EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR FIRST-ORDER IMPULSIVE DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. By using the classical fixed point index theorem for compact maps and the Leggett-Williams fixed point theorem respectively, in this paper, some results of single and multiple positive solutions to a class of nonlinear first-order periodic boundary value problems of impulsive dynamic equations on time scales are obtained. Two examples are given to illustrate the main results in this paper.

Keywords: Time scale; Boundary value problem; Positive solution; Fixed point; Impulsive dynamic equation

AMS (MOS) Subject Classification: 39A10, 34B15

1. INTRODUCTION

Let $T$ be a time scale, i.e., $T$ is a nonempty closed subset of $R$. Let $T > 0$ be fixed and 0, $T$ be points in $T$, an interval $(0, T)_T$ denote time scales interval, that is, $(0, T)_T := (0, T) \cap T$. Other types of intervals are defined similarly. Some definitions concerning time scales can be found in [1, 6, 7, 20, 23].

In this paper, we are concerned with the existence of positive solutions for the following nonlinear first-order periodic boundary value problem on time scale

\begin{equation}
\begin{aligned}
&x^\Delta(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in J := [0, T]_T, t \neq t_k, k = 1, 2, \ldots, m, \\
x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \\
x(0) = x(\sigma(T)),
\end{aligned}
\end{equation}

where $f \in C(J \times [0, \infty), [0, \infty))$, $I_k \in C([0, \infty) \times [0, \infty))$, $p : [0, T]_T \to (0, \infty)$ is right-dense continuous, $t_k \in (0, T)_T$, $0 < t_1 < \cdots < t_m < T$, and for each $k = 1, 2, \ldots, m$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$. 

Received February 10, 2009 1056-2176 $15.00 \copyright$Dynamic Publishers, Inc.
The theory of impulsive differential equations is emerging as an important area of investigation, since it is a lot richer than the corresponding theory of differential equations without impulse effects. Moreover, such equations may exhibit several real-world phenomena in physics, biology, engineering, etc. (see [3, 4, 27]). At the same time, the boundary value problems for impulsive differential equations and impulsive difference equations have received much attention [2, 13, 21, 22, 30–32, 34–37, 40, 43]. On the other hand, recently, the theory of dynamic equations on time scales has become a new important branch (See, for example, [1, 6, 7, 10–12, 20, 23, 38, 39, 42]). Naturally, some authors have focused their attention on the boundary value problems of impulsive dynamic equations on time scales [5, 8, 9, 16–19, 33, 41]. In particular, for the first order impulsive dynamic equations on time scales

\[
\begin{align*}
\left\{
\begin{array}{ll}
y(\Delta(t)) + p(t)y(\sigma(t)) &= f(t, y(t)), & t \in J := [a, b], t \neq t_k, k = 1, 2, \ldots, m, \\
y(t_k^+) &= I_k(y(t_k^-)), & k = 1, 2, \ldots, m, \\
y(a) &= \eta,
\end{array}
\right.
\end{align*}
\]

where \(T\) is a time scale which has at least finitely-many right-dense points, \([a, b] \subset T\), \(p\) is regressive and right-dense continuous, \(f : T \times \mathbb{R} \to \mathbb{R}\) is given function, \(I_k \in C(\mathbb{R}, \mathbb{R})\). The paper [8] obtained the existence of one solution to problem (1.2) by using the nonlinear alternative of Leray-Schauder type [15].

In [9], Benchohra et al considered the following impulsive boundary value problem on time scales

\[
\begin{align*}
\left\{
\begin{array}{ll}
-y(\Delta)(t) &= f(t, y(t)), & t \in J := [0, 1]_T, t \neq t_k, \\
y(t_k^+) - y(t_k^-) &= I_k(y(t_k^-)), \\
y(t_k^+) - y(t_k^-) &= I_k(y(t_k^-)), \\
y(0) &= y(1) = 0.
\end{array}
\right.
\end{align*}
\]

They proved the existence of one solution to problem (1.3) by applying Schaefer’s fixed point theorem [28] and the nonlinear alternative of Leray-Schauder type [15].

In [33], Li and Shen studied problem (1.3). Some existence results to problem (1.3) are established by using a fixed point theorem, which is due to Krasnoselskii and Zabreiko [25], and the Leggett-Williams fixed point theorem [14, 26].

In [41], the first author studied problem (1.1). The existence of positive solutions to problem (1.1) was obtained by means of the well-known Guo-Krasnoselskii fixed point theorem [14].
Recently, Feng et al [13] considered the following impulsive boundary value problem

\[
\begin{cases}
- \Phi_p(u'(t))' = f(t, u(t)), & t \neq t_k, t \in (0, 1), \\
- \Delta u \big|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \ldots, n, \\
u'(0) = 0, & u(1) = \int_0^1 g(t)u(t)dt.
\end{cases}
\]

By using the classical fixed point index theorem for compact maps [24, 29], some sufficient conditions for the existence of multiple positive solutions to problem (1.4) are obtained.

Motivated by the results mentioned above, in this paper, we shall show that problem (1.1) has at least one or two or three positive solutions by means of the classical fixed point index theorem for compact maps [24, 29] and the Leggett-Williams fixed point theorem [14, 26]. We note that for the case \( I_k(x) \equiv 0, k = 1, 2, \ldots, m \), problem (1.1) reduces to the problem studied by [38, 39].

In the remainder of this section, we state the following theorem, which are crucial to our proof.

**Theorem 1.1** ([24, 29]). Let \( K \) be a cone in a real Banach space \( X \). Let \( D \) be an bounded open subset of \( X \) with \( D_K = D \cap K \neq \emptyset \) and \( \overline{D}_K \neq K \). Assume that \( A : \overline{D}_K \rightarrow K \) is completely continuous such that \( x \neq Ax \) for \( x \in \partial D_K \). Then the following results hold:

(i) If \( \|Ax\| \leq \|x\| \), \( x \in \partial D_K \), then \( i_K(A, D_K) = 1 \).

(ii) If there exist \( e \in K \setminus \{0\} \) such that \( x \neq Ax + \lambda e \) for all \( x \in \partial D_K \) and all \( \lambda > 0 \), then \( i_K(A, D_K) = 0 \).

(iii) Let \( U \) be open in \( K \) such that \( \overline{U} \subset D_K \). If \( i_K(A, D_K) = 1 \) and \( i_K(A, U_K) = 0 \), then \( A \) has a fixed point in \( D_K \setminus \overline{U}_K \). The same result holds if \( i_K(A, D_K) = 0 \) and \( i_K(A, U_K) = 1 \).

**Remark 1.1.** In theorem 1.1, the use of (ii) give better results than use of the common assumption \( \|Ax\| \geq \|x\| \) for all \( x \in \partial D_K \).

Let \( E \) be a real Banach space and \( K \subset E \) be a cone. A function \( \beta : K \rightarrow [0, \infty) \) is called a nonnegative continuous concave functional if \( \beta \) is continuous and

\[ \beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y) \]

for all \( x, y \in K \) and \( t \in [0, 1] \).

Let \( a, b > 0 \) be constants, \( K_a = \{x \in K : \|x\| < \alpha \} \), \( K(\beta, a, b) = \{x \in K : a \leq \beta(x), \|x\| \leq b \} \).

**Theorem 1.2** ([14, 26]). Let \( A : \overline{K}_c \rightarrow \overline{K}_c \) be a completely continuous map and \( \beta \) be a nonnegative continuous concave functional on \( K \) such that \( \beta(x) \leq \|x\| \), \( \forall x \in \overline{K}_c \). Suppose there exist \( a, b, d \) with \( 0 < d < a < b \leq c \), such that:
Then \( A \) has at least three fixed points \( x_1, x_2, x_3 \) satisfying
\[
\|x_1\| < d, \quad a < \beta(x_2) \text{ and } \|x_3\| > d \text{ with } \beta(x_3) < a.
\]

2. PRELIMINARIES

Throughout the rest of this paper, we always assume that the points of impulse \( t_k \) are right-dense for each \( k = 1, 2, \ldots, m \).

We define
\[
PC = \{ x \in [0, \sigma(T)]_T \to R : x_k \in C(J_k, R), \quad k = 1, 2, \ldots, m \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k^-) = x(t_k), \quad k = 1, 2, \ldots, m \},
\]
where \( x_k \) is the restriction of \( x \) to \( J_k = (t_k, t_{k+1}]_T \subset (0, \sigma(T)]_T, \quad k = 1, 2, \ldots, m \) and \( J_0 = [0, t_1]_T, \quad J_{m+1} = \sigma(T) \).

Let
\[
X = \{ x(t) : x(t) \in PC, \ x(0) = x(\sigma(T)) \}
\]
with the norm \( \|x\| = \sup_{t \in [0, \sigma(T)]_T} |x(t)| \). Then \( X \) is a Banach space.

**Definition 2.1.** A function \( x \in PC \cap C^1(J \setminus \{t_1, t_2, \ldots, t_m\}, R) \) is said to be a solution of the problem (1.1) if and only if \( x \) satisfies the dynamic equation
\[
x^\Delta(t) + p(t)x(\sigma(t)) = f(t, x(\sigma(t))) \quad \text{everywhere on } J \setminus \{t_1, t_2, \ldots, t_m\},
\]
the impulsive conditions
\[
x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m,
\]
and the periodic boundary condition \( x(0) = x(\sigma(T)) \).

**Lemma 2.1** ([41]). Suppose \( h : [0, T]_T \to R \) is rd-continuous, then \( x \) is a solution of
\[
x(t) = \int_0^{\sigma(T)} G(t, s)h(s)\Delta s + \sum_{k=1}^m G(t, t_k)I_k(x(t_k)), \quad t \in [0, \sigma(T)]_T,
\]
where
\[
G(t, s) = \begin{cases} \frac{\sigma_p(s, t)_p}{\sigma_p(\sigma(T), 0)_p} & 0 \leq s \leq t \leq \sigma(T), \\ \frac{\sigma_p(\sigma(T), 0)_p - 1}{\sigma_p(\sigma(T), 0)_p - 1} & 0 \leq t < s \leq \sigma(T), \end{cases}
\]
if and only if \( x \) is a solution of the boundary value problem
\[
\begin{cases} x^\Delta(t) + p(t)x(\sigma(t)) = h(t), \quad t \in J := [0, T]_T, t \neq t_k, k = 1, 2, \ldots, m, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \\ x(0) = x(\sigma(T)). \end{cases}
\]
Lemma 2.2. Let $G(t,s)$ be defined as Lemma 2.1, then

\[
A \triangleq \frac{1}{e_{\rho}(\sigma(T),0)} - 1 \leq G(t,s) \leq \frac{e_{\rho}(\sigma(T),0)}{e_{\rho}(\sigma(T),0)} - 1 \triangleq B \text{ for all } t, s \in [0,\sigma(T)]_T.
\]

Proof. It is obvious, so we omit it here. \hfill \square

Let

\[
K = \{ x(t) \in X : x(t) \geq \delta \|x\| \},
\]

where \( \delta = \frac{4}{B} = \frac{1}{e_{\rho}(\sigma(T),0)} \in (0,1) \). It is not difficult to verify that \( K \) is a cone in \( X \).

We define an operator \( \Phi : K \to X \) by

\[
(\Phi x)(t) = \int_{0}^{\sigma(T)} G(t,s)f(s,x(s))\Delta s + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k)), \quad t \in [0,\sigma(T)]_T.
\]

Lemma 2.3 ([41]). \( \Phi : K \to K \) is completely continuous.

3. EXISTENCE OF ONE OR TWO POSITIVE SOLUTIONS

We define

\[
\Omega_\rho = \left\{ x \in K : \min_{t \in [0,\sigma(T)]_T} x(t) < \delta \rho \right\} = \left\{ x \in X : \delta \|x\| \leq \min_{t \in [0,\sigma(T)]_T} x(t) < \delta \rho \right\}.
\]

Similar to [29, Lemma 2.5], we have the following result:

Lemma 3.1 ([42]). \( \Omega_\rho \) has the following properties:

1. \( \Omega_\rho \) is open relative to \( K \).
2. \( K_\delta \rho \subset \Omega_\rho \subset K_\rho \).
3. \( x \in \partial \Omega_\rho \) if and only if \( \min_{t \in [0,\sigma(T)]_T} x(t) = \delta \rho \).
4. If \( x \in \partial \Omega_\rho \), then \( \delta \rho \leq x(t) \leq \rho \) for \( t \in [0,\sigma(T)]_T \).

For convenience, we denote:

\[
f_{\delta \rho} = \min \left\{ \min_{t \in [0,T]_T} \frac{f(t,x)}{\rho} : x \in [\delta \rho, \rho] \right\},
\]

\[
f_0 = \max \left\{ \max_{t \in [0,T]_T} \frac{f(t,x)}{\rho} : x \in [0, \rho] \right\}; \quad I_0^\rho(k) = \max \{ I_k(x) : x \in [0, \rho] \},
\]

\[
f^i = \lim_{x \to -i} \sup \max_{t \in [0,T]_T} \frac{f(t,x)}{x}, \quad I^i(k) = \lim_{x \to -i} \sup \frac{I_k(x)}{x},
\]

\[
f_i = \lim_{x \to -i} \inf \min_{t \in [0,T]_T} \frac{f(t,x)}{x}, \quad I_i(k) = \lim_{x \to -i} \inf \frac{I_k(x)}{x}, \quad \text{where } i = \infty, \text{ or } 0^+.
\]

\[
l = \frac{1}{B(\sigma(T)+m)}, \quad L = \frac{1}{A\sigma(T)}.
\]

Now we state our main results

Lemma 3.2. If there exists \( \rho > 0 \) such that \( f_0^\rho < l, \quad I_0^\rho(k) < \rho l \), then \( i_K(\Phi, K_\rho) = 1 \).
Proof. Since $f_0^p < l$, $I_0^p(k) < pl$, by (2.2) and Lemma 2.2 we have for $x \in \partial K_\rho$

$$
(\Phi x)(t) = \int_0^{\sigma(T)} G(t, s)f(s, x(\sigma(s)))\Delta s + \sum_{k=1}^{m} G(t, t_k)I_k(x(t_k))
$$

$$
\leq B \int_0^{\sigma(T)} f(s, x(\sigma(s)))\Delta s + B \sum_{k=1}^{m} I_k(x(t_k))
$$

$$
< B(\sigma(T) + m)pl \quad \text{i.e., } \|\Phi x\| < \|x\| \quad \text{for } x \in \partial K_\rho,
$$
then by (i) of Theorem 1.1 we have $i_K(\Phi, K_\rho) = 1$. □

Lemma 3.3. If there exists $\rho > 0$ such that $f_0^p > L$, then $i_K(\Phi, \Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$, $t \in [0, \sigma(T)]_T$, then $e(t) \in \partial K_1$. We claim that $x \neq \Phi x + \lambda e$, $x \in \partial \Omega_\rho$, $\lambda > 0$.

In fact, if not, there exist $x_0 \in \partial \Omega_\rho$ and $\lambda_0 > 0$ such that $x_0 = \Phi x_0 + \lambda_0 e$. Then by (2.2) and Lemma 2.2 we have

$$
x_0 = \Phi x_0 + \lambda_0 e \geq \delta \|\Phi x_0\| + \lambda_0 \geq A\delta \int_0^{\sigma(T)} f(s, x_0(\sigma(s)))\Delta s + \lambda_0
$$

$$
> A\delta \sigma(T)\rho L + \lambda_0 = \delta \rho + \lambda_0,
$$
then from (3) of Lemma 3.1 we get $\delta \rho > \delta \rho + \lambda_0$, which is a contradiction. Hence, by (ii) of Theorem 1.1 we have $i_K(\Phi, \Omega_\rho) = 0$. □

Theorem 3.1. Suppose one of the following conditions holds:

\[ \text{(H}_1\text{)} \quad \text{There exist } \rho_1, \rho_2, \rho_3 \in (0, \infty), \text{ with } \rho_1 < \delta \rho_2 \text{ and } \rho_2 < \rho_3 \text{ such that } \]

\[ f_0^{\rho_1} < l, \quad I_0^{\rho_1}(k) < \rho_1 l, \quad f_0^{\rho_2} > L, \quad f_0^{\rho_3} < l, \quad I_0^{\rho_3}(k) < \rho_3 l. \]

\[ \text{(H}_2\text{)} \quad \text{There exist } \rho_1, \rho_2, \rho_3 \in (0, \infty), \text{ with } \rho_1 < \rho_2 < \delta \rho_3 \text{ such that } \]

\[ f_0^{\rho_1} > L, \quad f_0^{\rho_2} < l, \quad I_0^{\rho_2}(k) < l \rho_2, \quad f_0^{\rho_3} > L. \]

Then problem (1.1) has at least two positive solutions $x_1, x_2$ with $x_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$.

Proof. Suppose (H$_1$) holds. We claim that $\Phi$ have two fixed points $x_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$. In fact, if (H$_1$) holds then by Lemma 3.2 and Lemma 3.3 we obtain:

\[ i_K(\Phi, K_{\rho_1}) = 1, \quad i_K(\Phi, \Omega_{\rho_2}) = 0, \quad i_K(\Phi, K_{\rho_3}) = 1. \]

Then from (2) of Lemma 3.1 and $\rho_1 < \delta \rho_2$, we have $K_{\rho_1} \subset K_{\delta \rho_2} \subset \Omega_{\rho_2}$. Therefore, (iii) of Theorem 1.1 implies that the $\Phi$ has two fixed points $x_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$ which are the positive solutions of problem (1.1).

If (H$_2$) holds, the proof is similar to that of the case when (H$_1$) holds, so we omit here. □
Corollary 3.1. If there exist \( \rho_1, \rho_2 \in (0, \infty) \), with \( \rho_1 < \delta \rho_2 \) such that one of the following conditions holds:

(H_3) \( f_0^\rho_1 < l, \ I_0^\rho_1(k) < \rho_1 l, \ f_0^\rho_2 > L, \ 0 \leq f^\infty < l, \ 0 \leq I^\infty(k) < l \).

(H_4) \( f_\delta^\rho_1 > L, \ f_0^\rho_2 < l, \ I_0^\rho_2(k) < \rho_2 l, \ L < f^\infty \leq \infty, \ L < I^\infty(k) \leq \infty \).

Then problem (1.1) has at least two positive solutions in \( K \).

Proof. Suppose (H_3) holds. We show that (H_3) implies (H_1). Let \( \eta \in (f^\infty, l) \). Then there exists \( \alpha > \eta \) such that \( \max_{t \in [0, T]} f(t, x) \leq \eta x \) for \( x \in [\alpha, \infty) \) since \( 0 \leq f^\infty < l \).

Taking

\[
\theta = \max \left\{ \max_{t \in [0, T]} f(t, x) : 0 \leq x \leq \alpha \right\} \quad \text{and} \quad \rho_3 > \left\{ \frac{\theta}{l - \eta}, \ \rho_2 \right\}.
\]

Then we get

\[
\max_{t \in [0, T]} f(t, x) \leq \eta x + \theta \leq \eta \rho_3 + \theta < l \rho_3, \quad x \in [0, \rho_3].
\]

This implies that \( f_0^\rho_3 < l \). Similarly, by \( 0 \leq I^\infty(k) < l \), we have \( I_0^\rho_2(k) < \rho_3 l \). So, (H_3) implies (H_1). Similarly (H_4) implies (H_2). \( \square \)

By an argument similar to that of Theorem 3.1 we obtain the following results.

Theorem 3.2. Suppose one of the following conditions holds:

(H_5) There exist \( \rho_1, \rho_2 \in (0, \infty) \), with \( \rho_1 < \delta \rho_2 \) such that

\[
f_0^\rho_1 \leq l, \quad I_0^\rho_1(k) \leq \rho_1 l, \quad f_\delta^\rho_2 \geq L.
\]

(H_6) There exist \( \rho_1, \rho_2 \in (0, \infty) \), with \( \rho_1 < \rho_2 \) such that

\[
f_\delta^\rho_1 \geq L, \quad f_0^\rho_2 \leq l, \quad I_0^\rho_2(k) < \rho_2 l.
\]

Then problem (1.1) has at least one positive solutions in \( K \).

As a special case of Theorem 3.2, we obtain the following result.

Corollary 3.2. Suppose one of the following conditions holds:

(H_7) \( 0 \leq f^0 < l, \ 0 \leq I^0(k) < l \) and \( L < f^\infty \leq \infty \).

(H_8) \( 0 \leq f^\infty < l, \ 0 \leq I^\infty(k) < l \) and \( L < f^0 \leq \infty \).

Then problem (1.1) has at least one positive solutions in \( K \).
4. EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS

For \( x \in K \), let
\[
\beta(x) = \min_{t \in [0, \sigma(T)]_T} x(t),
\]
then it is easy to see that \( \beta \) is a nonnegative continuous concave functional on \( K \) with \( \beta(x) \leq \|x\| \) for \( x \in K \).

**Theorem 4.1.** Suppose that the following conditions are hold:

(C1) There exist positive constants \( c_k \) such that \( I_k(x) \leq c_k \), for \( x \geq 0 \) and \( k = 1, 2, \ldots, m \).

(C2) There exist \( B \sum_{k=1}^{m} c_k < d < a \) such that
\[
\begin{align*}
\text{(4.1)} & \quad f(t, x) < \frac{d - B \sum_{k=1}^{m} c_k}{B \sigma(T)}, \quad \text{for } t \in [0, T]_T, \ x \in [0, d], \\
\text{and} & \\
\text{(4.2)} & \quad f(t, x) > \frac{a}{A \sigma(T)}, \quad \text{for } t \in [0, T]_T, \ x \in [a, \frac{1}{\delta} a].
\end{align*}
\]

(C3) Assume that one of the following conditions satisfies:

(a) \( \lim_{x \to \infty} \max_{t \in [0, T]_T} \frac{f(t, x)}{x} < \frac{1}{B \sigma(T)} \);

(b) There exists \( c > \frac{1}{\delta} a \) such that \( f(t, x) < \frac{c - B \sum_{k=1}^{m} c_k}{B \sigma(T)} \), \( t \in [0, T]_T \), \( x \in [0, c] \).

Then problem (1.1) has at least three positive solutions.

**Proof.** First, we assert that there exists \( h > \frac{1}{\delta} a \) such that \( \Phi : \overline{K}_h \to K_h \) if (a) holds.

In fact, if (a) holds, then there exist \( M > 0 \) and \( \varepsilon < \frac{1}{B \sigma(T)} \) such that
\[
f(t, x) < \varepsilon x, \quad x > M.
\]

Set
\[
\xi = \max \{ f(t, x) : t \in [0, T]_T, x \in [0, M] \}.
\]

It follows that \( f(t, x) \leq \varepsilon x + \xi \), for all \( x \in [0, \infty) \). Take
\[
h > \max \left\{ \frac{1}{\delta} a, \frac{B (\sigma(T) \xi + \sum_{k=1}^{m} c_k)}{1 - \varepsilon B \sigma(T)} \right\}.
\]

If \( x \in \overline{K}_h \), then
\[
(\Phi x)(t) = \int_{0}^{\sigma(T)} G(t, s) f(s, x(\sigma(s))) \Delta s + \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k))
\leq B \int_{0}^{\sigma(T)} f(s, x(\sigma(s))) \Delta s + B \sum_{k=1}^{m} I_k(x(t_k))
\leq B \sigma(T) (\varepsilon h + \xi) + B \sum_{k=1}^{m} c_k
< h.
\]
Thus \( \| \Phi x \| < h \), i.e., \( \Phi x \in K_h \).

Now we assert that \( \Phi : \overline{K}_h \to K_\pi \) if there exists \( \hat{h} > \frac{1}{\delta} a \) such that \( f(t, x) < \frac{\hat{h} - B \sum c_k}{B \sigma(T)} \), \( t \in [0, T]_T, x \in [0, \hat{h}] \) (that is, (b) holds).

Indeed, if \( x \in \overline{K}_h \), then
\[
(\Phi x)(t) = \int_0^{\sigma(T)} G(t, s)f(s, x(\sigma(s)))\Delta s + \sum_{k=1}^{m} G(t, t_k)I_k(x(t_k))
\]
\[
< B\sigma(T)\frac{\hat{h} - B \sum c_k}{B \sigma(T)} + B \sum c_k
\]
\[
= \hat{h}.
\]

So, we have proved that there exist positive number \( c > \frac{1}{\delta} a \) such that \( \Phi : \overline{K}_c \to K_c \) if condition (a) or (b) hold. Furthermore, by (4.1) we have \( \Phi : \overline{K}_d \to K_d \).

Second, we assert that \( \{ x \in K(\beta, a, \frac{1}{\delta} a) : \beta(x) > a \} \neq \emptyset \) and \( \beta(\Phi x) > a \) for all \( x \in K(\beta, a, \frac{1}{\delta} a) \).

In fact, take \( x \equiv \frac{a + \frac{1}{\delta} a}{2} \); so \( x \in \{ x \in K(\beta, a, \frac{1}{\delta} a) : \beta(x) > a \} \). Moreover, for \( x \in K(\beta, a, \frac{1}{\delta} a) \), then \( \beta(x) \geq a \) and we have

\[
\frac{1}{\delta} a \geq \| x \| \geq \min_{t \in [0, \sigma(T)]_T} x(t) = \beta(x) \geq a.
\]

Thus, in view of (4.2) we get
\[
\beta(\Phi x) = \min_{t \in [0, \sigma(T)]_T} \left[ \int_0^{\sigma(T)} G(t, s)f(s, x(\sigma(s)))\Delta s + \sum_{k=1}^{m} G(t, t_k)I_k(x(t_k)) \right]
\]
\[
\geq \min_{t \in [0, \sigma(T)]_T} \int_0^{\sigma(T)} G(t, s)f(s, x(\sigma(s)))\Delta s
\]
\[
> A\sigma(T)\frac{a}{A\sigma(T)}
\]
\[
= a.
\]

Finally, we assert that \( \beta(\Phi x) > a \) if \( x \in K(\beta, a, c) \) and \( \| \Phi x \| > \frac{1}{\delta} a \).

To do this, if \( x \in K(\beta, a, c) \) and \( \| \Phi x \| > \frac{1}{\delta} a \), then
\[
\beta(\Phi x) = \min_{t \in [0, \sigma(T)]_T} \left[ \int_0^{\sigma(T)} G(t, s)f(s, x(\sigma(s)))\Delta s + \sum_{k=1}^{m} G(t, t_k)I_k(x(t_k)) \right]
\]
\[
\geq \delta \| \Phi x \| > \delta \frac{1}{\delta} a = a.
\]

To sum up, all the hypotheses of Theorem 1.2 are satisfied by taking \( b = \frac{1}{\delta} a \). Hence \( \Phi \) has at least three fixed points, that is, problem (1.1) has at least three positive solutions \( x_1, x_2 \) and \( x_3 \) such that
\[
\| x_1 \| < d, \ a < \beta(x_2) \text{ and } \| x_3 \| > d \text{ with } \beta(x_3) < a. \]
5. EXAMPLES

Example 5.1. Let $T = [0, 1] \cup [2, 3]$. We consider the following problem

$$
\begin{aligned}
& x^{\infty}(t) + x(\sigma(t)) = f(t, x(\sigma(t))), \quad t \in [0, 3]_T, t \neq \frac{1}{2}, \\
& x(\frac{1}{2}^+) - x(\frac{1}{2}^-) = I(x(\frac{1}{2})), \\
& x(0) = x(3),
\end{aligned}
$$

(5.1)

where $p(t) \equiv 1$, $T = 3$, $m = 1$, $f(t, x(\sigma(t))) = (t + 1)(x(\sigma(t)))^2$, and $I(x) = x^2$.

It is easy to see that
\[
\delta = \frac{1}{2e^2}, \quad l = \frac{2e^2 - 1}{8e^2}, \quad L = \frac{2e^2 - 1}{3}.
\]

Taking $\rho_1 = \frac{2e^2 - 1}{40e^2}$, $\rho_2 = \frac{2e^2 - 1}{8e^2}$, then by simple calculation we have $\rho_1 < \delta \rho_2$,
\[
f^{(1)}_0 = 4\rho_1 = \frac{2e^2 - 1}{10e^2} < \frac{2e^2 - 1}{8e^2} = l, \quad f^{(1)}_0(1) = \rho_1^2 = \left(\frac{2e^2 - 1}{40e^2}\right)^2 < \frac{2e^2 - 1}{40e^2} \cdot \frac{2e^2 - 1}{8e^2} = \rho_1 l, \quad
\]
\[
f^{(2)}_0 = \delta^2 \rho_2 = \frac{1}{4e^2} \cdot (4e^2 - 2e^4) = \frac{2e^2 - 1}{2} > \frac{2e^2 - 1}{3} = L.
\]

Therefore, together with Theorem 3.2, it follows that problem (5.1) has at least one positive solution.

Example 5.2. Let $T = [0, 1] \cup [2, 3]$. We consider the following problem

$$
\begin{aligned}
& x^{\infty}(t) + x(\sigma(t)) = f(t, x(\sigma(t))), t \in [0, 3]_T, t \neq \frac{1}{2}, \\
& x(\frac{1}{2}^+) - x(\frac{1}{2}^-) = I(x(\frac{1}{2})), \\
& x(0) = x(3),
\end{aligned}
$$

(5.2)

where $p(t) \equiv 1$, $T = 3$, $m = 1$, $I(x) = \frac{1}{x^2 + 1}$ and

$$
f(t, x) \equiv f(x) = \begin{cases}
\frac{1}{8e^2}, & 0 \leq x \leq \frac{2e^2 + 1}{2e^2 - 1}, \\
g(x) & \frac{2e^2 + 1}{2e^2 - 1} \leq x \leq 2, \\
\frac{16e^4 - 10e^2 - 1}{6e^2} & 2 \leq x \leq 8e^2, \\
s(x) & x \geq 8e^2,
\end{cases}
$$

here $g(x)$ and $s(x)$ satisfy: $g(\frac{2e^2 + 1}{2e^2 - 1}) = \frac{1}{8e^2}$, $g(2) = \frac{16e^4 - 10e^2 - 1}{6e^2}$, $s(8e^2) = \frac{16e^4 - 10e^2 - 1}{6e^2}$, $g^{\infty}(x) = 0$ for $x \in (\frac{2e^2 + 1}{2e^2 - 1}, 2)$ and $s(x) : (-\infty, \infty) \to [0, \infty)$ is continuous.

Choose $d = \frac{2e^2 + 1}{2e^2 - 1}$, $a = 2$, $b = 4e^2$, $c = 8e^2$; then by $BC_1 = B = \frac{2e^2}{2e^2 - 1}$ (for $c_1 = 1$) and $\delta = \frac{1}{2e^2}$ we have $BC_1 < d < a < \frac{1}{3} a = b < c$, and then $f(x)$ satisfies

$$
f(x) = \begin{cases}
\frac{1}{8e^2} < \frac{1}{6e^2} = \frac{2e^2 + 1}{3} \cdot \frac{2e^2 - 1}{2e^2 - 1} = \frac{d - Bc}{B\sigma(T)}, & x \in [0, \frac{2e^2 + 1}{2e^2 - 1}] = [0, d]; \\
\frac{16e^4 - 10e^2 - 1}{6e^2} > \frac{4e^2 - 2}{3} = \frac{a}{A\sigma(T)}, & x \in [2, 4e^2] = [a, b]; \\
\frac{16e^4 - 10e^2 - 1}{6e^2} < \frac{16e^4 - 10e^2}{6e^2} = \frac{c - Bc}{B\sigma(T)}, & x \in [0, 8e^2] = [0, c].
\end{cases}
$$
Then by Theorem 4.1, problem (5.2) has at least three positive solutions $x_1$, $x_2$ and $x_3$ such that

$$\|x_1\| < \frac{2e^2 + 1}{2e^2 - 1}, \quad 2 < \beta(x_2) \quad \text{and} \quad \|x_3\| > \frac{2e^2 + 1}{2e^2 - 1} \quad \text{with} \quad \beta(x_3) < 2.$$ 

REFERENCES


