Dynamic Systems and Applications 19 (2010) 515-526

EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR FIRST-ORDER IMPULSIVE DYNAMIC EQUATIONS ON TIME SCALES

DA-BIN WANG[1] AND XIA-YI WANG[2]

[1]Department of Applied Mathematics, Lanzhou University of Technology
Lanzhou, Gansu, 730050, P. R. China wangdb@lut.cn
[2]College of Mathematics, Sichuan University, Chengdu, Sichuan, 610064
P. R. China

ABSTRACT. By using the classical fixed point index theorem for compact maps and the Leggett-Williams fixed point theorem respectively, in this paper, some results of single and multiple positive solutions to a class of nonlinear first-order periodic boundary value problems of impulsive dynamic equations on time scales are obtained. Two examples are given to illustrate the main results in this paper.

Keywords: Time scale; Boundary value problem; Positive solution; Fixed point; Impulsive dynamic equation

AMS (MOS) Subject Classification: 39A10, 34B15

1. INTRODUCTION

Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a nonempty closed subset of R. Let $T > 0$ be fixed and 0, T be points in $\mathbf{T}$, an interval $(0,T)_{\mathbf{T}}$ denote time scales interval, that is, $(0,T)_{\mathbf{T}} := (0,T) \cap \mathbf{T}$. Other types of intervals are defined similarly. Some definitions concerning time scales can be found in [1, 6, 7, 20, 23].

In this paper, we are concerned with the existence of positive solutions for the following nonlinear first-order periodic boundary value problem on time scale

(1.1)
$$\begin{cases} x^{\triangle}(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))), & t \in J := [0,T]_{\mathbf{T}}, t \neq t_k, k = 1,2,\ldots,m, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), & k = 1,2,\ldots,m, \\ x(0) = x(\sigma(T)), & \end{cases}$$

where $f \in C(J \times [0,\infty), [0,\infty))$, $I_k \in C([0,\infty),[0,\infty))$, $p : [0,T]_{\mathbf{T}} \to (0,\infty)$ is right-dense continuous, $t_k \in (0,T)_{\mathbf{T}}$, $0 < t_1 < \cdots < t_m < T$, and for each $k = 1,2,\ldots,m$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k+h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k+h)$ represent the right and left limits of $x(t)$ at $t = t_k$.

Received February 10, 2009

The theory of impulsive differential equations is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of differential equations without impulse effects. Moreover, such equations may exhibit several real world phenomena in physics, biology, engineering, etc. (see [3, 4, 27]). At the same time, the boundary value problems for impulsive differential equations and impulsive difference equations have received much attention [2, 13, 21, 22, 30–32, 34–37, 40, 43]. On the other hand, recently, the theory of dynamic equations on time scales has become a new important branch (See, for example, [1, 6, 7, 10–12, 20, 23, 38, 39, 42]). Naturally, some authors have focused their attention on the boundary value problems of impulsive dynamic equations on time scales [5, 8, 9, 16-19, 33, 41]. In particular, for the first order impulsive dynamic equations on time scales

$$\begin{cases} y^{\triangle}(t) + p(t)y(\sigma(t)) = f(t, y(t)), & t \in J := [a, b]\,, t \neq t_k, k = 1, 2, \ldots, m, \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, 2, \ldots, m, \\ y(a) = \eta, \end{cases} \tag{1.2}$$

where $\mathbf{T}$ is a time scale which has at least finitely-many right-dense points, $[a, b] \subset \mathbf{T}$, p is regressive and right-dense continuous, $f : \mathbf{T} \times R \to R$ is given function, $I_k \in C(R, R)$. The paper [8] obtained the existence of one solution to problem (1.2) by using the nonlinear alternative of Leray-Schauder type [15].

In [9], Benchohra et al considered the following impulsive boundary value problem on time scales

$$\begin{cases} -y^{\triangle\triangle}(t) = f(t, y(t)), & t \in J := [0, 1]_{\mathbf{T}}\,, t \neq t_k, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \\ y^{\triangle}(t_k^+) - y^{\triangle}(t_k^-) = \overline{I}_k(y(t_k^-)), \\ y(0) = y(1) = 0. \end{cases} \tag{1.3}$$

They proved the existence of one solution to problem (1.3) by applying Schaefer's fixed point theorem [28] and the nonlinear alternative of Leray-Schauder type [15].

In [33], Li and Shen studied problem (1.3). Some existence results to problem (1.3) are established by using a fixed point theorem, which is due to Krasnoselskii and Zabreiko [25], and the Leggett-Williams fixed point theorem [14, 26].

In [41], the first author studied problem (1.1). The existence of positive solutions to problem (1.1) was obtained by means of the well-known Guo-Krasnoselskii fixed point theorem [14].

Recently, Feng et al [13] considered the following impulsive boundary value problem

$$
\begin{cases}
-\left[\Phi_p(u'(t))\right]' = f(t, u(t)), & t \neq t_k, t \in (0,1), \\
-\triangle u \mid_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots, n, \\
u'(0) = 0, & u(1) = \int_0^1 g(t)u(t)dt.
\end{cases} \tag{1.4}
$$

By using the classical fixed point index theorem for compact maps [24, 29], some sufficient conditions for the existence of multiple positive solutions to problem (1.4) are obtained.

Motivated by the results mentioned above, in this paper, we shall show that problem (1.1) has at least one or two or three positive solutions by means of the classical fixed point index theorem for compact maps [24, 29] and the Leggett-Williams fixed point theorem [14, 26]. We note that for the case $I_k(x) \equiv 0, k = 1, 2, \dots, m$, problem (1.1) reduces to the problem studied by [38, 39].

In the remainder of this section, we state the following theorem, which are crucial to our proof.

Theorem 1.1 ([24, 29]). *Let K be a cone in a real Banach space X. Let D be an bounded open subset of X with $D_K = D \cap K \neq \phi$ and $\overline{D}_K \neq K$. Assume that $A : \overline{D}_K \to K$ is completely continuous such that $x \neq Ax$ for $x \in \partial D_K$. Then the following results hold:*

(i) *If $\|Ax\| \leq \|x\|$, $x \in \partial D_K$, then $i_K(A,\, D_K) = 1$.*
(ii) *If there exist $e \in K \backslash \{0\}$ such that $x \neq Ax + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(A, D_K) = 0$.*
(iii) *Let U be open in K such that $\overline{U} \subset D_K$. If $i_K(A, D_K) = 1$ and $i_K(A, U_K) = 0$, then A has a fixed point in $D_K \backslash \overline{U}_K$. The same result holds if $i_K(A, D_K) = 0$ and $i_K(A, U_K) = 1$.*

Remark 1.1. In theorem 1.1, the use of (ii) give better results than use of the common assumption $\|Ax\| \geq \|x\|$ for all $x \in \partial D_K$.

Let E be a real Banach space and $K \subset E$ be a cone. A function $\beta : K \to [0, \infty)$ is called a nonnegative continuous concave functional if β is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y)$$

for all x, $y \in K$ and $t \in [0, 1]$.

Let $a, b > 0$ be constants, $K_a = \{x \in K : \|x\| < a\}$, $K(\beta, a, b) = \{x \in K : a \leq \beta(x), \|x\| \leq b\}$.

Theorem 1.2 ([14, 26]). *Let $A : \overline{K}_c \to \overline{K}_c$ be a completely continuous map and β be a nonnegative continuous concave functional on K such that $\beta(x) \leq \|x\|$, $\forall x \in \overline{K}_c$. Suppose there exist a, b, d with $0 < d < a < b \leq c$, such that:*

(i) $\{x \in K(\beta, a, b) : \beta(x) > a\} \neq \phi$ *and* $\beta(Ax) > a$ *for all* $x \in K(\beta, a, b)$*;*

(ii) $\|Ax\| < d$ *for all* $\|x\| \leq d$*;*

(iii) $\beta(Ax) > a$*, for all* $x \in K(\beta, a, c)$ *with* $\|Ax\| > b$*.*

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\|x_1\| < d, \quad a < \beta(x_2) \text{ and } \|x_3\| > d \text{ with } \beta(x_3) < a.$$

2. PRELIMINARIES

Throughout the rest of this paper, we always assume that the points of impulse t_k are right-dense for each $k = 1, 2, \ldots, m$.

We define

$$PC = \big\{x \in [0, \sigma(T)]_{\mathbf{T}} \to R : x_k \in C(J_k, R), \quad k = 1, 2, \ldots, m \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k^-) = x(t_k), k = 1, 2, \ldots, m\big\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]_{\mathbf{T}} \subset (0, \sigma(T)]_{\mathbf{T}}$, $k = 1, 2, \ldots, m$ and $J_0 = [0, t_1]_{\mathbf{T}}$, $J_{m+1} = \sigma(T)$.

Let

$$X = \{x(t) : x(t) \in PC,\ x(0) = x(\sigma(T))\}$$

with the norm $\|x\| = \sup_{t \in [0,\sigma(T)]_{\mathbf{T}}} |x(t)|$. Then X is a Banach space.

Definition 2.1. A function $x \in PC \cap C^1(J \backslash \{t_1, t_2, \ldots, t_m\}, R)$ is said to be a solution of the problem (1.1) if and only if x satisfies the dynamic equation

$$x^{\triangle}(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))) \text{ everywhere on } J \backslash \{t_1, t_2, \ldots, t_m\},$$

the impulsive conditions

$$x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m,$$

and the periodic boundary condition $x(0) = x(\sigma(T))$.

Lemma 2.1 ([41])**.** *Suppose* $h : [0, T]_{\mathbf{T}} \to R$ *is rd-continuous, then* x *is a solution of*

$$x(t) = \int_0^{\sigma(T)} G(t, s)h(s)\triangle s + \sum_{k=1}^{m} G(t, t_k)I_k(x(t_k)), \quad t \in [0, \sigma(T)]_{\mathbf{T}}, \tag{2.1}$$

where $G(t, s) = \begin{cases} \frac{e_p(s,t)e_p(\sigma(T),0)}{e_p(\sigma(T),0)-1}, & 0 \leq s \leq t \leq \sigma(T), \\ \frac{e_p(s,t)}{e_p(\sigma(T),0)-1}, & 0 \leq t < s \leq \sigma(T), \end{cases}$ *if and only if* x *is a solution of the boundary value problem*

$$\begin{cases} x^{\triangle}(t) + p(t)x(\sigma(t)) = h(t), & t \in J := [0, T]_{\mathbf{T}}, t \neq t_k, k = 1, 2, \ldots, m, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), & k = 1, 2, \ldots, m, \\ x(0) = x(\sigma(T)). \end{cases}$$

Lemma 2.2. *Let* $G(t,s)$ *be defined as Lemma* 2.1*, then*

$$A \triangleq \frac{1}{e_p(\sigma(T),0)-1} \leq G(t,s) \leq \frac{e_p(\sigma(T),0)}{e_p(\sigma(T),0)-1} \triangleq B \text{ for all } t,s \in [0,\sigma(T)]_{\mathbf{T}}.$$

Proof. It is obvious, so we omit it here. □

Let

$$K = \{x(t) \in X : x(t) \geq \delta \|x\|\},$$

where $\delta = \frac{A}{B} = \frac{1}{e_p(\sigma(T),\, 0)} \in (0,1)$. It is not difficult to verify that K is a cone in X.

We define an operator $\Phi : K \to X$ by

$$(2.2)\quad (\Phi x)(t) = \int_0^{\sigma(T)} G(t,s) f(s, x(\sigma(s)))\triangle s + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)),\ t \in [0,\sigma(T)]_{\mathbf{T}}.$$

Lemma 2.3 ([41])**.** $\Phi : K \to K$ *is completely continuous.*

3. EXISTENCE OF ONE OR TWO POSITIVE SOLUTIONS

We define

$$\Omega_\rho = \left\{x \in K : \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x(t) < \delta\rho\right\} = \left\{x \in X : \delta\|x\| \leq \min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x(t) < \delta\rho\right\}.$$

Similar to [29, Lemma 2.5], we have the following result:

Lemma 3.1 ([42])**.** Ω_ρ *has the following properties:*

(1) Ω_ρ *is open relative to* K.
(2) $K_{\delta\rho} \subset \Omega_\rho \subset K_\rho$.
(3) $x \in \partial\Omega_\rho$ *if and only if* $\min_{t\in[0,\sigma(T)]_{\mathbf{T}}} x(t) = \delta\rho$.
(4) *If* $x \in \partial\Omega_\rho$*, then* $\delta\rho \leq x(t) \leq \rho$ *for* $t \in [0,\sigma(T)]_{\mathbf{T}}$.

For convenience, we denote:

$f_{\delta\rho}^{\rho} = \min\left\{\min_{t\in[0,T]_{\mathbf{T}}} \frac{f(t,x)}{\rho} : x \in [\delta\rho,\rho]\right\}$,
$f_0^{\rho} = \max\left\{\max_{t\in[0,T]_{\mathbf{T}}} \frac{f(t,x)}{\rho} : x \in [0,\rho]\right\}$, $I_0^{\rho}(k) = \max\{I_k(x) : x \in [0,\rho]\}$,
$f^i = \lim_{x\to i}\sup\max_{t\in[0,T]_{\mathbf{T}}} \frac{f(t,x)}{x}$, $I^i(k) = \lim_{x\to i}\sup \frac{I_k(x)}{x}$,
$f_i = \lim_{x\to i}\inf\min_{t\in[0,T]_{\mathbf{T}}} \frac{f(t,x)}{x}$, $I_i(k) = \lim_{x\to i}\inf \frac{I_k(x)}{x}$, *where* $i = \infty$, *or* 0^+.
$l = \frac{1}{B(\sigma(T)+m)}$, $L = \frac{1}{A\sigma(T)}$.

Now we state our main results

Lemma 3.2. *If there exists* $\rho > 0$ *such that* $f_0^{\rho} < l$, $I_0^{\rho}(k) < \rho l$*, then* $i_K(\Phi, K_\rho) = 1$.

Proof. Since $f_0^\rho < l$, $I_0^\rho(k) < \rho l$, by (2.2) and Lemma 2.2 we have for $x \in \partial K_\rho$

$$\begin{aligned}(\Phi x)(t) &= \int_0^{\sigma(T)} G(t,s) f(s, x(\sigma(s)))\triangle s + \sum_{k=1}^m G(t,t_k) I_k(x(t_k))\\ &\le B\int_0^{\sigma(T)} f(s,x(\sigma(s)))\triangle s + B\sum_{k=1}^m I_k(x(t_k))\\ &< B(\sigma(T)+m)\rho l\\ &= \rho = \|x\|,\end{aligned}$$

i.e., $\|\Phi x\| < \|x\|$ for $x \in \partial K_\rho$, then by (i) of Theorem 1.1 we have $i_K(\Phi, K_\rho) = 1$. □

Lemma 3.3. *If there exists $\rho > 0$ such that $f_{\delta\rho}^\rho > L$, then $i_K(\Phi, \Omega_\rho) = 0$.*

Proof. Let $e(t) \equiv 1$, $t \in [0, \sigma(T)]_{\mathbf{T}}$, then $e(t) \in \partial K_1$. We claim that $x \neq \Phi x + \lambda e$, $x \in \partial\Omega_\rho$, $\lambda > 0$.

In fact, if not, there exist $x_0 \in \partial\Omega_\rho$ and $\lambda_0 > 0$ such that $x_0 = \Phi x_0 + \lambda_0 e$. Then by (2.2) and Lemma 2.2 we have

$$\begin{aligned}x_0 = \Phi x_0 + \lambda_0 e \ge \delta\|\Phi x_0\| + \lambda_0 &\ge A\delta\int_0^{\sigma(T)} f(s, x_0(\sigma(s)))\triangle s + \lambda_0\\ &> A\delta\sigma(T)\rho L + \lambda_0 = \delta\rho + \lambda_0,\end{aligned}$$

then from (3) of Lemma 3.1 we get $\delta\rho > \delta\rho + \lambda_0$, which is a contradiction. Hence, by (ii) of Theorem 1.1 we have $i_K(\Phi, \Omega_\rho) = 0$. □

Theorem 3.1. *Suppose one of the following conditions holds:*

(H_1) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \delta\rho_2$ and $\rho_2 < \rho_3$ such that*

$$f_0^{\rho_1} < l, \quad I_0^{\rho_1}(k) < \rho_1 l, \quad f_{\delta\rho_2}^{\rho_2} > L, \quad f_0^{\rho_3} < l, \quad I_0^{\rho_3}(k) < \rho_3 l.$$

(H_2) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2 < \delta\rho_3$ such that*

$$f_{\delta\rho_1}^{\rho_1} > L, \quad f_0^{\rho_2} < l, \quad I_0^{\rho_2}(k) < l\rho_2, \quad f_{\delta\rho_3}^{\rho_3} > L.$$

Then problem (1.1) *has at least two positive solutions x_1, x_2 with $x_1 \in \Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3}\backslash\overline{\Omega}_{\rho_2}$.*

Proof. Suppose (H_1) holds. We claim that Φ have two fixed points $x_1 \in \Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3}\backslash\overline{\Omega}_{\rho_2}$. In fact, if ($\mathrm{H}_1$) holds then by Lemma 3.2 and Lemma 3.3 we obtain: $i_K(\Phi, K_{\rho_1}) = 1$, $i_K(\Phi, \Omega_{\rho_2}) = 0$, $i_K(\Phi, K_{\rho_3}) = 1$. Then from (2) of Lemma 3.1 and $\rho_1 < \delta\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\delta\rho_2} \subset \Omega_{\rho_2}$. Therefore, (iii) of Theorem 1.1 implies that the Φ has two fixed points $x_1 \in \Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3}\backslash\overline{\Omega}_{\rho_2}$ which are the positive solutions of problem (1.1).

If (H_2) holds, the proof is similar to that of the case when (H_1) holds, so we omit here. □

As a special case of Theorem 3.1, we obtain the following result.

Corollary 3.1. *If there exist $\rho_1,\ \rho_2 \in (0,\infty)$, with $\rho_1 < \delta\rho_2$ such that one of the following conditions holds:*

(H$_3$) $f_0^{\rho_1} < l,\ I_0^{\rho_1}(k) < \rho_1 l,\ f_{\delta\rho_2}^{\rho_2} > L,\ 0 \le f^\infty < l,\ 0 \le I^\infty(k) < l.$
(H$_4$) $f_{\delta\rho_1}^{\rho_1} > L,\ f_0^{\rho_2} < l,\ I_0^{\rho_2}(k) < l\rho_2,\ L < f_\infty \le \infty,\ L < I_\infty(k) \le \infty.$

Then problem (1.1) *has at least two positive solutions in K.*

Proof. Suppose (H$_3$) holds. We show that (H$_3$) implies (H$_1$). Let $\eta \in (f^\infty, l)$. Then there exists $\alpha > \eta$ such that $\max_{t\in[0,T]_{\mathbf{T}}} f(t,x) \le \eta x$ for $x \in [\alpha,\infty)$ since $0 \le f^\infty < l$.

Taking

$$\theta = \max\left\{\max_{t\in[0,T]_{\mathbf{T}}} f(t,x) : 0 \le x \le \alpha\right\} \text{ and } \rho_3 > \left\{\frac{\theta}{l-\eta},\ \rho_2\right\}.$$

Then we get

$$\max_{t\in[0,T]_{\mathbf{T}}} f(t,x) \le \eta x + \theta \le \eta\rho_3 + \theta < l\rho_3, \quad x \in [0,\rho_3].$$

This implies that $f_0^{\rho_3} < l$. Similarly, by $0 \le I^\infty(k) < l$, we have $I_0^{\rho_3}(k) < \rho_3 l$. So, (H$_3$) implies (H$_1$). Similarly (H$_4$) implies (H$_2$). ☐

By an argument similar to that of Theorem 3.1 we obtain the following results.

Theorem 3.2. *Suppose one of the following conditions holds:*

(H$_5$) *There exist $\rho_1,\ \rho_2 \in (0,\infty)$, with $\rho_1 < \delta\rho_2$ such that*

$$f_0^{\rho_1} \le l, \quad I_0^{\rho_1}(k) \le \rho_1 l, \quad f_{\delta\rho_2}^{\rho_2} \ge L.$$

(H$_6$) *There exist $\rho_1, \rho_2 \in (0,\infty)$, with $\rho_1 < \rho_2$ such that*

$$f_{\delta\rho_1}^{\rho_1} \ge L, \quad f_0^{\rho_2} \le l, \quad I_0^{\rho_2}(k) < l\rho_2.$$

Then problem (1.1) *has at least one positive solutions in K.*

As a special case of Theorem 3.2, we obtain the following result.

Corollary 3.2. *Suppose one of the following conditions holds:*

(H$_7$) $0 \le f^0 < l,\ 0 \le I^0(k) < l$ *and* $L < f_\infty \le \infty.$
(H$_8$) $0 \le f^\infty < l,\ 0 \le I^\infty(k) < l$ *and* $L < f_0 \le \infty.$

Then problem (1.1) *has at least one positive solutions in K.*

4. EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS

For $x \in K$, let

$$\beta(x) = \min_{t \in [0,\sigma(T)]_{\mathbf{T}}} x(t),$$

then it is easy to see that β is a nonnegative continuous concave functional on K with $\beta(x) \leq \|x\|$ for $x \in K$.

Theorem 4.1. *Suppose that the following conditions are hold:*

(C_1) *There exist positive constants c_k such that $I_k(x) \leq c_k$, for $x \geq 0$ and $k = 1, 2, \ldots, m$.*

(C_2) *There exist $B\sum_{k=1}^m c_k < d < a$ such that*

$$f(t,x) < \frac{d - B\sum_{k=1}^m c_k}{B\sigma(T)}, \quad \text{for } t \in [0,T]_{\mathbf{T}}, \quad x \in [0,d], \tag{4.1}$$

and

$$f(t,x) > \frac{a}{A\sigma(T)}, \quad \text{for } t \in [0,T]_{\mathbf{T}}, \quad x \in [a, \frac{1}{\delta}a]. \tag{4.2}$$

(C_3) *Assume that one of the following conditions satisfies:*

(a) $\lim_{x\to\infty} \max_{t\in[0,T]_{\mathbf{T}}} \frac{f(t,x)}{x} < \frac{1}{B\sigma(T)}$;

(b) *There exists $c > \frac{1}{\delta}a$ such that $f(t,x) < \frac{c - B\sum_{k=1}^m c_k}{B\sigma(T)}$, $t \in [0,T]_{\mathbf{T}}$, $x \in [0,c]$.*

Then problem (1.1) *has at least three positive solutions.*

Proof. First, we assert that there exists $h > \frac{1}{\delta}a$ such that $\Phi : \overline{K}_h \to K_h$ if (a) holds.

In fact, if (a) holds, then there exist $M > 0$ and $\varepsilon < \frac{1}{B\sigma(T)}$ such that

$$f(t,x) < \varepsilon x, \quad x > M.$$

Set

$$\xi = \max\left\{f(t,x) : t \in [0,T]_{\mathbf{T}}, x \in [0,M]\right\}.$$

It follows that $f(t,x) \leq \varepsilon x + \xi$, for all $x \in [0,\infty)$. Take

$$h > \max\left\{\frac{1}{\delta}a, \frac{B\left(\sigma(T)\xi + \sum_{k=1}^m c_k\right)}{1 - \varepsilon B\sigma(T)}\right\}.$$

If $x \in \overline{K}_h$, then

$$\begin{aligned}
(\Phi x)(t) &= \int_0^{\sigma(T)} G(t,s) f(s, x(\sigma(s)))\triangle s + \sum_{k=1}^m G(t,t_k) I_k(x(t_k)) \\
&\leq B \int_0^{\sigma(T)} f(s, x(\sigma(s)))\triangle s + B\sum_{k=1}^m I_k(x(t_k)) \\
&\leq B\sigma(T)(\varepsilon h + \xi) + B\sum_{k=1}^m c_k \\
&< h.
\end{aligned}$$

Thus $\|\Phi x\| < h$, i.e., $\Phi x \in K_h$.

Now we assert that $\Phi : \overline{K_{\overline{h}}} \to K_{\overline{h}}$ if there exists $\overline{h} > \frac{1}{\delta}a$ such that $f(t,x) < \frac{\overline{h} - B\sum_{k=1}^{m} c_k}{B\sigma(T)}$, $t \in [0,T]_{\mathbf{T}}$, $x \in [0,\overline{h}]$ (that is, (b) holds).

Indeed, if $x \in \overline{K_{\overline{h}}}$, then

$$\begin{aligned}
(\Phi x)(t) &= \int_0^{\sigma(T)} G(t,s) f(s, x(\sigma(s))) \triangle s + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)) \\
&< B\sigma(T) \frac{\overline{h} - B\sum_{k=1}^{m} c_k}{B\sigma(T)} + B\sum_{k=1}^{m} c_k \\
&= \overline{h}.
\end{aligned}$$

So, we have proved that there exist positive number $c > \frac{1}{\delta}a$ such that $\Phi : \overline{K}_c \to K_c$ if condition (a) or (b) hold. Furthermore, by (4.1) we have $\Phi : \overline{K}_d \to K_d$.

Second, we assert that $\left\{x \in K(\beta, a, \frac{1}{\delta}a) : \beta(x) > a\right\} \neq \phi$ and $\beta(\Phi x) > a$ for all $x \in K(\beta, a, \frac{1}{\delta}a)$.

In fact, take $x \equiv \frac{a + \frac{1}{\delta}a}{2}$, so $x \in \left\{x \in K(\beta, a, \frac{1}{\delta}a) : \beta(x) > a\right\}$. Moreover, for $x \in K(\beta, a, \frac{1}{\delta}a)$, then $\beta(x) \geq a$ and we have

$$\frac{1}{\delta}a \geq \|x\| \geq \min_{t \in [0,\sigma(T)]_{\mathbf{T}}} x(t) = \beta(x) \geq a.$$

Thus, in view of (4.2) we get

$$\begin{aligned}
\beta(\Phi x) &= \min_{t \in [0,\sigma(T)]_{\mathbf{T}}} \left[\int_0^{\sigma(T)} G(t,s) f(s, x(\sigma(s))) \triangle s + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)) \right] \\
&\geq \min_{t \in [0,\sigma(T)]_{\mathbf{T}}} \int_0^{\sigma(T)} G(t,s) f(s, x(\sigma(s))) \triangle s \\
&> A\sigma(T) \frac{a}{A\sigma(T)} \\
&= a.
\end{aligned}$$

Finally, we assert that $\beta(\Phi x) > a$ if $x \in K(\beta, a, c)$ and $\|\Phi x\| > \frac{1}{\delta}a$.

To do this, if $x \in K(\beta, a, c)$ and $\|\Phi x\| > \frac{1}{\delta}a$, then

$$\begin{aligned}
\beta(\Phi x) &= \min_{t \in [0,\sigma(T)]_{\mathbf{T}}} \left[\int_0^{\sigma(T)} G(t,s) f(s, x(\sigma(s))) \triangle s + \sum_{k=1}^{m} G(t,t_k) I_k(x(t_k)) \right] \\
&\geq \delta \|\Phi x\| > \delta \frac{1}{\delta} a = a.
\end{aligned}$$

To sum up, all the hypotheses of Theorem 1.2 are satisfied by taking $b = \frac{1}{\delta}a$. Hence Φ has at least three fixed points, that is, problem (1.1) has at least three positive solutions x_1, x_2 and x_3 such that

$$\|x_1\| < d, \ a < \beta(x_2) \text{ and } \|x_3\| > d \text{ with } \beta(x_3) < a. \qquad \square$$

5. EXAMPLES

Example 5.1. Let $\mathbf{T} = [0,1] \cup [2,3]$. We consider the following problem

$$(5.1)\qquad \begin{cases} x^{\triangle}(t) + x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in [0,3]_{\mathbf{T}}, t \neq \frac{1}{2}, \\ x\left(\frac{1}{2}^{+}\right) - x\left(\frac{1}{2}^{-}\right) = I(x(\frac{1}{2})), \\ x(0) = x(3), \end{cases}$$

where $p(t) \equiv 1$, $T = 3$, $m = 1$, $f(t, x(\sigma(t))) = (t+1)(x(\sigma(t)))^2$, and $I(x) = x^2$.

It is easy to see that

$$\delta = \frac{1}{2e^2}, \quad l = \frac{2e^2-1}{8e^2}, \quad L = \frac{2e^2-1}{3}.$$

Taking $\rho_1 = \frac{2e^2-1}{40e^2}$, $\rho_2 = 4e^6 - 2e^4$, then by simple calculation we have $\rho_1 < \delta\rho_2$, $f_0^{\rho_1} = 4\rho_1 = \frac{2e^2-1}{10e^2} < \frac{2e^2-1}{8e^2} = l$, $I_0^{\rho_1}(1) = \rho_1^2 = \left(\frac{2e^2-1}{40e^2}\right)^2 < \frac{2e^2-1}{40e^2} \cdot \frac{2e^2-1}{8e^2} = \rho_1 l$, $f_{\delta\rho_2}^{\rho_2} = \delta^2\rho_2 = \frac{1}{4e^4} \cdot (4e^6 - 2e^4) = \frac{2e^2-1}{2} > \frac{2e^2-1}{3} = L$.

Therefore, together with Theorem 3.2, it follows that problem (5.1) has at least one positive solution.

Example 5.2. Let $\mathbf{T} = [0,1] \cup [2,3]$. We consider the following problem

$$(5.2)\qquad \begin{cases} x^{\triangle}(t) + x(\sigma(t)) = f(t, x(\sigma(t))), t \in [0,3]_{\mathbf{T}}, t \neq \frac{1}{2}, \\ x\left(\frac{1}{2}^{+}\right) - x\left(\frac{1}{2}^{-}\right) = I(x(\frac{1}{2})), \\ x(0) = x(3), \end{cases}$$

where $p(t) \equiv 1$, $T = 3$, $m = 1$, $I(x) = \frac{1}{x^2+1}$ and

$$f(t,x) \equiv f(x) = \begin{cases} \frac{1}{8e^2}, & 0 \le x \le \frac{2e^2+1}{2e^2-1}, \\ g(x) & \frac{2e^2+1}{2e^2-1} \le x \le 2, \\ \frac{16e^4-10e^2-1}{6e^2} & 2 \le x \le 8e^2, \\ s(x) & x \ge 8e^2, \end{cases}$$

here $g(x)$ and $s(x)$ satisfy: $g(\frac{2e^2+1}{2e^2-1}) = \frac{1}{8e^2}$, $g(2) = \frac{16e^4-10e^2-1}{6e^2}$, $s(8e^2) = \frac{16e^4-10e^2-1}{6e^2}$, $g^{\triangle\triangle}(x) = 0$ for $x \in (\frac{2e^2+1}{2e^2-1}, 2)$ and $s(x) : (-\infty, \infty) \to [0, \infty)$ is continuous.

Choose $d = \frac{2e^2+1}{2e^2-1}$, $a = 2$, $b = 4e^2$, $c = 8e^2$; then by $Bc_1 = B = \frac{2e^2}{2e^2-1}$ (for $c_1 = 1$) and $\delta = \frac{1}{2e^2}$ we have $Bc_1 < d < a < \frac{1}{\delta}a = b < c$, and then $f(x)$ satisfies

$$f(x) = \frac{1}{8e^2} < \frac{1}{6e^2} = \frac{\frac{2e^2+1}{2e^2-1} - \frac{2e^2}{2e^2-1}}{3 \cdot \frac{2e^2}{2e^2-1}} = \frac{d - Bc_1}{B\sigma(T)}, \quad x \in [0, \frac{2e^2+1}{2e^2-1}] = [0, d];$$

$$f(x) = \frac{16e^4 - 10e^2 - 1}{6e^2} > \frac{4e^2-2}{3} = \frac{a}{A\sigma(T)}, \quad x \in [2, 4e^2] = [a, b];$$

$$f(x) \le \frac{16e^4 - 10e^2 - 1}{6e^2} < \frac{16e^4 - 10e^2}{6e^2} = \frac{c - Bc_1}{B\sigma(T)}, \quad x \in [0, 8e^2] = [0, c].$$

Then by Theorem 4.1, problem (5.2) has at least three positive solutions x_1, x_2 and x_3 such that

$$\|x_1\| < \frac{2e^2+1}{2e^2-1},\ 2 < \beta(x_2) \text{ and } \|x_3\| > \frac{2e^2+1}{2e^2-1} \text{ with } \beta(x_3) < 2.$$

REFERENCES

[1] R. P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.*, 35(1999)3–22.

[2] R. P. Agarwal and D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations, *Appl. Math. Comput.*, 114(2000)51–59.

[3] D. D. Bainov and P. S. Simeonov, *Systems with impulse effect: stability theorey and applications*, Horwood, Chicester, 1989.

[4] D. D. Bainov and P. S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, Harlow, 1993.

[5] A. Belarbi, M. Benchohra and A. Ouahab, Existence results for impulsive dynamic inclusions on time scales, *Electron. J. Qualitative Theory Differential Equations*, 12(2005)1–22.

[6] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhauser, Boston, 2001.

[7] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhauser, Boston, 2003.

[8] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, On first order impulsive dynamic equations on time scales, *J. Difference Equ. Appl.*, 6(2004)541–548.

[9] M. Benchohra, S. K. Ntouyas and A. Ouahab, Existence results for second-order bounary value problem of impulsive dynamic equations on time scales, *J. Math. Anal. Appl.*, 296(2004)65–73.

[10] M. Feng, X. Zhang and W. Ge, Positive solutions for a class of boundary value problems on time scales, *Comput. Math. Appl.*, 54(2007)467–475.

[11] M. Feng, X. Li and W. Ge, Triple Positive Solutions of Fourth-Order Four-Point Boundary Value Problems for p-Laplacian Dynamic Equations on Time Scales, *Advances in Difference Equations*, Volume 2008, Article ID 496078, 9 pages doi:10.1155/2008/496078.

[12] M. Feng, H. Feng, X. Zhang and W. Ge, Triple positive solutions for a class of m-point dynamic equations on time scales with p-Laplacian, *Math. Comput. Modelling*, 48(2008) 1213–1226.

[13] M. Feng, B. Du and W. Ge, Impulsive boundary value problems with integral boundary conditions and one-dimensional p-Laplacian, *Nonlinear Anal.*, 70(2009)3119–3126.

[14] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[15] J. Dugundji and A. Granas, *Fixed Point Theory*, in: Mongrafie Mat., PWN, Warsaw, 1982.

[16] F. Geng, D. Zhu and Q. Lu, A new existence result for impulsive dynamic equations on timescales, *Appl. Math. Lett.*, 20(2007)206–212.

[17] F. Geng, Y. Xu and D. Zhu, Periodic boundary value problems for first-order impulsive dynamic equations on time scales, *Nonlinear Anal.*, 69(2008)4074–4087.

[18] J. R. Graef and A. Ouahab, Extremal solutions for nonresonance impulsive functional dynamic equations on time scales, *Appl. Math. Comput.*, 196(2008)333–339.

[19] J. Henderson, Double solutions of impulsive dynamic boundary value problems on time scale, *J. Difference Equ. Appl.*, 8(2002)345–356.

[20] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, *Results Math.*, 18(1990)18–56.

[21] Z. He and J. S. Yu, Periodic boundary value problem for first order impulsive functional differential equations, *J. Comput. Appl. Math.*, 138(2002)205–217.

[22] Z. He and X. Zhang, Monotone iteative technique for first order impulsive differential equations with peroidic boundary conditions, *Appl. Math. Comput.*, 156(2004)605–620.

[23] B. Kaymakcalan, V. Lakshmikantham and S. Sivasundaram, *Dynamical Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.

[24] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[25] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Spriger-Verlag, New York, 1984.

[26] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operator on ordered Banach space, *Indiana Math. J.*, 28(1979)673–688.

[27] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[28] N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, 1978.

[29] K. Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. London Math. Soc.*, 63(2001)690–704.

[30] J. L. Li and J. H. Shen, Existence of positive periodic solutions to a class of functional differential equations with impulses, *Mathematica Applicata*, (17)(2004)456–463.

[31] J. L. Li, J. J. Nieto and J. Shen, Impulsive periodic boundary value problems of first-order differential equastions, *J. Math. Anal. Appl.*, 325(2007)226–236.

[32] J. L. Li and J. H. Shen, Positive solutions for first-order difference equation with impulses, *Int. J. Differ. Equ.*, 2(2006)225–239.

[33] J. L. Li and J. H. Shen, Existence results for second-order impulsive boundary value problems on time scales, *Nonlinear Anal.*, 70(2009)1648–1655.

[34] X. Liu, Nonlinear boundary value problems for firster order impulsive integro-diffeerential equations, *Appl. Anal.*, 36(1990)119–130.

[35] J. J. Nieto, Basic theory for nonresonance impulsive periodic problems of first order, *J. Math. Anal. Appl.*, 205(1997)423–433.

[36] J. J. Nieto, Impulsive resonance periodic problems of first order, *Appl. Math. Lett.*, 15(2002)489–493.

[37] J. J. Nieto, Periodic boundary value problems for first-order impulsive ordinary diffeerential equations, *Nonlinear Anal.*, 51(2002)1223–1232.

[38] J. P. Sun and W. T. Li, Existence of solutions to nonlinear first-order PBVPs on time scales, *Nonlinear Anal.*, 67(2007)883–888.

[39] J. P. Sun and W. T. Li, Existence and multiplicity of positive solutions to nonlinear first-order PBVPs on time scales, *Comput. Math. Appl.*, 54(2007)861–871.

[40] A. S. Vatsala and Y. Sun, Periodic boundary value problems of impulsive differential equations, *Appl. Anal.*, 44(1992)145–158.

[41] D. B. Wang, Positive solutions for nonlinear first-order periodic boundary value problems of impulsive dynamic equations on time scales, *Comput. Math. Appl.*, 56(2008)1496–1504.

[42] D. B. Wang, J. P. Sun and X. J. Li, Positive solutions for system of first-order dynamic equations, submitted.

[43] X. Zhang and W. Ge, Impulsive boundary value problems involving the one-dimensional p-Laplacian, *Nonlinear Anal.*, 70(2009)1692–1701.