0), z_0(1)) = g(-1, -1) = 1 > 0.$$

It proves that y_0, z_0 are lower and upper solutions of problem (3.1), respectively. Indeed, $K(t) = 2e^t$, $L(t) = (\sin t)e^{-2(\sqrt{t}-t)e}$, $M(t) = 0$, $m = 1$. Moreover,

$$\int_0^1 n(t)e^{\int_t^{\alpha(t)} m(s)ds}dt \leq \int_0^1 \sin t dt = 1 - \cos 1 < 1,$$

so assumption H_2 holds too. By Theorem 3.2, problem (3.1) has extremal solutions in the region $[-1, t]$.

4. SOME GENERALIZATIONS OF PROBLEM 1.1

In this section we consider a boundary value problem of the form

(4.1)
$$\begin{cases} x'(t) = f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_r(t)), \int_t^T k(t,s)x(s)ds) \equiv (\mathcal{G}x)(t), \quad t \in J, \\ \quad 0 \ = g(x(0), x(T)). \end{cases}$$

We formulate only corresponding results using the notions of lower and upper solutions of problem (4.1) which are the same as before with the operator $\mathcal{G}$ instead of operator $\mathcal{F}$. The next theorem is similar to Theorem 3.2 and therefore the proof is omitted.

Theorem 4.1. *Assume that* $f \in C(J \times \mathbb{R}^{r+2}, \mathbb{R})$, $k \in C(J \times J, \mathbb{R}_+)$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha_i \in C(J, J)$, $t \leq \alpha_i(t)$ *and* $\alpha_i(t) \not\equiv t$ *on* J *for* $i = 1, 2, \ldots, r$. *Let* $y_0, z_0 \in C^1(J, \mathbb{R})$ *be lower and upper solutions of* (4.1), *respectively and* $z_0(t) \leq y_0(t)$, $t \in J$. *We assume that there exists functions* $K \in C(J, \mathbb{R})$, $L_i, M \in C(J, \mathbb{R}_+)$, $i = 1, 2, \ldots r$ *such that*

$$f(t, u_0, u_1, \ldots, u_r, v) - f(t, v_0, v_1, \ldots, v_r, \bar{v}) \geq -K(t)[v_0 - u_0] - \sum_{i=1}^{r} L_i(t)[v_i - u_i] - M(t)[\bar{v} - v]$$

if $t \in J$, $z_0(\alpha_i(t)) \leq u_i \leq v_i \leq y_0(\alpha_i(t))$, $i = 0, 1, \ldots, r$ *with* $\alpha_0(t) = t$ *and* $\int_t^T k(t,s)z_0(s)ds \leq v \leq \bar{v} \leq \int_t^T k(t,s)y_0(s)ds$. *Moreover, we assume that assumption* H_2 *holds with*

$$\sum_{i=1}^{r} L_i(t)e^{\int_t^{\alpha_i(t)} K(s)ds} \quad \textit{instead of} \quad L(t)e^{\int_t^T K(s)ds}.$$

In addition, we assume that assumption H_4 *holds.*

Then problem (4.1) *has extremal solutions in the region* $[z_0, y_0]$.

REFERENCES

[1] S. Heikkila and V. Lakshmikantham, Monotone Iterative Technique for Discontinuous Nonlinear Differential Equations, Dekker, New York, 1994.

[2] T. Jankowski, Advanced differential equations with nonlinear boundary conditions, *J. Math. Anal. Appl.*, 304:490–503, 2005.

[3] G.S. Ladde, V. Lakshmikantham and A.S. Vatsala, Monotone Iterative Technique for Nonlinear Differential Equations, Pitman, Boston, 1985.

[4] W. Wang, X. Yang and J. Shen, Boundary value problems involving upper and lower solutions in reverse order, *J. Comput. Appl. Math.*, 230:1–7, 2009.