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OSCILLATION OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATION WITH SEVERAL COEFFICIENTS

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ABSTRACT. In this article, we show that the oscillation of all solutions to the neutral equation

$$[x(t) - R(t) N(x(t-\kappa))]' + \sum_{i=1}^{n} P_i(t) F_i(x(t-\tau_i)) - \sum_{j=1}^{m} Q_j(t) G_j(x(t-\sigma_j)) = 0$$

is implied by the oscillation of all solutions to the linear equation

$$[x(t) - rx(t-\kappa)]' + \sum_{i=1}^{n} p_i x(t-\tau_i) - \sum_{j=1}^{m} q_j x(t-\sigma_j) = 0.$$

In these equations, R,P_i, Q_j are positive and continuous functions, and κ, τ_i, σ_j are positive constants that represent delays.

Key Words. Oscillation, Nonlinear, Linearized, Neutral, Delay, Positive and Negative Coefficients

AMS (MOS) Subject Classification. 34K40, 34K99, 34C10

1. INTRODUCTION

There is a lot of interest in the oscillation of solutions to delay differential equations, mainly because of their applications in physics, ecology, biology, etc. In particular, we are interested in the oscillation of solutions to

$$[x(t) - R(t) N(x(t-\kappa))]' + \sum_{i=1}^{n} P_i(t) F_i(x(t-\tau_i)) - \sum_{j=1}^{m} Q_j(t) G_j(x(t-\sigma_j)) = 0, \tag{1.1}$$

which is a neutral delay equation with several positive and negative coefficients. This equation includes the equation studied by Q. Chuanxi and G. Ladas [2], L. Erbe et al [6, page 185] and Ö. Öcalan et al [12]. Their equation is

$$[x(t) - R(t) N(x(t-\kappa))]' + P(t) F(x(t-\tau)) - Q(t) G(x(t-\sigma)) = 0,$$

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which is the same as (1.1), only with $n = m = 1$. When N, F_i, G_j are identity functions in (1.1), we obtain

$$[x(t) - R(t)x(t-\kappa)]' + \sum_{i=1}^{n} P_i(t)x(t-\tau_i) - \sum_{j=1}^{m} Q_j(t)x(t-\sigma_j) = 0 \tag{1.2}$$

for which oscillation and non-oscillation has been studied by Z. Luo and J. Shen [10]. When $n = m = 1$ in (1.2), we have

$$[x(t) - R(t)x(t-\kappa)]' + P(t)x(t-\tau) - Q(t)x(t-\sigma) = 0$$

which was studied in [2, 7, 9, 11, 13, 14]. As far as we know, this is the first publication on the oscillation of solutions to (1.1); therefore our results are new.

As usual we call a function is oscillatory if it has arbitrary large zeros otherwise we call it nonoscillatory.

In *Section 2* we will associate (1.1) with

$$[x(t) - rx(t-\kappa)]' + \sum_{i=1}^{n} p_i x(t-\tau_i) - \sum_{j=1}^{m} q_j x(t-\sigma_j) = 0 \tag{1.3}$$

which is a linear autonomous neutral equation. In (1.3), coefficients are as follows:

$$r := \limsup_{t\to\infty} R(t), \quad p_i := \liminf_{t\to\infty} P_i(t) \quad \text{and} \quad q_j := \limsup_{t\to\infty} Q_j(t)$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Under the following hypothesis (1.1) is associated with (1.3).

(H_1) $R, P_i, Q_j \in C([t_0, \infty), \mathbb{R}^+)$ and $N, F_i, G_j \in C(\mathbb{R}, \mathbb{R})$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

(H_2) $\kappa > 0$ and $r, \tilde{r} \in (0, 1)$ where $\tilde{r} := \liminf_{t\to\infty} R(t)$.

(H_3) $0 \leq \dfrac{N(s)}{s} \leq 1$ for $s \neq 0$ and $\lim_{s\to\infty} \dfrac{N(s)}{s} = 1$.

(H_4) There exits J_i sets for $i = 1, 2, \ldots, u$, where $u \leq n$ such that

$$\bigcup_{i=1}^{u} J_i = \{1, 2, \ldots, m\} \text{ and } \bigcap_{j=1}^{u} J_i = \emptyset.$$

$(H_4; a)$ If $1 \leq i \leq u$ then $\tau_i \geq \sigma_j > 0$ for $j \in J_i$ else $\tau_i > 0$.

$(H_4; b)$ $q_j > 0$ for $j = 1, 2, \ldots, m$.

If $1 \leq i \leq u$ then $p_i - \sum_{j\in J_i} q_j > 0$ else $p_i > 0$.

$(H_4; c)$ There exists positive constants M_j such that

$$0 \leq \frac{G_j(s)}{s} \leq M_j \text{ for } s \neq 0,\ \lim_{s\to 0} \frac{G_j(s)}{s} = 1 \text{ for } j = 1, 2, \ldots, m.$$

If $1 \leq i \leq u$ then $F_i(s) \geq G_j(s)$ for $j \in J_i$ else $\lim_{s\to\infty} \dfrac{F_i(s)}{s} = 1$.

$(H_4; d)$ $1 > r + \sum_{i=1}^{u} \sum_{j\in J_i} M_j q_j (\tau_i - \sigma_j)$.

2. LINEARIZED OSCILLATION OF EQUATION (1.1)

In this section, we give two general lemmas and prove our objective.

Lemma 2.1. *Assume that* $r \in (0,1]$, $\kappa > 0$, $(H_4;a)$ *and* $(H_4;b)$ *holds. If every solution of*

$$[x(t) - rx(t-\kappa)]' + \sum_{i=1}^{n} p_i x(t-\tau_i) - \sum_{j=1}^{m} q_j x(t-\sigma_j) = 0 \tag{2.1}$$

oscillates then there exits a positive ε_0 *such that*

$$[x(t) - (r-\varepsilon)x(t-\kappa)]' + \sum_{i=1}^{n} (p_i - \xi_i) x(t-\tau_i) - \sum_{j=1}^{m} (q_j + \zeta_j) x(t-\sigma_j) = 0 \tag{2.2}$$

is oscillatory, where $\varepsilon, \xi_i, \zeta_j \in [0, \varepsilon_0)$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$.

Proof. To prove this lemma, it suffices to show that the characteristic equation of (2.2) has no real roots. The assumption that every solution of (2.1) oscillates implies that

$$f(\lambda) := \lambda - r\lambda e^{-\lambda\kappa} + \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} - \sum_{j=1}^{m} q_j e^{-\lambda\sigma_j} = 0$$

has no real roots. By $(H_4;a)$ and $(H_4;b)$ we have that

$$f(\lambda) = \lambda - r\lambda e^{-\lambda\kappa} + \sum_{i=1}^{u} \left(p_i e^{-\lambda\tau_i} - \sum_{j\in J_i} q_j e^{-\lambda\sigma_j} \right) + \sum_{i=u+1}^{n} p_i e^{-\lambda\tau_i}$$

so $\lim_{|\lambda|\to\infty} f(\lambda) = \infty$. Then $f(\lambda) > 0$ holds for $\lambda \in \mathbb{R}$. Now define

$$M := \min_{\lambda\in\mathbb{R}} f(\lambda), \ v_i := \begin{cases} p_i - \sum_{j\in J_i} q_j, & 1 \le i \le u \\ p_i, & u+1 \le i \le n \end{cases}$$

and

$$\delta := \frac{1}{(n+m+1)} \min_{1\le i\le n} \{r, v_i\} \text{ and } g(\lambda) := \delta \left(|\lambda| e^{-\lambda\kappa} + \sum_{i=1}^{n} e^{-\lambda\tau_i} + \sum_{j=1}^{m} e^{-\lambda\sigma_j} \right).$$

Clearly $\delta > 0$. Observing that

$$f(\lambda) - g(\lambda) = \lambda - (r\lambda + \delta|\lambda|) e^{-\lambda\kappa} + \sum_{i=1}^{n} (p_i - \delta) e^{-\lambda\tau_i} - \sum_{j=1}^{m} (q_j + \delta) e^{-\lambda\sigma_j},$$

we have $\lim_{|\lambda|\to\infty} (f(\lambda) - g(\lambda)) = \infty$. In particular there exits a $\lambda_0 > 0$ such that $f(\lambda) - g(\lambda) > \frac{M}{n+m}$ for $|\lambda| \ge \lambda_0$. Now set $\varepsilon_0 := \delta \min\left\{1, \frac{M}{(n+m)g(-\lambda_0)}\right\}$. To complete the proof, it suffices to show that for any $\varepsilon, \xi_i, \zeta_j \in [0, \varepsilon_0)$ for $i = 1, \ldots, n$ and for $j = 1, \ldots, m$

$$h(\lambda) := \lambda - (r-\varepsilon)\lambda e^{-\lambda\kappa} + \sum_{i=1}^{n} (p_i - \xi_i) e^{-\lambda\tau_i} - \sum_{j=1}^{m} (q_j + \zeta_j) e^{-\lambda\sigma_j} = 0$$

has no real roots which is the characteristic equation of (2.2). In fact, for $|\lambda| \geq \lambda_0$

$$\begin{aligned} h(\lambda) &= f(\lambda) + \varepsilon\lambda \mathrm{e}^{-\lambda\kappa} - \sum_{i=1}^{n} \xi_i \mathrm{e}^{-\lambda\tau_i} - \sum_{j=1}^{m} \zeta_j \mathrm{e}^{-\lambda\sigma_j} \\ &\geq f(\lambda) - \varepsilon|\lambda| \mathrm{e}^{-\lambda\kappa} - \sum_{i=1}^{n} \xi_i \mathrm{e}^{-\lambda\tau_i} - \sum_{j=1}^{m} \zeta_j \mathrm{e}^{-\lambda\sigma_j} \\ &\geq f(\lambda) - g(\lambda) > \frac{M}{n+m} > 0 \end{aligned}$$

and for $|\lambda| \leq \lambda_0$

$$\begin{aligned} h(\lambda) &\geq f(\lambda) - \varepsilon_0 \left(|\lambda_0| \mathrm{e}^{\lambda_0\kappa} + \sum_{i=1}^{n} \mathrm{e}^{\lambda_0\tau_i} + \sum_{j=1}^{m} \mathrm{e}^{\lambda_0\sigma_j} \right) \\ &= f(\lambda) - \varepsilon_0 g(-\lambda_0) \geq M - \delta\frac{M}{n+m} \\ &\geq M - \frac{r}{(n+m+1)}\frac{M}{(n+m)} \\ &\geq M - \frac{M}{(n+m+1)(n+m)} > 0. \end{aligned}$$

□

Lemma 2.2. *Assume that* $\kappa > 0, \alpha_j, \beta_j, \gamma_k \geq 0$, $(H_4; a)$, $R \in C([T,\infty),(0,\infty))$, $F_k, G_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ *for* $j = 1, \ldots, m$, $k = u+1, \ldots, n$. *And furthermore,* G_j, H_k *functions are nondecreasing in a neighborhood of the origin. Define* $M := \max_{1\leq i\leq n} \{\kappa, \tau_i\}$ *and suppose that the integral inequality*

$$\begin{aligned} z(t) \geq R(t) z(t-\kappa) &+ \sum_{k=u+1}^{n} \gamma_k \int_{t-\tau_k}^{\infty} F_k(z(s)) \mathrm{d}s \\ &+ \sum_{i=1}^{u} \sum_{j\in J_i} \left(\alpha_j \int_{t-\tau_i}^{t-\sigma_j} G_j(z(s)) \mathrm{d}s + \beta_j \int_{t-\tau_i}^{\infty} G_j(z(s)) \mathrm{d}s \right) \end{aligned} \tag{2.3}$$

has a solution $z \in C([T-M,\infty), \mathbb{R}^+)$ *such that* $\lim_{t\to\infty} z(t) = 0$. *Then the corresponding integral equality*

$$\begin{aligned} y(t) = R(t) y(t-\kappa) &+ \sum_{k=u+1}^{n} \gamma_k \int_{t-\tau_k}^{\infty} F_k(y(s)) \mathrm{d}s \\ &+ \sum_{i=1}^{u} \sum_{j\in J_i} \left(\alpha_j \int_{t-\tau_i}^{t-\sigma_j} G_j(y(s)) \mathrm{d}s + \beta_j \int_{t-\tau_i}^{\infty} G_j(y(s)) \mathrm{d}s \right) \end{aligned} \tag{2.4}$$

has a solution $y \in C([T-M,\infty), \mathbb{R}^+)$ *such that* $\lim_{t\to\infty} y(t) = 0$.

Proof. Choose a $T_1 \geq T$ and a $\delta > 0$ such that $0 < z(t) < \delta$ for $t \geq T_1 - M$ and $F_k(s), G_j(s)$ functions are nondecreasing for $s \in [0,\delta]$ for $j = 1, \ldots, m$, $k =$

$u+1,\ldots,n$. Now define a set of functions

$$\Omega := \left\{w \in C\left([T-M,\infty),\mathbb{R}^+\right) : 0 \leq w(t) \leq z(t),\ t \geq T-M\right\}$$

and a mapping Ψ on Ω as the following

$$(2.5)\quad (\Psi w)(t) := \begin{cases} R(t)\,w(t-\kappa) + \sum_{k=u+1}^{n} \gamma_k \int_{t-\tau_k}^{\infty} F_k(w(s))\,\mathrm{d}s \\ +\sum_{i=1}^{u}\sum_{j\in J_i}\Big(\alpha_j \int_{t-\tau_i}^{t-\sigma_j} G_j(w(s))\,\mathrm{d}s & , \qquad t \geq T_1 \\ +\beta_j \int_{t-\tau_i}^{\infty} G_j(w(s))\,\mathrm{d}s\Big) \\ (\Psi w)(T_1) + z(t) - z(T_1), \qquad T-M \leq t \leq T_1. \end{cases}$$

It is obvious that Ψ is continuous. Also, for $w_1, w_2 \in \Omega$ satisfying $w_1 \leq w_2$, satisfy $\Psi w_1 \leq \Psi w_2$. From (2.3), $z \geq \Psi z$ and so $w \in \Omega$ implies $0 \leq \Psi w \leq \Psi z \leq z$. Thus, $\Psi \in C(\Omega,\Omega)$. Now define a sequence of functions $\{y_n\}$ on Ω as follows:

$$y_n := \begin{cases} z, & n = 0 \\ \Psi y_{n-1}, & n = 1,2,\ldots. \end{cases}$$

By induction, we see that

$$0 \leq y_n \leq y_{n-1} \leq z \text{ for } t \geq T-M.$$

Set $y(t) := \lim_{n\to\infty} y_n(t)$ for $t \geq T-M$. Then $y(t)$ satisfies (2.4) by Lebesgue's dominant convergence theorem. From (2.5), $y(t) > 0$ for $T-M \leq t \leq T_1$. Consequently, $y(t) > 0$ for $t \geq T-M$. It is clear that $\lim_{t\to\infty} y(t) = 0$ holds. □

Theorem 2.3. *Assume that* $(H_1)-(H_4)$ *holds. If every solution of* (1.3) *is oscillatory then every solution of* (1.1) *is also oscillatory.*

Proof. Assume for the sake of contradiction that (1.1) has an eventually positive solution $x(t)$ and every solution of (1.3) oscillates. Then by the assumption, the characteristic equation of (1.3)

$$f(\lambda) := \lambda - r\lambda \mathrm{e}^{-\lambda\kappa} + \sum_{i=1}^{n} p_i \mathrm{e}^{-\lambda\tau_i} - \sum_{j=1}^{m} q_j \mathrm{e}^{-\lambda\sigma_j} = 0$$

has no real roots. From $(H_4;a)$ and $(H_4;b)$, we have $f(0) > 0$ as in Lemma 2.1, which implies $f(\lambda) > 0$ holds for $\lambda \in \mathbb{R}$. In the view of $(H_4;b)$ set v_j numbers for $j = 1,2,\ldots,m$ such

$$v_j - q_j > 0 \text{ and } \sum_{j\in J_i} v_j = p_i \text{ for } 1 \leq i \leq u$$

Pick $b_j, c_j > 0$ for $j = 1,2,\ldots,m$ arbitrary small such satisfies

$$(2.6)\qquad (v_j - c_j) - (q_j + b_j) > 0$$

and by considering $(H_4; d)$

$$1 > r + \sum_{i=1}^{u} \sum_{j \in J_i} M_j \left(q_j + b_j\right) \left(\tau_i - \sigma_j\right). \tag{2.7}$$

Set

$$a_i := \sum_{j \in J_i} c_j > 0 \text{ for } 1 \leq i \leq u$$

and pick arbitrary small $a_i > 0$ for $u+1 \leq i \leq n$ such that $(p_i - a_i) > 0$. By summing (2.6) for $j \in J_i$ $(i = 1, 2, \ldots, u)$, we have

$$(p_i - a_i) - \sum_{j \in J_i} (q_j + b_j) > 0.$$

By (1.1), (H_3) and $(H_4; c)$, for sufficiently large t_1, we have

$$\begin{aligned} 0 \geq {} & [x(t) - R(t) N(x(t-\kappa))]' + \sum_{i=1}^{n} (p_i - a_i) F_i(x(t-\tau_i)) \\ & - \sum_{j=1}^{m} (q_j + b_j) G_j(x(t-\sigma_j)) \end{aligned}$$

for $t \geq t_1$. Now set

$$z(t) := x(t) - R(t) N(x(t-\kappa)) - \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) \int_{t-\tau_i}^{t-\sigma_j} G_j(x(s))\, \mathrm{d}s. \tag{2.8}$$

By the assumption $(H_4; c)$

$$\begin{aligned} z'(t) = {} & [x(t) - R(t) N(x(t-\kappa))]' \\ & - \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) \left(G_j(x(t-\sigma_j)) - G_j(x(t-\tau_i))\right) \\ = {} & [x(t) - R(t) N(x(t-\kappa))]' - \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) G_j(x(t-\sigma_j)) \\ & + \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) G_j(x(t-\tau_i)) \\ = {} & [x(t) - R(t) N(x(t-\kappa))]' - \sum_{j=1}^{m} (q_j + b_j) G_j(x(t-\sigma_j)) \\ & + \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) G_j(x(t-\tau_i)). \end{aligned}$$

By (1.1) and (2.6)

$$z'(t) = -\sum_{i=1}^{n} (p_i - a_i) F_i(x(t-\tau_i)) + \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) G_j(x(t-\tau_i))$$

$$
\begin{aligned}
&= -\sum_{i=1}^{u} (p_i - a_i) F_i (x (t-\tau_i)) - \sum_{j \in J_i} (q_j + b_j) G_j (x (t-\tau_i)) \\
&\quad - \sum_{i=u+1}^{n} (p_i - a_i) F_i (x (t-\tau_i)) \\
&\leq -\sum_{i=1}^{u} \sum_{j \in J_i} (v_j - c_j) F_i (x (t-\tau_i)) - \sum_{j \in J_i} (q_j + b_j) G_j (x (t-\tau_i)) \\
&\quad - \sum_{i=u+1}^{n} (p_i - a_i) F_i (x (t-\tau_i)) \\
&\leq -\sum_{i=1}^{u} \sum_{j \in J_i} (v_j - c_j) G_j (x (t-\tau_i)) - \sum_{j \in J_i} (q_j + b_j) G_j (x (t-\tau_i)) \\
&\quad - \sum_{i=u+1}^{n} (p_i - a_i) F_i (x (t-\tau_i))
\end{aligned}
$$

so we have

$$
\begin{aligned}
z'(t) = &-\sum_{i=1}^{u} \sum_{j \in J_i} ((v_j - c_j) - (q_j + b_j)) G_j (x (t-\tau_i)) \\
&- \sum_{i=u+1}^{n} (p_i - a_i) F_i (x (t-\tau_i)) < 0,
\end{aligned} \tag{2.9}
$$

which implies that $z(t)$ is decreasing. Now we claim that $x(t)$ is bounded. Otherwise there exits a sequence $\{t_n\}$ such $\lim_{n\to\infty} t_n = \infty$, $x(t_n) = \max_{s \leq t_n} x(s)$ and $\lim_{n\to\infty} x(t_n) = \infty$. Then by hypotheses $(H_2), (H_3), (H_4; c)$ and (2.7)

$$
\begin{aligned}
z(t_n) &= x(t_n) - R(t_n) N(x(t_n - \kappa)) - \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) \int_{t_n-\tau_i}^{t_n-\sigma_j} G_j (x(s)) \, \mathrm{d}s \\
&\geq x(t_n) - R(t_n) \frac{N(x(t_n-\kappa))}{x(t_n-\kappa)} x(t_n-\kappa) \\
&\quad - \sum_{i=1}^{u} \sum_{j \in J_i} (q_j + b_j) \int_{t_n-\tau_i}^{t_n-\sigma_j} \frac{G_j (x(s))}{x(s)} x(s) \, \mathrm{d}s \\
&\geq x(t_n) \left(1 - r - \sum_{i=1}^{u} \sum_{j \in J_i} M_j (q_j + b_j)(\tau_i - \sigma_j) \right) \to \infty \text{ as } n \to \infty,
\end{aligned}
$$

which contradicts (2.7). Therefore, $x(t)$ is bounded and so is $z(t)$. Now define $\ell := \lim_{t\to\infty} z(t)$ and by integrating (2.7), we have

$$
\begin{aligned}
\ell - z(t_1) \leq &-\sum_{i=1}^{u} \sum_{j \in J_i} ((v_j - c_j) - (q_j + b_j)) \int_{t_1}^{\infty} G_j (x(s-\tau_i)) \, \mathrm{d}s \\
&- \sum_{i=u+1}^{n} (p_i - a_i) \int_{t_1}^{\infty} F_i (x(s-\tau_i)) \, \mathrm{d}s.
\end{aligned}
$$

We claim that $\liminf_{t\to\infty} x(t) = 0$ otherwise from $(H_4; c)$ there would exist a $t_2 > t_1$ and $\varepsilon_1 > 0$ such $x(t-\tau_i) > \varepsilon_1$ for all $t \geq t_2$, $i = 1, \ldots, n$. Then

$$\begin{aligned} z(t_1) &\geq \ell + \sum_{i=1}^{u} \sum_{j\in J_i} ((v_j - c_j) - (q_j + b_j)) \int_{t_1}^{\infty} G_j(x(s-\tau_i))\,\mathrm{d}s \\ &\quad + \sum_{i=u+1}^{n} (p_i - a_i) \int_{t_1}^{\infty} F_i(x(s-\tau_i))\,\mathrm{d}s \\ &\geq \ell + \sum_{i=1}^{u} \sum_{j\in J_i} ((v_j - c_j) - (q_j + b_j)) \int_{t_2}^{\infty} G_j(\varepsilon_1)\,\mathrm{d}s \\ &\quad + \sum_{i=u+1}^{n} (p_i - a_i) \int_{t_2}^{\infty} F_i(\varepsilon_1)\,\mathrm{d}s, \end{aligned} \tag{2.10}$$

which contradicts decreasing behavior of $z(t)$. Thus $\liminf_{t\to\infty} x(t) = 0$ holds and this means $\ell \leq 0$ because from (2.8), $z(t) \leq x(t)$ for $t \geq t_1$. Now we claim $\lim_{t\to\infty} x(t) = 0$ and define $\mu := \limsup_{t\to\infty} x(t)$. So there exists a sequence $\{t_n\}$ such $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = \mu$. On the other hand, for arbitrary small $\varepsilon_2 > 0$

$$\begin{aligned} z(t_n) &\geq x(t_n) - R(t_n) N(x(t_n - \kappa)) - \sum_{i=1}^{u} \sum_{j\in J_i} (q_j + b_j) \int_{t_n-\tau_i}^{t_n-\sigma_j} G_j(x(s))\,\mathrm{d}s \\ &\geq x(t_n) - R(t_n) x(t_n - \kappa) - \sum_{i=1}^{u} \sum_{j\in J_i} M_j (q_j + b_j) \int_{t_n-\tau_i}^{t_n-\sigma_j} x(s)\,\mathrm{d}s \\ &\geq x(t_n) - R(t_n)(\mu + \varepsilon_2) - \sum_{i=1}^{u} \sum_{j\in J_i} M_j (q_j + b_j)(\tau_i - \sigma_j)(\mu + \varepsilon_2). \end{aligned}$$

Letting $n \to \infty$ we obtain

$$\ell \geq \mu - (\mu + \varepsilon_2)\left(r + \sum_{i=1}^{u} \sum_{j\in J_i} M_j (q_j + b_j)(\tau_i - \sigma_j)\right).$$

Since, in the view of b_j $(j = 1, 2, \ldots, m)$ numbers are arbitrary small, ε_2 can be sufficiently small. So we have

$$\ell \geq \mu \left(1 - r - \sum_{i=1}^{u} \sum_{j\in J_i} M_j (q_j + b_j)(\tau_i - \sigma_j),\right),$$

which together $\ell \leq 0$ and (2.7), implies $\mu = \ell = 0$. Hence $\lim_{t\to\infty} x(t) = 0$ and so $\lim_{t\to\infty} z(t) = 0$. Together with (2.9) and (2.10), we obtain

$$\begin{aligned} x(t) &\geq R(t) N(x(t-\kappa)) + \sum_{i=u+1}^{n} (p_i - a_i) \int_{t-\tau_i}^{\infty} F_i(x(s))\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{u} \sum_{j\in J_i} ((v_j - c_j) - (q_j + b_j)) \int_{t-\tau_i}^{\infty} G_j(x(s))\,\mathrm{d}s \end{aligned}$$

$$+\sum_{i=1}^{u}\sum_{j\in J_i}+(q_j+b_j)\int_{t-\tau_i}^{t-\sigma_j}G_j(x(s))\,\mathrm{d}s$$

or equivalently

$$\begin{aligned}x(t)\geq{}&R(t)\frac{N(x(t-\kappa))}{x(t-\kappa)}x(t-\kappa)+\sum_{i=u+1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}\frac{F_i(x(s))}{x(s)}x(s)\,\mathrm{d}s\\&+\sum_{i=1}^{u}\sum_{j\in J_i}((v_j-c_j)-(q_j+b_j))\int_{t-\tau_i}^{\infty}\frac{G_j(x(s))}{x(s)}x(s)\,\mathrm{d}s\\&+\sum_{i=1}^{u}\sum_{j\in J_i}(q_j+b_j)\int_{t-\tau_i}^{t-\sigma_j}\frac{G_j(x(s))}{x(s)}x(s)\,\mathrm{d}s.\end{aligned}$$

Pick $0<\varepsilon_3<\frac{1}{(n+m)}\min_{1\leq j\leq m}\left\{r,\frac{b_j}{q_j+b_j}\right\}$. Then by assumptions (H_2), (H_3) and $(H_4;c)$ and for sufficiently large $t_3\geq t_1$

$$\begin{aligned}x(t)\geq{}&(r-\varepsilon_3)x(t-\kappa)+(1-\varepsilon_3)\sum_{i=u+1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\\&+(1-\varepsilon_3)\sum_{i=1}^{u}\sum_{j\in J_i}((v_j-c_j)-(q_j+b_j))\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s+(q_j+b_j)\int_{t-\tau_i}^{t-\sigma_j}x(s)\,\mathrm{d}s\\={}&(r-\varepsilon_3)x(t-\kappa)+(1-\varepsilon_3)\sum_{i=u+1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\\&+(1-\varepsilon_3)\sum_{i=1}^{u}\left((p_i-a_i)-\sum_{j\in J_i}(q_j+b_j)\right)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\\&+\sum_{j\in J_i}(q_j+b_j)\int_{t-\tau_i}^{t-\sigma_j}x(s)\,\mathrm{d}s\\={}&(r-\varepsilon_3)x(t-\kappa)+(1-\varepsilon_3)\sum_{i=u+1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\\&+(1-\varepsilon_3)\sum_{i=1}^{u}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\\&+\sum_{j\in J_i}(q_j+b_j)\left(\int_{t-\tau_i}^{t-\sigma_j}x(s)\,\mathrm{d}s-\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\right)\\={}&(r-\varepsilon_3)x(t-\kappa)+(1-\varepsilon_3)\sum_{i=u+1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\\&+(1-\varepsilon_3)\sum_{i=1}^{u}\left((p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s-\sum_{j\in J_i}(q_j+b_j)\int_{t-\sigma_j}^{\infty}x(s)\,\mathrm{d}s\right)\\={}&(r-\varepsilon_3)x(t-\kappa)+(1-\varepsilon_3)\sum_{i=u+1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s\end{aligned}$$

$$+ (1-\varepsilon_3)\sum_{i=1}^{u}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s-(1-\varepsilon_3)\sum_{i=1}^{u}\sum_{j\in J_i}(q_j+b_j)\int_{t-\sigma_j}^{\infty}x(s)\,\mathrm{d}s$$

$$=(r-\varepsilon_3)\,x(t-\kappa)+(1-\varepsilon_3)\sum_{i=1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}x(s)\,\mathrm{d}s$$

$$-(1-\varepsilon_3)\sum_{j=1}^{m}(q_j+b_j)\int_{t-\sigma_j}^{\infty}x(s)\,\mathrm{d}s$$

for $t \geq t_3$. By Lemma 2.2, the equation

$$y(t)=(r-\varepsilon_3)\,y(t-\kappa)+(1-\varepsilon_3)\sum_{i=1}^{n}(p_i-a_i)\int_{t-\tau_i}^{\infty}y(s)\,\mathrm{d}s$$

$$-(1-\varepsilon_3)\sum_{j=1}^{m}(q_j+b_j)\int_{t-\sigma_j}^{\infty}y(s)\,\mathrm{d}s$$

has a solution $y \in C([T-M,\infty),\mathbb{R}^+)$, where $M := \max_{1\leq i\leq n}\{\kappa,\tau_i\}$ and $T \geq t_3+M$. Thus, y is a positive solution of the neutral equation

$$(y(t)-(r-\varepsilon_3)\,y(t-\kappa))'$$

$$+(1-\varepsilon_3)\left(\sum_{i=1}^{n}(p_i-a_i)\,y(t-\tau_i)-\sum_{j=1}^{m}(q_j+b_j)\,y(t-\sigma_j)\right)=0.$$

By defining $\xi_i := a_i+\varepsilon_3(p_i-a_i)$ $(i=1,2,\ldots,n)$ and $\zeta_j := b_j-\varepsilon_3(q_j+b_j)$ $(j=1,2,\ldots,m)$ which are arbitrary small positive numbers, we have

$$(y(t)-(r-\varepsilon_3)\,y(t-\kappa))'+\sum_{i=1}^{n}(p_i-\xi_i)\,y(t-\tau_i)-\sum_{j=1}^{m}(q_j+\zeta_j)\,y(t-\sigma_j)=0.$$

Hence by Lemma 2.1, (1.3) has a positive solution. This is the contradiction completing the proof. □

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