Dynamic Systems and Applications 19 (2010) 249-270

FOURTH-ORDER M-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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ABSTRACT. Let $\mathbb{T}$ be a time scale with $[a,b]\subset\mathbb{T}$. We establish criteria for existence of one or more than one positive solutions of the non-eigenvalue problem

$$(0.1)\qquad \begin{cases} y^{\triangle^4}(t)-q(t)y^{\Delta^2}(\sigma(t))=f(t,y(t)), & t\in[a,b]\subset\mathbb{T},\\ y(a)=\sum_{i=1}^{m-2}a_iy(\xi_i), & y(\sigma^2(b))=\sum_{i=1}^{m-2}b_iy(\xi_i),\\ y^{\Delta^2}(a)=\sum_{i=1}^{m-2}a_iy^{\Delta^2}(\xi_i), & y^{\Delta^2}(\sigma^2(b))=\sum_{i=1}^{m-2}b_iy^{\Delta^2}(\xi_i), \end{cases}$$

where $\xi_i\in(a,b)$, a_i, $b_i\in[0,\infty)$ (for $i\in\{1,2,\dots,m-2\}$) are given constants. Later, we consider the existence and multiplicity of positive solutions for the eigenvalue problem $y^{\triangle^4}(t)-q(t)y^{\Delta^2}(\sigma(t))=\lambda f(t,y(t))$ with the same boundary conditions. We shall also obtain criteria which lead to nonexistence of positive solutions. In both problems, we will use Krasnoselskii fixed point theorem.

AMS (MOS) Subject Classification. 34B10, 39A10

1. INTRODUCTION

We are concerned with the following fourth-order m-point boundary value problem (BVP)

$$(1.1)\qquad \begin{cases} y^{\triangle^4}(t)-q(t)y^{\Delta^2}(\sigma(t))=f(t,y(t)), & t\in[a,b]\subset\mathbb{T},\\ y(a)=\sum_{i=1}^{m-2}a_iy(\xi_i), & y(\sigma^2(b))=\sum_{i=1}^{m-2}b_iy(\xi_i),\\ y^{\Delta^2}(a)=\sum_{i=1}^{m-2}a_iy^{\Delta^2}(\xi_i), & y^{\Delta^2}(\sigma^2(b))=\sum_{i=1}^{m-2}b_iy^{\Delta^2}(\xi_i), \end{cases}$$

and the eigenvalue problem $y^{\triangle^4}(t)-q(t)y^{\Delta^2}(\sigma(t))=\lambda f(t,y(t))$ with the same boundary conditions where λ is a positive parameter, $\xi_i\in(a,b)$, a_i, $b_i\in[0,\infty)$ (for $i\in\{1,2,\dots,m-2\}$) are given constants.

We will assume that the following conditions are satisfied.

(H1) $f:[a,\sigma^2(b)]\times\mathbb{R}\to\mathbb{R}$ is continuous with respect to y and $f(t,y)\geq 0$ for $y\in\mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

(H2) $q(t)\geq 0$.

Received March 24, 2009

Throughout this work we let $\mathbb{T}$ be any time scale (nonempty closed subset of $\mathbb{R}$), and $[a,b] = \{t \in \mathbb{T} : a \le t \le b\}$. The study of dynamic equations on time scales goes back to its founder Stefan Hilger [6]. Some preliminary definitions and theorems on time scales can be found in books [2, 3] which are excellent references for calculus of time scales.

It is well known that many authors have given considerable attention to the second-order boundary value problems for dynamic equations on time scales [1, 5, 7]. See also Ma [11] and Ma and Thompson [12] for related results when $\mathbb{T} = \mathbb{R}$. There are fewer results in the literature on boundary value problems for fourth-order ordinary differential equations when $\mathbb{T} = \mathbb{R}$ [4, 9, 10]. A few papers can be found in the literature on BVPs for fourth-order dynamic equations on time scales.

Wang and Sun [13] obtained criteria for a solution and a positive solution to the fourth-order two-point boundary value problem on time scale $\mathbb{T}$:

$$\begin{cases} u^{\Delta\Delta\Delta\Delta}(t) - f(t, u(t), u^{\Delta\Delta}(t)) = 0, & t \in [a, \rho^2(b)], \\ u(a) = A, \quad u(\sigma^2(b)) = B, \quad u^{\Delta\Delta}(a) = C, & u^{\Delta\Delta}(b) = D. \end{cases} \tag{1.2}$$

Their arguments are based on the Leray-Schauder fixed point theorem. Our results include criteria for existence of one or more than one positive solutions of our non-eigenvalue problem. Moreover we also determine values of λ for at least one positive solution of our eigenvalue problem and obtain criteria for existence of one or two positive solutions of this problem in terms of superlinear or sublinear behavior of $f(t,y)$. Finally, we obtain criteria which lead to nonexistence of positive solutions. In this article, the main tool is the following well-known Krasnoselskii fixed point theorem in a cone [8].

Theorem 1.1 ([8])**.** *Let B be a Banach space, and let $P \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega_1} \subset \Omega_2$, and let*

$$A : P \cap (\bar{\Omega_2} \setminus \Omega_1) \to P$$

be a completely continuous operator such that, either

(i) $\|Ay\| \le \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \ge \|y\|$, $y \in P \cap \partial\Omega_2$*; or*
(ii) $\|Ay\| \ge \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \le \|y\|$, $u \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\bar{\Omega_2} \setminus \Omega_1)$.

2. THE PRELIMINARY LEMMAS

Denote by φ and ψ, the solutions of the corresponding homogeneous equation

$$y^{\Delta^2}(t) - q(t)y(\sigma(t)) = 0, \quad t \in [a,b] \tag{2.1}$$

under the initial conditions

$$\varphi(a)=0,\quad \varphi^{\Delta}(a)=1,\quad \psi(\sigma^2(b))=0,\quad \psi^{\Delta}(\sigma^2(b))=-1. \tag{2.2}$$

Define the number D by

$$D:=\psi(a)=\varphi(\sigma^2(b)). \tag{2.3}$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for φ and ψ the following equations:

$$\varphi(t)=t-a+\int_a^t\int_a^\tau q(s)\varphi(\sigma(s))\Delta s\Delta\tau, \tag{2.4}$$

$$\psi(t)=\sigma^2(b)-t+\int_t^{\sigma^2(b)}\int_\tau^{\sigma^2(b)} q(s)\psi(\sigma(s))\Delta s\Delta\tau. \tag{2.5}$$

Lemma 2.1. *Under the condition* (H2) *the following inequalities*

$$\begin{cases} \varphi(t)\ge 0,\ t\in[a,\sigma^2(b)]; & \psi(t)\ge 0,\ t\in[a,\sigma^2(b)];\\ \varphi^{\Delta}(t)\ge 0,\ t\in[a,\sigma(b)]; & \psi^{\Delta}(t)\le 0,\ t\in[a,\sigma^2(b)] \end{cases} \tag{2.6}$$

yield.

Proof. Using the induction method on time scales as in [1] one can easily see these inequalities in (2.6) hold. □

Lemma 2.2. *Under the condition* (H2) *the inequality $D>0$ holds.*

Proof. By (2.3) and (2.4) we have

$$D=\sigma^2(b)-a+\int_a^{\sigma^2(b)}\int_a^\tau q(s)\varphi(\sigma(s))\Delta s\Delta\tau. \tag{2.7}$$

Since $\varphi(t)\ge 0$ for $t\in[a,\sigma^2(b)]$, we have

$$D\ge\sigma^2(b)-a \tag{2.8}$$

by (2.7). This completes the proof. □

We use the following assumptions in the rest of the paper.

(H3) $\sum_{i=1}^{m-2}a_i\psi(\xi_i)<1$, $\sum_{i=1}^{m-2}b_i\varphi(\xi_i)<1$,
(H4) $\sum_{i=1}^{m-2}a_i(\sigma^2(b)-\xi_i)<1$, $\sum_{i=1}^{m-2}b_i(\xi_i-a)<1$.

Set

$$\Delta_1=\begin{vmatrix} \sum_{i=1}^{m-2}a_i\varphi(\xi_i) & \sum_{i=1}^{m-2}a_i\psi(\xi_i)-\psi(a)\\ \sum_{i=1}^{m-2}b_i\varphi(\xi_i)-\varphi(\sigma^2(b)) & \sum_{i=1}^{m-2}b_i\psi(\xi_i)\end{vmatrix}. \tag{2.9}$$

$$\Delta_2=\begin{vmatrix} \sum_{i=1}^{m-2}a_i(\xi_i-a) & \sum_{i=1}^{m-2}a_i(\sigma^2(b)-\xi_i)-(\sigma^2(b)-a)\\ \sum_{i=1}^{m-2}b_i(\xi_i-a)-(\sigma^2(b)-a) & \sum_{i=1}^{m-2}b_i(\sigma^2(b)-\xi_i)\end{vmatrix}. \tag{2.10}$$

Lemma 2.3. *Let* (H2) *hold and assume that*

(H5) $\Delta_1 \neq 0$.

Then for any $g \in \mathcal{C}[a, \sigma^2(b)]$, *the problem*

$$\begin{cases} -y^{\Delta^2}(t) + q(t)y(\sigma(t)) = g(t), \quad t \in [a,b] \\ y(a) = \sum_{i=1}^{m-2} a_i y(\xi_i), \quad y(\sigma^2(b)) = \sum_{i=1}^{m-2} b_i y(\xi_i) \end{cases} \tag{2.11}$$

has a unique solution

$$y(t) = \int_a^{\sigma^2(b)} G_1(t,s)g(s)\Delta s + A(g)\varphi(t) + B(g)\psi(t), \tag{2.12}$$

where

$$G_1(t,s) = \frac{1}{D}\begin{cases} \varphi(t)\psi(\sigma(s)), & t \leq s, \\ \varphi(\sigma(s))\psi(t), & \sigma(s) \leq t, \end{cases} \tag{2.13}$$

$$A(g) := \frac{1}{\Delta_1}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i \int_a^{\sigma^2(b)} G_1(\xi_i,s)g(s)\Delta s & \sum_{i=1}^{m-2} a_i\psi(\xi_i) - \psi(a) \\ -\sum_{i=1}^{m-2} b_i \int_a^{\sigma^2(b)} G_1(\xi_i,s)g(s)\Delta s & \sum_{i=1}^{m-2} b_i\psi(\xi_i) \end{vmatrix} \tag{2.14}$$

and

$$B(g) := \frac{1}{\Delta_1}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\varphi(\xi_i) & -\sum_{i=1}^{m-2} a_i \int_a^{\sigma^2(b)} G_1(\xi_i,s)g(s)\Delta s \\ \sum_{i=1}^{m-2} b_i\varphi(\xi_i) - \varphi(\sigma^2(b)) & -\sum_{i=1}^{m-2} b_i \int_a^{\sigma^2(b)} G_1(\xi_i,s)g(s)\Delta s \end{vmatrix}. \tag{2.15}$$

Proof. Now we show that the function defined by (2.12) is a solution of (2.11) only if $G_1(t,s)$, $A(g)$, and $B(g)$ are as in (2.13), (2.14) and (2.15), respectively. Let $y(t) = \int_a^{\sigma^2(b)} G(t,s)g(s)\Delta s + A(g)\varphi(t) + B(g)\psi(t)$ be a solution of (2.11), then we have that

$$\begin{aligned} y(t) &= \int_a^t \frac{1}{D}\varphi(\sigma(s))\psi(t)g(s)\Delta s + \int_t^{\sigma^2(b)} \frac{1}{D}\psi(\sigma(s))\varphi(t)g(s)\Delta s \\ &\quad + A(g)\varphi(t) + B(g)\psi(t), \\ y^{\Delta}(t) &= \psi^{\Delta}(t)\int_a^t \frac{1}{D}\varphi(\sigma(s))g(s)\Delta s + \varphi^{\Delta}(t)\int_t^{\sigma^2(b)} \frac{1}{D}\psi(\sigma(s))g(s)\Delta s \\ &\quad + A(g)\varphi^{\Delta}(t) + B(g)\psi^{\Delta}(t) \end{aligned}$$

and

$$\begin{aligned} y^{\Delta^2}(t) &= \psi^{\Delta^2}(t)\int_a^t \frac{1}{D}\varphi(\sigma(s))g(s)\Delta s + \varphi^{\Delta^2}(t)\int_t^{\sigma^2(b)} \frac{1}{D}\psi(\sigma(s))g(s)\Delta s \\ &\quad + A(g)\varphi^{\Delta^2}(t) + B(g)\psi^{\Delta^2}(t) - g(t), \end{aligned}$$

so that

$$\begin{aligned} -y^{\Delta^2}(t) + q(t)y(\sigma(t)) &= \frac{1}{D}[-\psi^{\Delta^2}(t) + q(t)\psi(\sigma(t))]\int_a^t \varphi(\sigma(s))g(s)\Delta s \\ &\quad + \frac{1}{D}[-\varphi^{\Delta^2}(t) + q(t)\varphi(\sigma(t))]\int_t^{\sigma^2(b)} \psi(\sigma(s))g(s)\Delta s \\ &\quad + A(g)(-\varphi^{\Delta^2}(t) + q(t)\varphi(\sigma(t))) \end{aligned}$$

$$+ B(g)(-\psi^{\Delta^2}(t) + q(t)\psi(\sigma(t))) + g(t)$$
$$= g(t).$$

Since

$$y(a) = B(g)\psi(a), \quad y(\sigma^2(b)) = A(g)\varphi(\sigma^2(b)),$$

we have that

$$B(g)\psi(a) = \sum_{i=1}^{m-2} a_i \left[\int_a^{\sigma^2(b)} G(\xi_i, s)g(s)\Delta s + A(g)\varphi(\xi_i) + B(g)\psi(\xi_i) \right],$$

$$A(g)\varphi(\sigma^2(b)) = \sum_{i=1}^{m-2} b_i \left[\int_a^{\sigma^2(b)} G(\xi_i, s)g(s)\Delta s + A(g)\varphi(\xi_i) + B(g)\psi(\xi_i) \right].$$

Then we get that

$$\left[-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \right] A(g) + \left[\psi(a) - \sum_{i=1}^{m-2} a_i\psi(\xi_i) \right] B(g) = \sum_{i=1}^{m-2} a_i \int_a^{\sigma^2(b)} G(\xi_i, s)g(s)\Delta s$$

$$\left[\varphi(\sigma^2(b)) - \sum_{i=1}^{m-2} b_i\varphi(\xi_i) \right] A(g) - \left[\sum_{i=1}^{m-2} b_i\psi(\xi_i) \right] B(g) = \sum_{i=1}^{m-2} b_i \int_a^{\sigma^2(b)} G(\xi_i, s)g(s)\Delta s$$

which implies that $A(g)$ and $B(g)$ satisfy (2.14) and (2.15), respectively. □

Lemma 2.4. *Let* (H2), (H3) *hold and assume that*

(H6) $\Delta_1 < 0$.

Then for any $g \in \mathcal{C}[a, \sigma^2(b)]$ *and* $g \geq 0$, *the unique solution* y *of problem* (2.11) *satisfies*

$$y(t) \geq 0, \quad t \in [a, \sigma^2(b)].$$

Proof. From Lemma 2.1 and Lemma 2.2, the Green's function (2.13) satisfies $G_1(t, s) \geq 0$ on $[a, \sigma^2(b)] \times [a, \sigma(b)]$. By hypotheses (H3) and (H6), it is clear that $A(g)$ and $B(g)$ are nonnegative. Thus the result follows. □

Lemma 2.5. *Let* (H2) *hold and assume that*

(H7) $\Delta_2 \neq 0$.

Then for any $g \in \mathcal{C}[a, \sigma^2(b)]$, *the problem*

$$\begin{cases} -y^{\Delta^2}(t) = g(t), \quad t \in [a, \sigma^2(b)] \\ y(a) = \sum_{i=1}^{m-2} a_i y(\xi_i), \quad y(\sigma^2(b)) = \sum_{i=1}^{m-2} b_i y(\xi_i) \end{cases} \tag{2.16}$$

has a unique solution

$$y(t) = \int_a^{\sigma^2(b)} G_2(t, s)g(s)\Delta s + C(g)(t - a) + D(g)(\sigma^2(b) - t),$$

where

$$G_2(t,s)=\frac{1}{\sigma^2(b)-a}\begin{cases}(t-a)(\sigma^2(b)-\sigma(s)), & t\le s,\\ (\sigma(s)-a)(\sigma^2(b)-t), & \sigma(s)\le t,\end{cases}\tag{2.17}$$

$$C(g):=\frac{1}{\Delta_2}\begin{vmatrix}-\sum_{i=1}^{m-2}a_i\int_a^{\sigma^2(b)}G_2(\xi_i,s)g(s)\Delta s & X\\ -\sum_{i=1}^{m-2}b_i\int_a^{\sigma^2(b)}G_2(\xi_i,s)g(s)\Delta s & \sum_{i=1}^{m-2}b_i(\sigma^2(b)-\xi_i)\end{vmatrix},\tag{2.18}$$

$$D(g):=\frac{1}{\Delta_2}\begin{vmatrix}\sum_{i=1}^{m-2}a_i(\xi_i-a) & -\sum_{i=1}^{m-2}a_i\int_a^{\sigma^2(b)}G_2(\xi_i,s)g(s)\Delta s\\ Y & -\sum_{i=1}^{m-2}b_i\int_a^{\sigma^2(b)}G_2(\xi_i,s)g(s)\Delta s\end{vmatrix},\tag{2.19}$$

$$X=\sum_{i=1}^{m-2}a_i(\sigma^2(b)-\xi_i)-(\sigma^2(b)-a),$$

and

$$Y=\sum_{i=1}^{m-2}b_i(\sigma^2(b)-\xi_i)-(\sigma^2(b)-a).$$

Lemma 2.6. *Let* (H2), (H4) *hold and assume that*

(H8) $\Delta_2<0$.

Then for any $g\in\mathcal{C}[a,\sigma^2(b)]$ *and* $g\ge 0$, *the unique solution* y *of problem* (2.16) *satisfies*

$$y(t)\ge 0,\quad t\in[a,\sigma^2(b)].$$

Lemma 2.7. *Let* (H2), (H6) *and* (H8) *hold. Then for any* $g\in\mathcal{C}[a,\sigma^2(b)]$, *the problem*

$$\begin{cases}y^{\Delta^4}(t)-q(t)y^{\Delta^2}(\sigma(t))=g(t),\\ y(a)=\sum_{i=1}^{m-2}a_iy(\xi_i),\quad y(\sigma^2(b))=\sum_{i=1}^{m-2}b_iy(\xi_i),\\ y^{\Delta^2}(a)=\sum_{i=1}^{m-2}a_iy^{\Delta^2}(\xi_i),\quad y^{\Delta^2}(\sigma^2(b))=\sum_{i=1}^{m-2}b_iy^{\Delta^2}(\xi_i),\end{cases}\tag{2.20}$$

has a unique solution

$$\begin{aligned}y(t)=&\int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)}G_2(t,\tau)G_1(\tau,s)g(s)\Delta s\Delta\tau+\int_a^{\sigma^2(b)}G_2(t,\tau)A(g)\varphi(\tau)\Delta\tau\\ &+\int_a^{\sigma^2(b)}G_2(t,\tau)B(g)\psi(\tau)\Delta\tau+C(h)(t-a)+D(h)(\sigma^2(b)-t),\end{aligned}\tag{2.21}$$

where G_1, G_2, $A(g)$, $B(g)$, $C(g)$, $D(g)$ *are defined as in* (2.13), (2.17), (2.14), (2.15), (2.18), (2.19) *respectively and*

$$h(t)=\int_a^{\sigma^2(b)}G_1(t,s)g(s)\Delta s+A(g)\varphi(t)+B(g)\psi(t).$$

In addition, if (H3), (H4) *hold and* $g\ge 0$, *then*

$$y(t)\ge 0,\quad t\in[a,\sigma^2(b)].$$

Proof. Let us consider the following BVP:

$$\begin{cases} -y^{\Delta^2} = \displaystyle\int_a^{\sigma^2(b)} G_1(t,s)g(s)\Delta s + A(g)\varphi(t) + B(g)\psi(t), \quad t \in [a, \sigma^2(b)] \\ y(a) = \sum_{i=1}^{m-2} a_i y(\xi_i), \quad y(\sigma^2(b)) = \sum_{i=1}^{m-2} b_i y(\xi_i) \end{cases} \tag{2.22}$$

The Green's function associated with the BVP (2.22) is $G_2(t,s)$. This completes the proof. □

Lemma 2.8. *Let* (H2) *hold.* $G_i(t,s)$ $(i = 1, 2)$ *has the following properties:*

(i) $G_i(t,s) > 0$, $\forall (t,s) \in (a, \sigma^2(b)) \times (a, \sigma(b))$.

(ii) $G_i(t,s) \le G_i(\sigma(s), s)$, $\forall (t,s) \in [a, \sigma^2(b)] \times [a, \sigma(b)]$.

(iii) $G_i(t,s) \ge \delta_i G_i(t,t) G_i(\sigma(s), s)$, $\forall (t,s) \in [a, \sigma(b)] \times [a, \sigma(b)]$, *where* $\delta_i > 0$ *is a constant.*

Proof. We can easily see that $G_i(t,s) > 0$ for all $(t,s) \in (a, \sigma^2(b)) \times (a, \sigma(b))$ and

$$\sup_{(t,s)\in(a,\sigma^2(b))\times(a,\sigma(b))} \frac{G_i(t,s)}{G_i(\sigma(s),s)} = 1 < +\infty,$$

$$\inf_{(t,s)\in(a,\sigma(b))\times(a,\sigma(b))} \frac{G_i(t,s)}{G_i(t,t)G_i(\sigma(s),s)} = \delta_i > 0,$$

where $\delta_1 = \min\{\frac{1}{\psi(\sigma(a))}, \frac{1}{\varphi(\sigma(b))}\}$, and $\delta_2 = \min\{\frac{1}{\sigma^2(b)-\sigma(a)}, \frac{1}{\sigma(b)-a}\}$. Hence the result holds. □

Lemma 2.9. *Let* (H2)-(H4), (H6) *and* (H8) *hold. Then for* $g \in \mathcal{C}[a, \sigma^2(b)]$ *with* $g \ge 0$, *the unique solution of boundary value problem* (2.20) *satisfies* $y(t) \ge \Gamma \|y\|$ *for* $t \in [\frac{\sigma(b)+3a}{4}, \frac{3\sigma(b)-a}{4}]$, *where* $\|y\| = \max_{a\le t\le\sigma^2(b)} |y(t)|$ *and*

$$\Gamma := \delta_2 \min_{t\in[\frac{\sigma(b)+3a}{4}, \frac{3\sigma(b)-a}{4}]} G_2(t,t). \tag{2.23}$$

Proof. From (2.21) and Lemma 2.8, we get

$$\begin{aligned} y(t) \ge{}& \delta_2 \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t,t) G_2(\sigma(\tau),\tau) G_1(\tau,s) g(s) \Delta s \Delta\tau \\ &+ \delta_2 \int_a^{\sigma^2(b)} G_2(t,t) G_2(\sigma(\tau),\tau) A(g)\varphi(\tau)\Delta\tau \\ &+ \delta_2 \int_a^{\sigma^2(b)} G_2(t,t) G_2(\sigma(\tau),\tau) B(g)\psi(\tau)\Delta\tau \\ &+ C(h)(t-a) + D(h)(\sigma^2(b)-t), \end{aligned}$$

for all $t \in [a, \sigma(b)]$. Since the inequalities $t - a \ge \delta_2 G_2(t,t)(\sigma^2(b) - a)$ for $t \in [a, \sigma(b)]$, $\sigma^2(b) - t \ge \delta_2 G_2(t,t)(\sigma^2(b) - a)$ for $t \in [a, \sigma(b)]$, and

$$y(t) \le \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau) G_1(\tau,s) g(s) \Delta s \Delta\tau$$

$$+\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)A(g)\varphi(\tau)\Delta\tau$$
$$+\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)B(g)\psi(\tau)\Delta\tau$$
$$+C(h)(\sigma^2(b)-a)+D(h)(\sigma^2(b)-a)$$

for $t\in[a,\sigma^2(b)]$ hold, for $t\in[\frac{\sigma(b)+3a}{4},\frac{3\sigma(b)-a}{4}]$, we have

$$\begin{aligned} y(t) \geq \delta_2 G_2(t,t)\bigg\{ &\int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)G_1(\tau,s)g(s)\Delta s\Delta\tau \\ &+\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)A(g)\varphi(\tau)\Delta\tau \\ &+\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)B(g)\psi(\tau)\Delta\tau \\ &+C(h)(\sigma^2(b)-a)+D(h)(\sigma^2(b)-a)\bigg\} \\ \geq \Gamma\|y\|, \end{aligned}$$

where Γ is defined as in (2.23). The proof is complete. □

3. EXISTENCE OF ONE OR MORE POSITIVE SOLUTIONS

Denote

$$\begin{aligned} \Theta := \Bigg[\max_{a\leq t\leq\sigma^2(b)}\bigg\{ &\int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(t,\tau)G_1(\tau,s)\Delta s\Delta\tau \\ &+\int_a^{\sigma^2(b)} G_2(t,\tau)A\varphi(\tau)\Delta\tau. \\ &+\int_a^{\sigma^2(b)} G_2(t,\tau)B\psi(\tau)\Delta\tau \\ &+C\Big(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\varphi(t)+B\psi(t)\Big)(t-a) \\ &+D\Big(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\varphi(t)+B\psi(t)\Big)(\sigma^2(b)-t)\bigg\}\Bigg]^{-1}, \end{aligned}$$

$$\Theta^* := \left[\max_{a\leq t\leq\sigma^2(b)}\left\{\int_a^{\sigma^2(b)}\int_\zeta^{\sigma(\omega)} G_2(t,\tau)G_1(\tau,s)\Delta s\Delta\tau\right\}\right]^{-1},$$

where

$$A := \frac{1}{\Delta_1}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\int_a^{\sigma^2(b)} G_1(\xi_i,s)\Delta s & \sum_{i=1}^{m-2} a_i\psi(\xi_i)-\psi(a) \\ -\sum_{i=1}^{m-2} b_i\int_a^{\sigma^2(b)} G_1(\xi_i,s)\Delta s & \sum_{i=1}^{m-2} b_i\psi(\xi_i)\end{vmatrix}, \tag{3.1}$$

and

$$B := \frac{1}{\Delta_1}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\varphi(\xi_i) & -\sum_{i=1}^{m-2} a_i \int_a^{\sigma^2(b)} G_1(\xi_i, s)\Delta s \\ \sum_{i=1}^{m-2} b_i\varphi(\xi_i) - \varphi(\sigma^2(b)) & -\sum_{i=1}^{m-2} b_i \int_a^{\sigma^2(b)} G_1(\xi_i, s)\Delta s \end{vmatrix}. \tag{3.2}$$

We further assume that the set $[a, \sigma(b)]$ is such that

$$\zeta := \min\left\{\tau \in \mathbb{T} : \tau \geq \frac{\sigma(b) + 3a}{4}\right\}, \quad \omega := \max\left\{\tau \in \mathbb{T} : \tau \leq \frac{3\sigma(b) - a}{4}\right\}$$

exist and satisfy

$$\frac{\sigma(b) + 3a}{4} \leq \zeta < \omega \leq \frac{3\sigma(b) - a}{4}.$$

We also assume that if $\sigma(\omega) = b$, then $\sigma(\omega) < \sigma(b)$. Let

$$\Gamma^* := \min\left\{\Gamma, \min_{s\in[\zeta,\omega]} \frac{G_2(\sigma(\omega), s)}{G_2(\sigma(s), s)}, \frac{\sigma(\omega) - a}{\sigma^2(b) - a}, \frac{\sigma^2(b) - \sigma(\omega)}{\sigma^2(b) - a}\right\} \tag{3.3}$$

where Γ is as in (2.23). Then for $\eta > 0$, set

$$F(\eta) = \max\{f(t, v) : a \leq t \leq \sigma^2(b), 0 \leq v \leq \eta\},$$

$$H(\eta) = \min\{f(t, v) : \zeta \leq t \leq \sigma(\omega), \Gamma^*\eta \leq v \leq \eta\}.$$

Theorem 3.1. *Let* (H1)–(H4), (H6) *and* (H8) *hold. Assume there exist two positive numbers η_1 and η_2 with $\eta_1 \neq \eta_2$ such that*

$$F(\eta_1) \leq \eta_1\Theta, \quad H(\eta_2) \geq \eta_2\Theta^*.$$

Then the BVP (1.1) *has at least one positive solution y satisfying*

$$\min\{\eta_1, \eta_2\} \leq \|y\| \leq \max\{\eta_1, \eta_2\}.$$

Proof. We only show the case $\eta_1 < \eta_2$. The other case can be treated by the same method.

We work in the Banach space $\mathcal{B} = \mathcal{C}[a, \sigma^2(b)]$ with the norm

$$\|y\| := \max_{a\leq t\leq\sigma^2(b)} |y(t)|.$$

Then define a cone K in $\mathcal{B}$ by

$$K := \{y \in \mathcal{B} : y(t) \geq 0 \text{ on } [a, \sigma^2(b)], \text{ and } y(t) \geq \Gamma^*\|y\|, \quad t \in [\zeta, \sigma(\omega)]\}, \tag{3.4}$$

where Γ^* is as in (3.3). For each $y \in K$, denote

$$\begin{aligned} Ty(t) = &\int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t, \tau)G_1(\tau, s)f(s, y(s))\Delta s\Delta\tau \\ &+ \int_a^{\sigma^2(b)} G_2(t, \tau)A(f)\varphi(\tau)\Delta\tau \\ &+ \int_a^{\sigma^2(b)} G_2(t, \tau)B(f)\psi(\tau)\Delta\tau + C(h)(t - a) + D(h)(\sigma^2(b) - t) \end{aligned}$$

where G_1, G_2, $A(f)$, $B(f)$, $C(f)$, $D(f)$ are defined as in (2.13), (2.17), (2.14), (2.15), (2.18), (2.19) respectively and

$$h(t) = \int_a^{\sigma^2(b)} G_1(t,s) f(s,y(s))\Delta s + A(g)\varphi(t) + B(g)\psi(t).$$

We now show that $T : K \to K$. First, note that $y \in K$ implies that $Ty(t) \geq 0$ on $[a, \sigma^2(b)]$, and

$$\begin{aligned}
\min_{t\in[\zeta,\omega]} Ty(t) \geq \Gamma\Bigg\{ & \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau) G_1(\tau,s) f(s,y(s))\Delta s \Delta\tau \\
& + \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau) A(f)\varphi(\tau)\Delta\tau \\
& + \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau) B(f)\psi(\tau)\Delta\tau \\
& + C(h)(\sigma^2(b)-a) + D(h)(\sigma^2(b)-a)\Bigg\} \\
\geq{} & \Gamma\|Ty\|
\end{aligned}$$

by Lemma 2.9. It follows that

$$\min_{t\in[\zeta,\omega]} Ty(t) \geq \Gamma^*\|Ty\|.$$

Also

$$\begin{aligned}
Ty(\sigma(\omega)) \geq \Gamma^*\Bigg\{ & \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau) G_1(\tau,s) f(s,y(s))\Delta s \Delta\tau \\
& + \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau) A(f)\varphi(\tau)\Delta\tau \\
& + \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau) B(f)\psi(\tau)\Delta\tau \\
& + C(h)(\sigma^2(b)-a) + D(h)(\sigma^2(b)-a)\Bigg\} \\
\geq{} & \Gamma^*\|Ty\|.
\end{aligned}$$

Hence $Ty \in K$ and so $T : K \to K$. Applying Arzelà-Ascoli theorem, we can easily check that T is completely continuous.

For $y \in K$ with $\|y\| = \eta_1$, we have

$$f(t,y) \leq F(\eta_1) \leq \eta_1\Theta.$$

Hence

$$\|Ty\| = \max_{a\leq t\leq\sigma^2(b)} \Bigg\{ \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t,\tau) G_1(\tau,s) f(s,y(s))\Delta s \Delta\tau$$

$$
\begin{aligned}
&+\int_a^{\sigma^2(b)} G_2(t,\tau)A(f)\varphi(\tau)\Delta\tau \\
&+\int_a^{\sigma^2(b)} G_2(t,\tau)B(f)\psi(\tau)\Delta\tau + C(h)(t-a)+D(h)(\sigma^2(b)-t)\Bigg\} \\
\le{}& \max_{a\le t\le\sigma^2(b)}\Bigg\{\int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(t,\tau)G_1(\tau,s)\Delta s\Delta\tau \\
&+\int_a^{\sigma^2(b)} G_2(t,\tau)A\varphi(\tau)\Delta\tau \\
&+\int_a^{\sigma^2(b)} G_2(t,\tau)B\psi(\tau)\Delta\tau \\
&+C\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\varphi(t)+B\psi(t)\right)(t-a) \\
&+D\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\varphi(t)+B\psi(t)\right)(\sigma^2(b)-t)\Bigg\}F(\eta_1) \\
\le{}& \eta_1 = \|y\|,
\end{aligned}
$$

where A, B are given as in (3.1), (3.2), respectively.

For $y\in K$ with $\|y\|=\eta_2$, we have that

$$\Gamma^*\eta_2 \le y(t) \le \eta_2$$

and

$$\min\{f(t,v):\zeta\le t\le\sigma(\omega),\Gamma^*\eta_2\le v\le\eta_2\} = H(\eta_2)\ge\eta_2\Theta^*,$$

so that

$$
\begin{aligned}
\|Ty\| &\ge \max_{a\le t\le\sigma^2(b)}\int_a^{\sigma^2(b)}\int_\zeta^{\sigma(\omega)} G_2(t,\tau)G_1(\tau,s)f(s,y(s))\Delta s\Delta\tau \\
&\ge \max_{a\le t\le\sigma^2(b)}\int_a^{\sigma^2(b)}\int_\zeta^{\sigma(\omega)} G_2(t,\tau)G_1(\tau,s)\Delta s\Delta\tau H(\eta_2) \\
&\ge \eta_2 = \|y\|.
\end{aligned}
$$

Therefore, by the first part of Theorem 1.1, it follows that T has a fixed point y with $\eta_1\le\|y\|\le\eta_2$. □

Theorem 3.2. *Let* (H1)–(H4), (H6), *and* (H8) *hold. If there exist* $j+1$ *positive numbers* $\eta_1,\eta_2,\ldots,\eta_{j+1}$ *with* $\eta_1<\eta_2<\cdots<\eta_{j+1}$ *such that either*

$$
\text{(3.5)}\qquad \begin{cases} F(\eta_{2k-1}) < \eta_{2k-1}\Theta \text{ for all } 2k-1\in\{1,2,\ldots,j+1\}, \\ H(\eta_{2k}) > \eta_{2k}\Theta^* \text{ for all } 2k\in\{1,2,\ldots,j+1\}; \end{cases}
$$

or

$$
\text{(3.6)}\qquad \begin{cases} H(\eta_{2k-1}) > \eta_{2k-1}\Theta \text{ for all } 2k-1\in\{1,2,\ldots,j+1\}, \\ F(\eta_{2k}) < \eta_{2k}\Theta^* \text{ for all } 2k\in\{1,2,\ldots,j+1\}. \end{cases}
$$

Then (1.1) *has at least* j *positive solutions.*

Proof. We only prove the result under (3.5). In the case that (3.6) holds, the results can be proved by the same method. Since F and H are continuous, $0 < \Theta \leq \Theta^*$, there exist θ_i and τ_i with $\eta_i < \theta_i < \tau_i < \eta_{i+1}$, $i = 1, 2, \ldots, j$ such that

$$F(\theta_{2k-1}) \leq \theta_{2k-1}\Theta \quad H(\tau_{2k-1}) \geq \tau_{2k-1}\Theta^* \text{ for all } 2k-1 \in \{1, 2, \ldots, j+1\},$$

$$H(\theta_{2k}) \geq \theta_{2k}\Theta^* \quad F(\tau_{2k}) \leq \tau_{2k}\Theta \text{ for all } 2k \in \{1, 2, \ldots, j+1\}.$$

From Theorem 3.1, for each $i \in \{1, 2, \ldots, j\}$, the BVP (1.1) has a positive solution y_i satisfying

$$\eta_i < \theta_i < \|y_i\| < \tau_i < \theta_{j+1}.$$

□

Corollary 3.3. Let (H1)–(H4), (H6), and (H8) hold. Assume that there exist two sequences $\{\eta_i\}$, $\{\theta_i\}$ of $(0, +\infty)$ such that

(i) $\lim_{i\to+\infty} \eta_i = +\infty$,
(ii) $\lim_{i\to+\infty} \theta_i = +\infty$,
(iii) $\lim_{i\to+\infty} \frac{F(\eta_i)}{\eta_i} < \Theta$,
(iv) $\lim_{i\to+\infty} \frac{H(\theta_i)}{\theta_i} > \Theta^*$.

Then the BVP (1.1) has a sequence of positive solutions $\{y_k\}$ such that $\|y_k\| \to \infty$ as $k \to \infty$.

4. BOUNDARY VALUE PROBLEM WITH A PARAMETER

In this section we consider the following BVP with parameter λ,

$$\text{(4.1)} \qquad \begin{cases} Ly(t) = y^{\triangle^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = \lambda f(t, y(t)), \quad t \in [a, b], \\ y(a) = \sum_{i=1}^{m-2} a_i y(\xi_i), \quad y(\sigma^2(b)) = \sum_{i=1}^{m-2} b_i y(\xi_i), \\ y^{\Delta^2}(a) = \sum_{i=1}^{m-2} a_i y^{\Delta^2}(\xi_i), \quad y^{\Delta^2}(\sigma^2(b)) = \sum_{i=1}^{m-2} b_i y^{\Delta^2}(\xi_i). \end{cases}$$

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0 := \liminf_{y\to0^+} \min_{t\in[a,\sigma^2(b)]} \frac{f(t,y)}{y}, \quad f^0 := \limsup_{y\to0^+} \max_{t\in[a,\sigma^2(b)]} \frac{f(t,y)}{y},$$

$$f_\infty := \liminf_{y\to\infty} \min_{t\in[a,\sigma^2(b)]} \frac{f(t,y)}{y}, \quad f^\infty := \limsup_{y\to\infty} \max_{t\in[a,\sigma^2(b)]} \frac{f(t,y)}{y},$$

respectively. These numbers can be regarded as generalized super or sublinear conditions on the function $f(t, y)$ at $y = 0$ and $y = \infty$. Thus, if $f_0 = f^0 = 0$ $(+\infty)$, then $f(t, y)$ is superlinear (sublinear) at $y = 0$ and if $f_\infty = f^\infty = 0$ $(+\infty)$, then $f(t, y)$ is sublinear (superlinear) at $y = +\infty$. The BVP (4.1) has a solution $y = y(t)$ if and only if y solves the operator equation

$$y(t) = (T_\lambda y)(t) = \lambda\bigg\{ \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t, \tau)G_1(\tau, s)f(s, y(s))\Delta s\Delta\tau$$

$$+\int_a^{\sigma^2(b)} G_2(t,\tau)A(f)\varphi(\tau)\Delta\tau$$
$$\left. + \int_a^{\sigma^2(b)} G_2(t,\tau)B(f)\psi(\tau)\Delta\tau + C(h)(t-a) + D(h)(\sigma^2(b)-t)\right\},$$

where G_1, G_2, $A(f)$, $B(f)$, $C(f)$, $D(f)$ are defined by (2.13), (2.17), (2.14), (2.15), (2.18), (2.19), respectively and

$$h(t) = \int_a^{\sigma^2(b)} G_1(t,s)f(s,y(s))\Delta s + A(f)\varphi(t) + B(f)\psi(t).$$

Define the cone K as in (3.4). It is clear that $T_\lambda K \subset K$ and T_λ is completely continuous. Define

$$\begin{aligned} M := &\int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)G_1(\tau,s)\Delta s\Delta\tau \\ &+ \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)A\|\varphi\|\Delta\tau \\ &+ \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)B\|\psi\|\Delta\tau \\ &+ C\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)(\sigma^2(b)-a) \\ &+ D\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)(\sigma^2(b)-a). \end{aligned} \tag{4.2}$$

Theorem 4.1. *Assume that* (H1)–(H4), (H6), *and* (H8) *are satisfied. Then, for each* λ *satisfying*

$$\frac{1}{\Gamma^* \int_a^{\sigma^2(b)}\int_\zeta^{\sigma(\omega)} G_2(t_0,\tau)G_1(\tau,s)\Delta s\Delta\tau f_\infty} < \lambda < \frac{1}{Mf_0}, \tag{4.3}$$

where $t_0 \in [a,\sigma^2(b)]$ *and* Γ^* *is a constant as in* (3.3), *there exists at least one positive solution of the BVP* (4.1).

Proof. Clearly,

$$\begin{aligned} A(f(s,y(s))) &\le \frac{1}{\Delta_1}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\int_a^{\sigma^2(b)} G_1(\xi_i,s)\Delta s & Z \\ -\sum_{i=1}^{m-2} b_i\int_a^{\sigma^2(b)} G_1(\xi_i,s)\Delta s & \sum_{i=1}^{m-2} b_i\psi(\xi_i)\end{vmatrix}\|f\| \\ &= A\|f\|, \end{aligned}$$

$$\begin{aligned} B(f(s,y(s))) &\le \frac{1}{\Delta_1}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\varphi(\xi_i) & -\sum_{i=1}^{m-2} b_i\int_a^{\sigma^2(b)} G_1(\xi_i,s)\Delta s \\ V & -\sum_{i=1}^{m-2} a_i\int_a^{\sigma^2(b)} G_1(\xi_i,s)\Delta s\end{vmatrix}\|f\| \\ &= B\|f\|, \end{aligned}$$

$$C(h) \leq \frac{1}{\Delta_2}\Phi_1\|f\| = C\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)\|f\|,$$

and

$$D(h) \leq \frac{1}{\Delta_2}\Phi_2\|f\| = D\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)\|f\|,$$

where

$$\Phi_1 = \begin{vmatrix} W & \sum_{i=1}^{m-2} a_i(\sigma^2(b)-\xi_i) - (\sigma^2(b)-a) \\ \Lambda & \sum_{i=1}^{m-2} b_i(\sigma^2(b)-\xi_i) \end{vmatrix},$$

$$\Phi_2 = \begin{vmatrix} \sum_{i=1}^{m-2} a_i(\xi_i - a) & W \\ \sum_{i=1}^{m-2} b_i(\xi_i-a) - (\sigma^2(b)-a) & \Lambda \end{vmatrix},$$

$$Z = \sum_{i=1}^{m-2} a_i\psi(\xi_i) - (\sigma^2(b)-a),$$

$$V = \sum_{i=1}^{m-2} b_i\varphi(\xi_i) - (\sigma^2(b)-a),$$

$$W = -\sum_{i=1}^{m-2} a_i \int_a^{\sigma^2(b)} G_2(\xi_i,s)\left(\int_a^{\sigma^2(b)} G_1(s,\tau)\Delta\tau + A\|\varphi\| + B\|\psi\|\right)\Delta s,$$

$$\Lambda = -\sum_{i=1}^{m-2} b_i \int_a^{\sigma^2(b)} G_2(\xi_i,s)\left(\int_a^{\sigma^2(b)} G_1(s,\tau)\Delta\tau + A\|\varphi\| + B\|\psi\|\right)\Delta s.$$

Let λ be given as in (4.3). Let $\epsilon > 0$ be chosen such that

$$\frac{1}{\Gamma^* \int_a^{\sigma^2(b)} \int_\zeta^{\sigma(\omega)} G_2(t_0,\tau)G_1(\tau,s)\Delta s\Delta\tau(f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{M(f_0+\epsilon)}. \tag{4.4}$$

Now, turning to f_0, there exists an $H_1 > 0$ such that $f(s,y) \leq (f_0+\epsilon)y$ for $0 < y \leq H_1$. So, from (4.4) and Lemma 2.8, for $y \in K$ with $\|y\| = H_1$, we have

$$\begin{aligned} T_\lambda y(t) \leq \lambda\Bigg\{ & \int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)G_1(\tau,s)\Delta s\Delta\tau \\ & + \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)A\|\varphi\|\Delta\tau \\ & + \int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)B\|\psi\|\Delta\tau \\ & + C\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)(\sigma^2(b)-a) \\ & + D\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)(\sigma^2(b)-a)\Bigg\}\|f\| \\ \leq \lambda\Bigg\{ & \int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)G_1(\tau,s)\Delta s\Delta\tau \end{aligned}$$

$$
\begin{aligned}
&+\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)A\|\varphi\|\Delta\tau \\
&+\int_a^{\sigma^2(b)} G_2(\sigma(\tau),\tau)B\|\psi\|\Delta\tau \\
&+C\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)(\sigma^2(b)-a) \\
&+D\left(\int_a^{\sigma^2(b)} G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right)(\sigma^2(b)-a)\Bigg\}(f_0+\epsilon)\|y\| \\
\le{}& \|y\|.
\end{aligned}
$$

Next, considering f_∞, there exists $\hat{H}_2 > 0$ such that $f(s,y) \ge (f_\infty - \epsilon)y$ for $y \ge \hat{H}_2$. Let $H_2 = \max\{2H_1, \frac{\hat{H}_2}{\Gamma^*}\}$. Then $y \in K$ and $\|y\| = H_2$ implies

$$\min_{t\in[\zeta,\sigma(\omega)]} y(t) \ge \Gamma^*\|y\| \ge \hat{H}_2$$

and so

$$
\begin{aligned}
T_\lambda y(t_0) &\ge \lambda \int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(t_0,\tau)G_1(\tau,s)f(s,y(s))\Delta s\Delta\tau \\
&\ge \lambda \int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(t_0,\tau)G_1(\tau,s)(f_\infty-\epsilon)y(s)\Delta s\Delta\tau \\
&\ge \Gamma^*\lambda \int_a^{\sigma^2(b)}\int_\zeta^{\sigma(\omega)} G_2(t_0,\tau)G_1(\tau,s)(f_\infty-\epsilon)\|y\|\Delta s\Delta\tau \\
&\ge \|y\|.
\end{aligned}
$$

Therefore, by first part of Theorem 1.1, it follows that T_λ has a fixed point y satisfying $H_1 \le \|y\| \le H_2$. The proof is complete. □

Theorem 4.2. *Assume that* (H1)–(H4), (H6), *and* (H8) *are satisfied. Then, for each* λ *satisfying*

$$\frac{1}{\Gamma^*\int_a^{\sigma^2(b)}\int_\zeta^{\sigma(\omega)} G_2(t_0,\tau)G_1(\tau,s)\Delta s\Delta\tau f_0} < \lambda < \frac{1}{Mf_\infty}, \tag{4.5}$$

where $t_0 \in [a,\sigma^2(b)]$ *and* Γ^* *is a constant as in* (3.3), *there exists at least one positive solution of the BVP* (4.1).

Proof. Let λ be given as in (4.5) and choose $\epsilon > 0$ such that

$$\frac{1}{\Gamma^*\int_a^{\sigma^2(b)}\int_\zeta^{\sigma(\omega)} G_2(t_0,\tau)G_1(\tau,s)\Delta s\Delta\tau (f_0-\epsilon)} \le \lambda \le \frac{1}{M(f_\infty+\epsilon)}. \tag{4.6}$$

Beginning with f_0, there exists an $H_1 > 0$ such that $f(s,y) \ge (f_0-\epsilon)y$ for $0 < y \le H_1$. Thus, from (4.6), for $y \in K$ with $\|y\| = H_1$, we have

$$T_\lambda y(t_0) \ge \lambda \int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(t_0,\tau)G_1(\tau,s)f(s,y(s))\Delta s\Delta\tau$$

$$\geq \lambda \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t_0,\tau)G_1(\tau,s)(f_0-\epsilon)y(s)\Delta s\Delta\tau$$
$$\geq \Gamma^*\lambda \int_a^{\sigma^2(b)} \int_\zeta^{\sigma(\omega)} G_2(t_0,\tau)G_1(\tau,s)(f_0-\epsilon)\|y\|\Delta s\Delta\tau$$
$$\geq \|y\|.$$

It remains to consider f_∞. There exists an $\hat{H}_2 > 0$ such that $f(s,y) \leq (f_\infty + \epsilon)y$, for all $y \geq \hat{H}_2$. There are two cases:

(a) f is bounded, and
(b) f is unbounded.

For case (a), suppose $N > 0$ such that $f(s,y) \leq N$, for all $0 \leq y \leq \infty$. Let

$$H_2 = \{2H_1, \lambda NM\}.$$

Then, for $y \in K$ with $\|y\| = H_2$, we have

$$T_\lambda y(t) \leq \lambda MN \leq \|y\|,$$

so that $\|T_\lambda y\| \leq \|y\|$.

For case (b), let $g(r) = \max\{f(t,y) : t \in [a,\sigma^2(b)], 0 \leq y \leq r\}$. The function g is nondecreasing and $\lim_{r\to\infty} g(r) = \infty$. Choose $H_2 = \max\{2H_1, \hat{H}_2\}$ so that $g(H_2) \geq g(r)$ for $0 \leq r \leq H_2$. From (4.6), for $y \in K$ with $\|y\| = H_2$, we obtain

$$T_\lambda y(t) \leq \lambda M g(H_2) \leq \lambda M(f_\infty + \epsilon)H_2 \leq H_2 = \|y\|$$

so that $\|T_\lambda y\| \leq \|y\|$. It follows from Theorem 1.1 that T_λ has a fixed point. Thus the problem (4.1) has a positive solution. The proof is complete. □

Theorem 4.3. *In addition to* (H1)–(H4), (H6) *and* (H8) *assumptions, assume* $f(s,y(s)) > 0$ *on* $[a,\sigma^2(b)] \times \mathbb{R}^+$.

(a) *If* $f^0 = 0$ *or* $f^\infty = 0$*, then there is a* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$ *the eigenvalue problem* (4.1) *has a positive solution.*
(b) *If* $f_0 = \infty$ *or* $f_\infty = \infty$*, then there is a* $\lambda_0 > 0$ *such that for all* $0 < \lambda \leq \lambda_0$ *the eigenvalue problem* (4.1) *has a positive solution.*
(c) *If* $f^0 = f^\infty = 0$*, then there is a* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$ *the eigenvalue problem* (4.1) *has two positive solutions.*
(d) *If* $f_0 = f_\infty = \infty$*, then there is a* $\lambda_0 > 0$ *such that for all* $0 < \lambda \leq \lambda_0$ *the eigenvalue problem* (4.1) *has two positive solutions.*

Proof. We now prove the part (a) of Theorem 4.3. Let $t_0 \in [a,\ \sigma^2(b)]$ and for all $r > 0$, define

$$m(r) = \min\left\{ \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t_0,\tau)G_1(\tau,s)f(s,y(s))\Delta s\Delta\tau \right.$$

$$+\int_a^{\sigma^2(b)} G_2(t_0,\tau)A(f)\varphi(\tau)\Delta\tau$$
$$\left.+\int_a^{\sigma^2(b)} G_2(t_0,\tau)B(f)\psi(\tau)\Delta\tau + C(h)(t-t_0) + D(h)(\sigma^2(b)-t_0)\right\}$$

for $y \in K$ with $\|y\| = r$. It can be shown that $m(r) > 0$ for all $r > 0$. We now show that for any $r_0 > 0$ and for all $\lambda \geq \lambda_0$, where $\lambda_0 := \frac{r_0}{m(r_0)}$, we have that if $y \in K$ with $\|y\| = r_0$, then $\|T_\lambda y\| \geq \|y\|$. To prove this let $y \in K$ with $\|y\| = r_0$. Then for $\lambda \geq \lambda_0$,

$$\begin{aligned}
T_\lambda y(t_0) = \lambda\Big\{ &\int_a^{\sigma^2(b)}\int_a^{\sigma^2(b)} G_2(t_0,\tau)G_1(\tau,s)f(s,y(s))\Delta s\Delta\tau \\
&+\int_a^{\sigma^2(b)} G_2(t_0,\tau)A(f)\varphi(\tau)\Delta\tau \\
&+\int_a^{\sigma^2(b)} G_2(t_0,\tau)B(f)\psi(\tau)\Delta\tau \\
&+C(h)(t_0-a) + D(h)(\sigma^2(b)-t_0)\Big\} \\
\geq \lambda m(r_0) &\geq \lambda_0 m(r_0) = r_0 = \|y\|.
\end{aligned}$$

Hence it follows that $\|T_\lambda y\| \geq \|y\|$ for all $y \in K$ with $\|y\| = r_0$ and $\lambda \geq \lambda_0$.

We now show that the condition $f^0 = 0$ implies that given any $r_0 > 0$ there exists p_0 such that $0 < p_0 < r_0$ and for any $y \in K$ with $\|y\| = p_0$ it follows that $\|T_\lambda y\| \leq \|y\|$ for all $\lambda \geq \lambda_0$. To prove this fix $\lambda \geq \lambda_0$ and pick $\epsilon_0 > 0$ so that

$$\lambda M \epsilon_0 \leq 1. \tag{4.7}$$

Since $f^0 = 0$, there exists $p_0 < r_0$ such that

$$\max_{t\in[a,\sigma^2(b)]} \frac{f(t,y)}{y} \leq \epsilon_0$$

for $0 < y \leq p_0$. Hence we have that

$$f(t,y) \leq \epsilon_0 y$$

for $t \in [a,\sigma^2(b)]$, $0 \leq y \leq p_0$. For $y \in K$ with $\|y\| = p_0$, we obtain

$$T_\lambda y(t) \leq \lambda M \epsilon_0 \|y\| \leq \|y\|$$

by (4.7). It follows that if $y \in K$ with $\|y\| = p_0$, then $\|T_\lambda y\| \leq \|y\|$ and hence, the problem (4.1) has a positive solution and the first part of (a) has been proven.

We now prove the second part of (a) of this theorem. Fix $\lambda \geq \lambda_0$, where $\lambda_0 := \frac{r_0}{m(r_0)}$. Pick ϵ_0 so that (4.7) holds. Since $f^\infty = 0$, there is $R > r_0$ so that

$$\max_{t\in[a,\sigma^2(b)]} \frac{f(t,y)}{y} \leq \epsilon_0$$

for $y \geq R$. Hence we have that

$$f(t,y) \leq \epsilon_0 y$$

for $t \in [a, \sigma^2(b)]$. We consider two cases. The first case is that $f(t,y)$ is bounded on $[a, \sigma^2(b)] \times \mathbb{R}^+$. In this case there is a positive number N such that

$$|f(t,y)| \leq N$$

for $t \in [a,\ \sigma^2(b)]$, $y \in \mathbb{R}^+$. Choose $R_1 \geq R$ so that

$$\lambda M N \leq R_1.$$

Then for $y \in K$ with $\|y\| = R_1$, we have

$$T_\lambda y(t) \leq \lambda M N \leq R_1 = \|y\|.$$

It follows that if $y \in K$ with $\|y\| = R_1$, then $\|T_\lambda y\| \leq \|y\|$. Since at the beginning of the proof of this theorem we proved that if $\|y\| = r_0$, then $\|T_\lambda y\| \geq \|y\|$, and since $r_0 < R_1$, it follows from Theorem 1.1 that T_λ has a fixed point and hence the BVP (4.1) has a positive solution.

Next we consider the case where $f(t,y)$ is unbounded on $[a, \sigma^2(b)] \times \mathbb{R}^+$. Let

$$g(h) := \max\{f(t,y) : t \in [a, \sigma^2(b)], 0 \leq y \leq h\}.$$

The function g is nondecreasing and

$$\lim_{h \to \infty} g(h) = \infty.$$

Choose $R_0 \geq R$ so that

$$g(R_0) \geq g(h) \quad \text{ for } 0 \leq h \leq R_0.$$

Then for $y \in K$ with $\|y\| = R_0$, we get

$$T_\lambda y(t) \leq \lambda M g(R_0) \leq \lambda M \epsilon_0 \|y\| \leq \|y\|$$

by (4.7). It follows that the problem (4.1) has a positive solution y_0 satisfying $r_0 \leq \|y_0\| \leq R_0$. Hence the proof of part (a) of this theorem is complete. The proof of part (b) can be made analogous way.

Now we show (c). Clearly if $f^0 = f^\infty = 0$, then by the proof of part (a) we get for any $r_0 > 0$ that for each fixed $\lambda \geq \lambda_0 := \frac{r_0}{m(r_0)}$ there are numbers $p_0 < r_0 < R_0$ such that there are two positive solutions of problem (4.1) with $p_0 \leq \|y_1\| \leq r_0 \leq \|y_2\| \leq R_0$. The proof of part (d) is similar. □

Theorem 4.4. *Under the hypotheses of Theorem* 4.3*, the following assertions hold.*

(a) *If there is a constant $c > 0$ such that $f(t,y) \geq cy$ for $y \geq 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem* (4.1) *has no positive solutions for $\lambda \geq \lambda_0$.*

(b) *If there is a constant $c > 0$ such that $f(t,y) \leq cy$ for $y \geq 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem* (4.1) *has no positive solutions for $0 < \lambda \leq \lambda_0$.*

Proof. We now prove the part (a) of this theorem. Assume there is a constant $c > 0$ such that $f(t,y) \geq cy$ for $y \geq 0$ and $y(t)$ is a positive solution of the eigenvalue problem (4.1). Since $T_\lambda y(t) = y(t)$ for $t \in [a, \sigma^2(b)]$, we have for $t_0 \in [a, \sigma^2(b)]$

$$\begin{aligned} y(t_0) = \lambda \Bigg\{ & \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t_0,\tau) G_1(\tau,s) f(s,y(s)) \Delta s \Delta \tau \\ & + \int_a^{\sigma^2(b)} G_2(t_0,\tau) A(f) \varphi(\tau) \Delta \tau \\ & + \int_a^{\sigma^2(b)} G_2(t_0,\tau) B(f) \psi(\tau) \Delta \tau + C(h)(t_0 - a) + D(h)(\sigma^2(b) - t_0) \Bigg\} \\ \geq {} & c\lambda \int_a^{\sigma^2(b)} \int_a^{\sigma^2(b)} G_2(t_0,\tau) G_1(\tau,s) y(s) \Delta s \Delta \tau \\ \geq {} & c\lambda \int_a^{\sigma^2(b)} \int_\zeta^{\sigma(\omega)} G_2(t_0,\tau) G_1(\tau,s) y(s) \Delta s \Delta \tau \\ \geq {} & c\Gamma^* \lambda \int_a^{\sigma^2(b)} \int_\zeta^{\sigma(\omega)} G_2(t_0,\tau) G_1(\tau,s) \Delta s \Delta \tau \|y\|. \end{aligned}$$

If we pick λ_0 sufficiently large so that

$$c\Gamma^* \lambda \int_a^{\sigma^2(b)} \int_\zeta^{\sigma(\omega)} G_2(t_0,\tau) G_1(\tau,s) \Delta s \Delta \tau > 1$$

for all $\lambda \geq \lambda_0$, then $y(t_0) > \|y\|$ which is a contradiction.

The proof of part (b) is similar. □

5. EXAMPLES

Example 5.1. Let $\mathbb{T} = \{\frac{n}{2} : n \in \mathbb{N}_0\}$. Consider the following problem on $\mathbb{T}$:

$$\left\{ \begin{array}{l} y^{\Delta^4}(t) = 100y^3/y^2 + 1, \quad t \in [0,1], \\ y(0) = 1/2y(1/2), \quad y(2) = 1/2y(1/2), \\ y^{\Delta^2}(0) = 1/2y^{\Delta^2}(1/2), \quad y^{\Delta^2}(2) = 1/2y^{\Delta^2}(1/2). \end{array} \right. \tag{5.1}$$

When taking $q(t) = 0$, $a = 0$, $b = 1$, $a_1 = b_1 = \frac{1}{2}$, $\xi_1 = \frac{1}{2}$ and $f(t, y(t)) = \frac{100y^3}{y^2+1}$, we prove the solvability of the problem (3.5) by means of Theorem 3.1.

Since $q(t) = 0$, we get

$$G_1(t,s) = G_2(t,s) = \left\{ \begin{array}{ll} \dfrac{t(2-\sigma(s))}{2}, & t \leq s, \\ \dfrac{\sigma(s)(2-t)}{2}, & \sigma(s) \leq t. \end{array} \right.$$

By using (2.9), (2.10), (3.1), (3.2), (2.18), (2.19) and (3.3), we obtain

$$\Delta_1 = \Delta_2 = -2,$$

$$A = B = 3/16,$$

$$C\left(\int_0^2 G_1(t,s)\Delta s + 3/8\right) = D\left(\int_0^2 G_1(t,s)\Delta s + 3/8\right) = 27/256,$$

$$\Gamma^* = 1/6,$$

respectively. Hence we get

$$\Theta = 16/9, \quad \Theta^* = 32/3.$$

There exist two positive numbers 1/100 and 12 such that

$$F(1/100) = 0.00009999000100 \leq 0.01777777778 = 1/100\Theta,$$

$$H(12) = 160 \geq 128 = 12\Theta^*.$$

Then, from Theorem 3.1 the problem (5.1) has least one positive solution y satisfying

$$1/100 \leq \|y\| \leq 12.$$

Example 5.2. Let us introduce an example to illustrate the usage of Theorem 4.1. Let $\mathbb{T} = [-1,2] \cup [3,5]$.

Consider the BVP:

(5.2)
$$\begin{cases} y^{\triangle^4}(t) - y^{\triangle^2}(\sigma(t)) = ye^y, \quad t \in [0,4], \\ y(0) = 1/20y(1) + 1/10y(2), \quad y(4) = 1/50y(1) + 1/300y(2), \\ y^{\Delta^2}(0) = 1/20y^{\Delta^2}(1) + 1/10y^{\Delta^2}(2), \quad y^{\Delta^2}(4) = 1/50y^{\Delta^2}(1) + 1/300y^{\Delta^2}(2). \end{cases}$$

Then $a = 0$, $b = 4$, $\xi_1 = 1$, $\xi_2 = 2$, $a_1 = 1/20$, $a_2 = 1/10$, $b_1 = 1/50$, $b_2 = 1/300$, and

$$q(t) = 1, \quad f(t,y) = f(y) = ye^y, \quad t \in [0,4].$$

It is clear that (H1)–(H4), (H6), and (H8) are satisfied, and $f_0 = 1$, $f_\infty = \infty$.

We can also see that

$$G_1(t,s) = \frac{1}{D}\begin{cases} \varphi(t)\psi(\sigma(s)), & t \leq s, \\ \varphi(\sigma(s))\psi(t), & \sigma(s) \leq t, \end{cases}$$

$$G_2(t,s) = \frac{1}{4}\begin{cases} t(4-\sigma(s)), & t \leq s, \\ \sigma(s)(4-t), & \sigma(s) \leq t, \end{cases}$$

where $D = e^3 + \sinh 1 \cosh 2$,

$$\varphi(t) = \begin{cases} \sinh t, & 0 \leq t \leq 2, \\ e^{t-1} + \cosh 2 \sinh(t-3), & 3 \leq t \leq 4, \end{cases}$$

$$\psi(t) = \begin{cases} e^{3-t} + \sinh 1 \cosh(2-t), & 0 \leq t \leq 2, \\ \sinh(4-t), & 3 \leq t \leq 4. \end{cases}$$

Hence we get
$$\int_0^4 \int_1^3 G_2(3,\tau)G_1(\tau,s)\Delta s\Delta\tau = 0.7135480189.$$
From (2.9), (2.10), (3.1), (3.2), (2.18), (2.19), (3.3), and (4.2) we obtain
$$\Delta_1 = -578.8438568 \quad \Delta_2 = -14.486,$$
$$A = 0.0005469299304, \quad B = 0.002863069162,$$
$$C\left(\int_0^4 G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right) = 0.006460923847,$$
$$D\left(\int_0^4 G_1(t,s)\Delta s + A\|\varphi\| + B\|\psi\|\right) = 0.04664575747,$$
$$\Gamma^* = 1/8, \quad M = 1.926834049$$
respectively. Thus, for each λ satisfying
$$0 < \lambda < 0.5189860541,$$
there exists at least one positive solution of BVP (5.2).

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