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EXISTENCE OF POSITIVE SOLUTIONS TO A SYSTEM OF SINGULAR BOUNDARY VALUE PROBLEMS

NASEER AHMAD ASIF[1], RAHMAT ALI KHAN[1], AND JOHNNY HENDERSON[2]

[1] Centre for Advanced Mathematics and Physics
National University of Sciences and Technology
Campus of College of Electrical and Mechanical Engineering
Peshawar Road, Rawalpindi, Pakistan
E-mail: naseerasif@yahoo.com; rahmat_alipk@yahoo.com
[2]Department of Mathematics, Baylor University
Waco, TX 76798-7328 USA
E-mail: Johnny_Henderson@baylor.edu

ABSTRACT. Existence results for positive solutions of a coupled system of nonlinear singular two point boundary value problems of the type

$$\begin{aligned} -x''(t) &= p(t)f(t, y(t), x'(t)), \quad t \in (0, 1), \\ -y''(t) &= q(t)g(t, x(t), y'(t)), \quad t \in (0, 1), \\ a_1x(0) - b_1x'(0) &= x'(1) = 0, \\ a_2y(0) - b_2y'(0) &= y'(1) = 0, \end{aligned}$$

are established. The nonlinearities $f, g : [0, 1] \times [0, \infty) \times (0, \infty) \to [0, \infty)$ are allowed to be singular at $x' = 0$ and $y' = 0$. The functions $p, q \in C(0, 1)$ are positive on $(0, 1)$ and the constants a_i, b_i $(i = 1, 2) > 0$. An example is included to show the applicability of our result.

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1. INTRODUCTION

Existence theory for nonlinear boundary value problems (BVPs) has attracted the attention of many researchers; see for example, [9, 10, 20] for scalar equations, and for system of BVPs, see [5, 16]. Recently, the study of singular BVPs has also attracted some attention, see for example, [3, 11, 12, 13, 15, 19] and the references therein. An excellent resource with an extensive bibliography was produced by Agarwal and O'Regan [1].

In [1], Agarwal and O'Regan studied existence of at least one positive solution for the following BVP

$$\begin{aligned} -y''(t) &= q(t)f(t, y(t), y'(t)), \quad t \in (0, 1), \\ y(0) &= y'(1) = 0, \end{aligned} \tag{1.1}$$

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where $f : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)$ is continuous and allowed to be singular at $y' = 0$, $q \in C(0,1)$ and is positive on $(0,1)$ and the real constants $\alpha, \beta > 0$. Under similar assumptions, existence of at least one positive solution for (1.1) is also studied in [17, 18]. Existence of multiple positive solutions of the following BVP (1.1) is studied by Yan et al. [17] by using fixed point index theory.

Yan et al. [18] extended the results studied in [17] to the following two-point singular BVP

$$-y''(t) = q(t)f(t, y(t), y'(t)), \quad t \in (0,1),$$
$$\alpha y(0) - \beta y'(0) = y'(1) = 0,$$

where $f : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)$ is continuous and allowed to be singular at $y' = 0$, $q \in C(0,1)$ and is positive on $(0,1)$ and the real constants $\alpha, \beta > 0$. They used fixed point index theory in cone of an ordered Banach space to prove existence of multiple positive solutions.

Inspired by the above mentioned works, in this paper, we study existence of C^1-positive solutions for the following coupled system of two-point singular BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t, y(t), x'(t)), \quad t \in (0,1),\\ -y''(t) &= q(t)g(t, x(t), y'(t)), \quad t \in (0,1),\\ a_1 x(0) - b_1 x'(0) &= x'(1) = 0,\\ a_2 y(0) - b_2 y'(0) &= y'(1) = 0, \end{aligned} \tag{1.2}$$

where the nonlinearities $f, g : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous and are allowed to be singular at $x' = 0$, $y' = 0$, $p, q \in C(0,1)$ are positive on $(0,1)$ and the real constants a_i $(i = 1,2) > 0$, b_i $(i = 1,2) > 0$. There are several results on existence of positive solutions for system of BVPs; see for example, [6, 7, 11, 12, 13, 8, 16]. However, there are only a few results on the existence of positive solutions for systems with nonlinearities dependent on first derivative. The present manuscript is an attempt in this regard.

By a C^1-positive solution of the system (1.2), we mean $(x, y) \in (C^1[0,1] \cap C^2(0,1)) \times (C^1[0,1] \cap C^2(0,1))$ satisfying (1.2), $x > 0$, $y > 0$ on $[0,1]$, and $x' > 0$, $y' > 0$ on $[0,1)$. By singularity we mean that the functions $f(t,x,y)$ or $g(t,x,y)$ are allowed to be unbounded at $y = 0$.

Throughout the paper, assume the following conditions hold:

(A_1) $p, q \in C(0,1)$, $p, q > 0$ on $(0,1)$, $\int_0^1 p(t)dt < +\infty$ and $\int_0^1 q(t)dt < +\infty$;

(A_2) $f, g : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous and are positive on $[0,1] \times (0,\infty) \times (0,\infty)$;

(A_3) $f(t,x,y) \le k_1(x)(u_1(y) + v_1(y))$ and $g(t,x,y) \le k_2(x)(u_2(y) + v_2(y))$, where u_i $(i = 1,2) > 0$ are continuous and nonincreasing on $(0,\infty)$, k_i $(i = 1,2)$, v_i $(i = 1,2) \ge 0$ are continuous and nondecreasing on $[0,\infty)$;

(A_4)

$$\sup_{x\in(0,\infty)} \frac{x}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(x)\int_0^1 q(t)dt))\int_0^1 p(t)dt)} > 1,$$

$$\sup_{x\in(0,\infty)} \frac{x}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(x)\int_0^1 p(t)dt))\int_0^1 q(t)dt)} > 1,$$

where $I(z) = \int_0^z \frac{dy}{u_1(y)+v_1(y)}$, $J(z) = \int_0^z \frac{dy}{u_2(y)+v_2(y)}$, for $z \in (0,\infty)$;

(A_5) $I(\infty) = \infty$ and $J(\infty) = \infty$;

(A_6) for real constant $E > 0$ and $F > 0$, there exist continuous functions φ_{EF} and ψ_{EF} defined on $[0,1]$ and positive on $(0,1)$, and constants $0 \le \delta_1, \delta_2 < 1$ such that

$$f(t,x,y) \ge \varphi_{EF}(t)x^{\delta_1}, g(t,x,y) \ge \psi_{EF}(t)x^{\delta_2} \text{ on } [0,1]\times[0,E]\times[0,F];$$

(A_7) $\int_0^1 p(t)u_1(C\int_t^1 p(s)\varphi_{EF}(s)ds)dt < +\infty$ and $\int_0^1 q(t)u_2(C\int_t^1 q(s)\psi_{EF}(s)ds)dt < +\infty$ for any real constant $C > 0$.

Remark 1.1. Since I, J are continuous, $I(0) = 0$, $I(\infty) = \infty$, $J(0) = 0$, $J(\infty) = \infty$, and they are monotone increasing. Hence, I and J are invertible. Moreover, I^{-1} and J^{-1} are also monotone increasing.

2. MAIN RESULT: EXISTENCE OF AT LEAST ONE POSITIVE SOLUTION

Theorem 2.1. *Assume that* (A$_1$)–(A$_7$) *hold. Then, the system* (1.2) *has at least one* C^1*-positive solution.*

Proof. In view of (A_4), we can choose real constants $M_1 > 0$ and $M_2 > 0$ such that

$$\frac{M_1}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_1)\int_0^1 q(t)dt))\int_0^1 p(t)dt)} > 1, \tag{2.1}$$

$$\frac{M_2}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(M_2)\int_0^1 p(t)dt))\int_0^1 q(t)dt)} > 1. \tag{2.2}$$

From the continuity of k_1, k_2, I and J, we choose $\varepsilon > 0$ small enough such that

$$\frac{M_1}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_1)\int_0^1 q(t)dt + J(\varepsilon)))\int_0^1 p(t)dt + I(\varepsilon))} > 1, \tag{2.3}$$

$$\frac{M_2}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(M_2)\int_0^1 p(t)dt + I(\varepsilon)))\int_0^1 q(t)dt + J(\varepsilon))} > 1. \tag{2.4}$$

Choose real constants $L_1 > 0$ and $L_2 > 0$ such that

$$I(L_1) > k_1(M_2)\int_0^1 p(t)dt + I(\varepsilon), \tag{2.5}$$

$$J(L_2) > k_2(M_1)\int_0^1 q(t)dt + J(\varepsilon). \tag{2.6}$$

Choose $n_0 \in \{1, 2, \dots\}$ such that $\frac{1}{n_0} < \varepsilon$. For each fixed $n \in \{n_0, n_0+1, \dots\}$, define retractions $\theta_i : \mathbb{R} \to [0, M_i]$ and $\rho_i : \mathbb{R} \to [\frac{1}{n}, L_i]$ by

$$\theta_i(x) = \max\{0, \min\{x, M_i\}\} \text{ and } \rho_i(x) = \max\left\{\frac{1}{n}, \min\{x, L_i\}\right\}, \quad i = 1, 2.$$

Consider the modified system of BVPs

$$\begin{aligned}
-x''(t) &= p(t)f(t, \theta_2(y(t)), \rho_1(x'(t))), \quad t \in (0,1),\\
-y''(t) &= q(t)g(t, \theta_1(x(t)), \rho_2(y'(t))), \quad t \in (0,1),\\
a_1x(0) - b_1x'(0) &= 0, \quad x'(1) = \frac{1}{n},\\
a_2y(0) - b_2y'(0) &= 0, \quad y'(1) = \frac{1}{n}.
\end{aligned} \tag{2.7}$$

Since $f(t, \theta_2(y(t)), \rho_1(x'(t)))$, $g(t, \theta_1(x(t)), \rho_2(y'(t)))$ are continuous and bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$, by Schauder's fixed point theorem, it follows that the modified system of BVPs (2.7) has a solution $(x_n, y_n) \in (C^1[0,1] \cap C^2(0,1)) \times (C^1[0,1] \cap C^2(0,1))$.

Using (2.7) and (A_2), we obtain

$$x_n''(t) \le 0 \text{ and } y_n''(t) \le 0 \text{ for } t \in (0,1),$$

which on integration from t to 1, and using the boundary conditions (BCs), yields

$$x_n'(t) \ge \frac{1}{n} \quad \text{and} \quad y_n'(t) \ge \frac{1}{n} \text{ for } t \in [0,1]. \tag{2.8}$$

Integrating (2.8) from 0 to t, using the BCs and (2.8), we have

$$x_n(t) \ge \left(t + \frac{b_1}{a_1}\right)\frac{1}{n} \quad \text{and} \quad y_n(t) \ge \left(t + \frac{b_2}{a_2}\right)\frac{1}{n} \quad \text{for } t \in [0,1]. \tag{2.9}$$

From (2.8) and (2.9), it follows that

$$\|x_n\| = x_n(1) \text{ and } \|y_n\| = y_n(1), \text{ where } \|z\| = \max_{t\in[0,1]} |z(t)|.$$

Now, we show that

$$x_n'(t) < L_1, y_n'(t) < L_2, \quad t \in [0,1]. \tag{2.10}$$

First, we prove $x_n'(t) < L_1$ for $t \in [0,1]$. Suppose $x_n'(t_1) \ge L_1$ for some $t_1 \in [0,1]$. Using (2.7) and (A_3), we have

$$-x_n''(t) \le p(t)k_1(\theta_2(y_n(t)))(u_1(\rho_1(x_n'(t))) + v_1(\rho_1(x_n'(t)))), \quad t \in (0,1),$$

which implies that

$$\frac{-x_n''(t)}{u_1(\rho_1(x_n'(t))) + v_1(\rho_1(x_n'(t)))} \le k_1(M_2)p(t), \quad t \in (0,1).$$

Integrating from t_1 to 1, using the BCs, we obtain

$$\int_{\frac{1}{n}}^{x'_n(t_1)} \frac{dz}{u_1(\rho_1(z)) + v_1(\rho_1(z))} \le k_1(M_2) \int_{t_1}^1 p(t)dt,$$

which can also be written as

$$\int_{\frac{1}{n}}^{L_1} \frac{dz}{u_1(\rho_1(z)) + v_1(\rho_1(z))} + \int_{L_1}^{x'_n(t_1)} \frac{dz}{u_1(\rho_1(z)) + v_1(\rho_1(z))} \le k_1(M_2) \int_0^1 p(t)dt.$$

Using the increasing property of I, we obtain

$$I(L_1) + \frac{x'_n(t_1) - L_1}{u_1(L_1) + v_1(L_1)} \le k_1(M_2) \int_0^1 p(t)dt + I(\varepsilon),$$

a contradiction to (2.5). Hence, $x'_n(t) < L_1$ for $t \in [0,1]$.

Similarly, we can show that $y'_n(t) < L_2$ for $t \in [0,1]$.

Now, we show that

$$x_n(t) < M_1, y_n(t) < M_2, \quad t \in [0,1]. \tag{2.11}$$

Suppose $x_n(t_2) \ge M_1$ for some $t_2 \in [0,1]$. From (2.7), (2.10) and (A_3), it follows that

$$-x''_n(t) \le p(t)k_1(\theta_2(y_n(t)))(u_1(x'_n(t)) + v_1(x'_n(t))), \quad t \in (0,1),$$
$$-y''_n(t) \le q(t)k_2(\theta_1(x_n(t)))(u_2(y'_n(t)) + v_2(y'_n(t))), \quad t \in (0,1),$$

which implies that

$$\frac{-x''_n(t)}{u_1(x'_n(t)) + v_1(x'_n(t))} \le k_1(\theta_2(\|y_n\|))p(t), \quad t \in (0,1),$$
$$\frac{-y''_n(t)}{u_2(y'_n(t)) + v_2(y'_n(t))} \le k_2(M_1)q(t), \qquad t \in (0,1).$$

Integrating from t to 1, using the BCs, we obtain

$$\int_{\frac{1}{n}}^{x'_n(t)} \frac{dz}{u_1(z) + v_1(z)} \le k_1(\theta_2(\|y_n\|)) \int_t^1 p(s)ds, \quad t \in [0,1],$$
$$\int_{\frac{1}{n}}^{y'_n(t)} \frac{dz}{u_2(z) + v_2(z)} \le k_2(M_1) \int_t^1 q(s)ds, \qquad t \in [0,1],$$

which implies that

$$I(x'_n(t)) - I(\frac{1}{n}) \le k_1(\theta_2(\|y_n\|)) \int_0^1 p(s)ds, \quad t \in [0,1],$$
$$J(y'_n(t)) - J(\frac{1}{n}) \le k_2(M_1) \int_0^1 q(s)ds, \qquad t \in [0,1].$$

The increasing property of I and J leads to

$$x'_n(t) \le I^{-1}(k_1(\theta_2(\|y_n\|)) \int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0,1], \tag{2.12}$$

$$y'_n(t) \le J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon)), \qquad t \in [0,1]. \tag{2.13}$$

Integrating (2.12) from 0 to t_2 and (2.13) from 0 to 1, using the BCs, (2.12) and (2.13), we obtain

$$M_1 \leq x_n(t_2) \leq \left(1+\frac{b_1}{a_1}\right) I^{-1}(k_1(\theta_2(\|y_n\|)) \int_0^1 p(s)ds + I(\varepsilon)), \tag{2.14}$$

$$\|y_n\| \leq \left(1+\frac{b_2}{a_2}\right) J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon)). \tag{2.15}$$

Either we have $\|y_n\| < M_2$ or $\|y_n\| \geq M_2$. If $\|y_n\| < M_2$, then from (2.14), we have

$$M_1 \leq \left(1+\frac{b_1}{a_1}\right) I^{-1}(k_1(\|y_n\|) \int_0^1 p(s)ds + I(\varepsilon)), \tag{2.16}$$

Now, by using (2.15) in (2.16) and the increasing property of k_1 and I^{-1}, we obtain

$$\begin{aligned} M_1 \leq \left(1+\frac{b_1}{a_1}\right) I^{-1}\Bigg(k_1\Bigg(\left(1+\frac{b_2}{a_2}\right) \\ \times J^{-1}\left(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)\right)\Bigg)\int_0^1 p(s)ds + I(\varepsilon)\Bigg), \end{aligned}$$

which implies that

$$\frac{M_1}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (2.3).

On the other hand, if $\|y_n\| \geq M_2$, then from (2.14) and (2.15), we have

$$M_1 \leq \left(1+\frac{b_1}{a_1}\right) I^{-1}\left(k_1(M_2)\int_0^1 p(s)ds + I(\varepsilon)\right), \tag{2.17}$$

$$M_2 \leq \left(1+\frac{b_2}{a_2}\right) J^{-1}\left(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)\right). \tag{2.18}$$

In view of (2.18) in (2.17) and the increasing property of k_1 and I^{-1}, we obtain

$$\begin{aligned} M_1 \leq \left(1+\frac{b_1}{a_1}\right) I^{-1}\Bigg(k_1\Bigg(\left(1+\frac{b_2}{a_2}\right) \\ \times J^{-1}\left(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)\right)\Bigg)\int_0^1 p(s)ds + I(\varepsilon)\Bigg), \end{aligned}$$

which implies that

$$\frac{M_1}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (2.3). Hence, $x_n(t) < M_1$ for $t \in [0,1]$.

Similarly, we can show that $y_n(t) < M_2$ for $t \in [0,1]$.

Thus, (x_n, y_n) is a solution of the following coupled system of BVPs

$$
\begin{aligned}
-x''(t) &= p(t)f(t, y(t), x'(t)), \quad t \in (0,1),\\
-y''(t) &= q(t)g(t, x(t), y'(t)), \quad t \in (0,1),\\
a_1x(0) - b_1x'(0) &= 0, \quad x'(1) = \frac{1}{n},\\
a_2y(0) - b_2y'(0) &= 0, \quad y'(1) = \frac{1}{n},
\end{aligned}
\tag{2.19}
$$

satisfying

$$
\begin{aligned}
\left(t + \frac{b_1}{a_1}\right)\frac{1}{n} \le x_n(t) < M_1, \quad \frac{1}{n} \le x_n'(t) < L_1, \quad t \in [0,1],\\
\left(t + \frac{b_2}{a_2}\right)\frac{1}{n} \le y_n(t) < M_2, \quad \frac{1}{n} \le y_n'(t) < L_2, \quad t \in [0,1].
\end{aligned}
\tag{2.20}
$$

Now, in view of (A_6), there exist continuous functions $\varphi_{M_2L_1}$ and $\psi_{M_1L_2}$ defined on $[0,1]$ and positive on $(0,1)$, and real constants $0 \le \delta_1, \delta_2 < 1$ such that

(2.21)

$$
\begin{aligned}
f(t, y_n(t), x_n'(t)) \ge \varphi_{M_2L_1}(t)(y_n(t))^{\delta_1}, \quad (t, y_n(t), x_n'(t)) \in [0,1] \times [0, M_2] \times [0, L_1],\\
g(t, x_n(t), y_n'(t)) \ge \psi_{M_1L_2}(t)(x_n(t))^{\delta_2}, \quad (t, x_n(t), y_n'(t)) \in [0,1] \times [0, M_1] \times [0, L_2].
\end{aligned}
$$

We claim that

$$
x_n'(t) \ge C_2^{\delta_1}\int_t^1 p(s)\varphi_{M_2L_1}(s)ds, \tag{2.22}
$$

$$
y_n'(t) \ge C_1^{\delta_2}\int_t^1 q(s)\psi_{M_1L_2}(s)ds, \tag{2.23}
$$

where

$$
C_1 = \left(\frac{b_1}{a_1}\right)^{\frac{1}{1-\delta_1\delta_2}}\left(\frac{b_2}{a_2}\right)^{\frac{\delta_1}{1-\delta_1\delta_2}}\left(\int_0^1 p(t)\varphi_{M_2L_1}(t)dt\right)^{\frac{1}{1-\delta_1\delta_2}}\left(\int_0^1 q(t)\psi_{M_1L_2}(t)dt\right)^{\frac{\delta_1}{1-\delta_1\delta_2}},
$$

$$
C_2 = \left(\frac{b_1}{a_1}\right)^{\frac{\delta_2}{1-\delta_1\delta_2}}\left(\frac{b_2}{a_2}\right)^{\frac{1}{1-\delta_1\delta_2}}\left(\int_0^1 p(t)\varphi_{M_2L_1}(t)dt\right)^{\frac{\delta_2}{1-\delta_1\delta_2}}\left(\int_0^1 q(t)\psi_{M_1L_2}(t)dt\right)^{\frac{1}{1-\delta_1\delta_2}}.
$$

To prove (2.22), consider the following relation

$$
\begin{aligned}
x_n(t) = \left(t + \frac{b_1}{a_1}\right)\frac{1}{n} + \frac{1}{a_1}\int_0^t (a_1s + b_1)p(s)f(s, y_n(s), x_n'(s))ds\\
+ \frac{1}{a_1}\int_t^1 (a_1t + b_1)p(s)f(s, y_n(s), x_n'(s))ds, \quad t \in [0,1],
\end{aligned}
\tag{2.24}
$$

which implies that

$$
x_n(0) = \frac{b_1}{a_1}\frac{1}{n} + \frac{b_1}{a_1}\int_0^1 p(s)f(s, y_n(s), x_n'(s))ds.
$$

Using (2.21) and (2.20), we obtain

$$\begin{aligned} x_n(0) &\geq \frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_2L_1}(s)(y_n(s))^{\delta_1}ds \\ &\geq (y_n(0))^{\delta_1}\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_2L_1}(s)ds. \end{aligned} \tag{2.25}$$

Similarly, using (2.21) and (2.20), we obtain

$$y_n(0) \geq (x_n(0))^{\delta_2}\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_1L_2}(s)ds, \tag{2.26}$$

which in view of (2.25) implies that

$$y_n(0) \geq (y_n(0))^{\delta_1\delta_2}\left(\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_2L_1}(s)ds\right)^{\delta_2}\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_1L_2}(s)ds.$$

Hence,

$$y_n(0) \geq C_2. \tag{2.27}$$

Now, from (2.24), it follows that

$$x_n'(t) \geq \int_t^1 p(s)f(s, y_n(s), x_n'(s))ds,$$

and using (2.21) and (2.27), we obtain (2.22).

Similarly, we can prove (2.23).

Now, using (2.19), (A_3), (2.20), (2.22) and (2.23), we have

(2.28)

$$\begin{aligned} 0 \leq -x_n''(t) &\leq k_1(M_2)p(t)\left(u_1\left(C_2^{\delta_1}\int_t^1 p(s)\varphi_{M_2L_1}(s)ds\right) + v_1(L_1)\right), \quad t \in (0,1), \\ 0 \leq -y_n''(t) &\leq k_2(M_1)q(t)\left(u_2\left(C_1^{\delta_2}\int_t^1 q(s)\psi_{M_1}(s)ds\right) + v_2(L_2)\right), \quad t \in (0,1). \end{aligned}$$

In view of (2.20), (2.28), (A_1) and (A_7), it follows that the sequences $\{(x_{n,1}^{(j)}, y_{n,1}^{(j)})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0,1]$. Hence, by the Arzelà-Ascoli theorem there exist subsequences $\{(x_{n_k}^{(j)}, y_{n_k}^{(j)})\}$ $(j = 0,1)$ of $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j = 0, 1)$ and $(x, y) \in C^1[0, 1] \times C^1[0, 1]$ such that $(x_{n_k}^{(j)}, y_{n_k}^{(j)})$ converges uniformly to $(x^{(j)}, y^{(j)})$ on $[0, 1]$ $(j = 0, 1)$. Also, $a_1x(0) - b_1x'(0) = a_2y(0) - b_2y'(0) = x'(1) = y'(1) = 0$. Moreover, from (2.22) and (2.23), with n_k in place of n and taking $\lim_{n_k\to+\infty}$, we have

$$\begin{aligned} x_n'(t) &\geq C_2^{\delta_1}\int_t^1 p(s)\varphi_{M_2L_1}(s)ds, \\ y_n'(t) &\geq C_1^{\delta_2}\int_t^1 q(s)\psi_{M_1L_2}(s)ds, \end{aligned}$$

which shows that $x' > 0$ and $y' > 0$ on $[0,1)$, $x > 0$ and $y > 0$ on $[0,1]$. Further, (x_{n_k}, y_{n_k}) satisfy

$$x'_{n_k}(t) = x'_{n_k}(0) - \int_0^t p(s)f(s, y_{n_k}(s), x'_{n_k}(s))ds, \quad t \in [0,1],$$
$$y'_{n_k}(t) = y'_{n_k}(0) - \int_0^t q(s)f(s, x_{n_k}(s), y'_{n_k}(s))ds, \quad t \in [0,1].$$

Passing to the limit as $n_k \to \infty$, we obtain

$$x'(t) = x'(0) - \int_0^t p(s)f(s, y(s), x'(s))ds, \quad t \in [0,1],$$
$$y'(t) = y'(0) - \int_0^t q(s)f(s, x(s), y'(s))ds, \quad t \in [0,1],$$

which implies that

$$-x''(t) = p(t)f(t, y(t), x'(t)), \quad t \in (0,1),$$
$$-y''(t) = q(t)f(t, x(t), y'(t)), \quad t \in (0,1).$$

Hence, (x,y) is a C^1-positive solution of the system (1.2). □

Example 2.2. Consider the following coupled system of singular BVPs

$$\begin{aligned} -x''(t) &= \mu(y(t))^{\alpha_1}(x'(t))^{-\beta_1}, \quad t \in (0,1),\\ -y''(t) &= \mu(x(t))^{\alpha_2}(y'(t))^{-\beta_2}, \quad t \in (0,1),\\ x(0) - x'(0) &= x'(1) = 0,\\ y(0) - y'(0) &= y'(1) = 0, \end{aligned} \tag{2.29}$$

where $0 \le \alpha_1 < 1$, $0 \le \alpha_2 < 1$, $0 < \beta_1 < 1$, $0 < \beta_2 < 1$ and $\mu > 0$.

Take $p(t) = q(t) = 1$, $k_1(x) = \mu x^{\alpha_1}$, $k_2(x) = \mu x^{\alpha_2}$, $u_1(x) = x^{-\beta_1}$, $u_2(x) = x^{-\beta_2}$ and $v_1(x) = v_2(x) = 0$. Then, $I(z) = \frac{z^{\beta_1+1}}{\beta_1+1}$, $J(z) = \frac{z^{\beta_2+1}}{\beta_2+1}$, $I^{-1}(z) = (\beta_1+1)^{\frac{1}{\beta_1+1}} z^{\frac{1}{\beta_1+1}}$ and $J^{-1}(z) = (\beta_2+1)^{\frac{1}{\beta_2+1}} z^{\frac{1}{\beta_2+1}}$.

Choose $\varphi_{EF}(t) = F^{-\beta_1}$, $\psi_{EF}(t) = F^{-\beta_2}$, $\delta_1 = \alpha_1$ and $\delta_2 = \alpha_2$. Then,

$$\sup_{c\in(0,\infty)} \frac{c}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(c)\int_0^1 q(t)dt))\int_0^1 p(t)dt)} =$$
$$\sup_{c\in(0,\infty)} \frac{c}{2^{1+\frac{\alpha_1}{\beta_1+1}}\mu^{(1+\frac{\alpha_1}{\beta_2+1})\frac{1}{\beta_1+1}}(\beta_1+1)^{\frac{1}{\beta_1+1}}(\beta_2+1)^{\frac{\alpha_1}{(\beta_1+1)(\beta_2+1)}}c^{\frac{\alpha_1\alpha_2}{(\beta_1+1)(\beta_2+1)}}} = \infty$$

and

$$\sup_{x\in(0,\infty)} \frac{c}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(c)\int_0^1 p(t)dt))\int_0^1 q(t)dt)} =$$
$$\sup_{c\in(0,\infty)} \frac{c}{2^{1+\frac{\alpha_2}{\beta_2+1}}\mu^{(1+\frac{\alpha_2}{\beta_1+1})\frac{1}{\beta_2+1}}(\beta_2+1)^{\frac{1}{\beta_2+1}}(\beta_1+1)^{\frac{\alpha_2}{(\beta_1+1)(\beta_2+1)}}c^{\frac{\alpha_1\alpha_2}{(\beta_1+1)(\beta_2+1)}}} = \infty.$$

Clearly, (A_1)–(A_7) are satisfied. Hence, by Theorem 2.1, the system (2.29) has at least one C^1-positive solution.

REFERENCES

[1] R. P. Agarwal and D. O'Regan, *Singular Differential and Integral Equations with Applications*, Kluwer Academic Publishers, Dordrecht, 2003.

[2] R. P. Agarwal and D. O'Regan, Right focal singular boundary value problems, *ZAMM* **79** (1999), 363–373.

[3] N. A. Asif and R. A. Khan, Positive solutions for a class of coupled system of singular three point boundary value problems, *Boundary Value Problems*, Article ID 273063 (accepted).

[4] Y. Chen, B. Yan and L. Zhang, Positive solutions for singular three-point boundary-value problems with sign changing nonlinearities depending on x', *Elec. Jour. Diff. Equa.,* Vol. 2007 (2007), No. **63**, pp. 1–9.

[5] X. Cheng and C. Zhong, Existence of positive solutions for a second-order ordinary differential system, *J. Math. Anal. Appl.* **312** (2005), 14–23.

[6] J. Henderson, S. K. Ntouyas and I. K. Ntouyas, Positive solutions for systems of second order four-point nonlinear boundary value problems, *Comm. Appl. Anal.* **12** (2008), 29–40.

[7] J. Henderson and S. K. Ntouyas, Positive solutions for systems of nonlinear boundary value problems, *Nonlin. Studies* **15** (2008), 51–60.

[8] P. Kang and Z. Wei, Three positive solutions of singular nonlocal boundary value problems for systems of nonlinear second-order ordinary differential equations, *Nonlinear Anal.* **70** (2009), 444–451.

[9] R. A. Khan and J. R. L. Webb, Existence of at least three solutions of a second-order three-point boundary value problem, *Nonlinear Anal.* **64** (2006), 1356–1366.

[10] R. A. Khan and N. A. Asif, Positive solutions for a class of singular two point boundary value problems, *J. Nonlinear Sci. Appl.* **2** (2009), 126–135.

[11] Y. Liu and B. Yan, Multiple solutions of singular boundary value problems for differential systems, *J. Math. Anal. Appl.* **287** (2003), 540–556.

[12] B. Liu, L. Liu and Y. Wu, Positive solutions for singular systems of three-point boundary value problems, *Comput. Math. Appl.* **53** (2007), 1429–1438.

[13] H. Lü, H. Yu and Y. Liu, Positive solutions for singular boundary value problems of a coupled system of differential equations, *J. Math. Anal. Appl.* **302** (2005), 14–29.

[14] D. O'Regan and R. P. Agarwal, Singular problems: an upper and lower solution approach, *J. Math. Anal. Appl.* **251** (2000), 230–250.

[15] Z. Wei, Positive solution of singular Dirichlet boundary value problems for second order differential equation system, *J. Math. Anal. Appl.*, **328** (2007) 1255–1267.

[16] Xu Xian, Existence and multiplicity of positive solutions for multi-parameter three-point differential equations system, *J. Math. Anal. Appl.* **324** (2006) 472–490.

[17] B. Yan, D. O'Regan and R. P. Agarwal, Multiple positive solutions via index theory for singular boundary value problems with derivative dependence, *Positivity* **11** (2007), 687–720.

[18] B. Yan, D. O'Regan and R. P. Agarwal, Multiple positive solutions of singular second order boundary value problems with derivative dependence, *Aequationes Mathematicae* **74** (2007), 62–89.

[19] Y. Yuan, C. Zhao and Y. Liu, Positive solutions for systems of singular differential equations, *Elec. Jour. Diff. Equa.,* Vol. 2008 (2008), No. **74**, pp. 1–14.

[20] G. Zhang, Positive solutions of two-point boundary value problems for second-order differential equations with the nonlinearity dependent on the derivative, *Nonlinear Anal.* **69** (2008), 222–229.