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LIDSTONE BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEAR TERMS AND HIGHER ORDER DERIVATIVES

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ABSTRACT. By employing topological degree theory, this paper investigates the existence of at least one nontrivial solutions for Lidstone boundary value problems with sign-changing nonlinear term and higher order derivatives. Meanwhile, one example is worked out to demonstrate the main result.

Keywords: Nontrivial solutions, Lidstone boundary value problems, Sign-changing nonlinear term, Topological degree theory, Higher order derivatives.

AMS (MOS) Subject Classification. 34B15.

1. INTRODUCTION

This paper is concerned with the following Lidstone boundary value problem (BVP, for short)

$$(1.1)\qquad \begin{cases} (-1)^n u^{2n}(t) = f(t, u(t), u''(t), \ldots, u^{(2(n-1))}(t)), & t \in (0,1); \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, \ldots, n-1, \end{cases}$$

where $n \geq 1$, f is a given sign-changing function satisfying some assumptions that will be specified later. For convenience, we first give some notations.

Let $\mathbf{R}^+ = [0, +\infty)$, $\mathbf{R}^- = (-\infty, 0]$, $U = (u_0, u_1, \ldots, u_{n-1}) \in \mathbf{R}^n$, $|U| = \max\{|u_0|, |u_1|, \ldots, |u_{n-1}|\}$, $\mathbf{R}_i^n = \prod_{i=0}^{n-1} (-1)^i \mathbf{R}^+$, $\mathbf{R}_+^n = \prod_{i=0}^{n-1} \mathbf{R}^+$, where

$$(-1)^i \mathbf{R}^+ = \begin{cases} \mathbf{R}^+, & i \text{ is even;} \\ \mathbf{R}^-, & i \text{ is odd.} \end{cases}$$

In last few years, many authors have studied the existence and multiplicity of positive solutions for Lidstone boundary value problem (for details, see [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13] and references therein) since it arises in many different areas of applied mathematics and physics. In particular, if $n = 2$, it describes the deformation of an elastic beam whose both ends are simply supported. Some authors

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even considered singular case (for instance, see, [1, 7, 8, 9]). The approaches used in references are mainly the monotone iterative and upper-lower solutions methods (see, [3, 11]), Leray-Schauder continuation theorem(see, [1]) and topological degree (see, [7, 8, 9, 12]), or Leggett-Williams theorem and the five functional fixed point theorem(see, [2, 4, 13]). It is required that the nonlinearity f does not depend on any derivatives of u in some references (see, for instance, [12]). Very recently, in paper [10], Ma investigated Lidstone BVP(1.1) by using global bifurcation techniques. The main result is the following:

Theorem A Suppose that

(A1) $f: [0,1]\times \mathbf{R}_i^n \to \mathbf{R}^+$ is continuous and there exist $A=(a_0,a_1,\ldots,a_{n-1})$, $B=(b_0,b_1,\ldots,b_{n-1})\in \mathbf{R}_+^{\mathbf{n}}\setminus\{(0,0,\ldots,0)\}$ such that

$$f(t,U)=\sum_{i=0}^{n-1}(-1)^i a_i u_i + o(|U|),\quad |U|\to 0,$$

$$f(t,U)=\sum_{i=0}^{n-1}(-1)^i b_i u_i + o(|U|),\quad |U|\to \infty,$$

uniformly in $t\in[0,1]$.

(A2) $f(t,U)>0$ for any $t\in[0,1]$ and $U\not\equiv 0$.

(A3) There exists $C=(c_0,c_1,\ldots,c_{n-1})\in \mathbf{R}_+^{\mathbf{n}}\setminus\{(0,0,\ldots,0)\}$ such that

$$f(t,U)\geq \sum_{i=0}^{n-1}(-1)^i c_i u_i,\quad (t,U)\in[0,1]\times \mathbf{R}_i^n.$$

(A4) $\lambda_1(B)<1<\lambda_1(A)$ or $\lambda_1(A)<1<\lambda_1(B)$, where

$$\lambda_1(A)=\frac{\pi^{2n}}{\sum\limits_{i=0}^{n-1} a_i\pi^{2i}},\quad \lambda_1(B)=\frac{\pi^{2n}}{\sum\limits_{i=0}^{n-1} b_i\pi^{2i}},$$

where $U=(u_0,u_1,\ldots,u_{n-1})\in \mathbf{R}_i^n$.

Then BVP(1.1) has at least one positive solution.

To our best knowledge, there is no paper which considers the case with sign-changing nonlinear terms by using topological degree theory. We try to fill this gap. It is showed in this paper that BVP(1.1) has at least one nontrivial solution under more general and extensive assumptions such as the nonlinear term f may be sign-changing and unbounded from below. Furthermore we extend Theorem A.

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 is devoted to the existence of positive solutions for BVP(1.1). Finally, one example is worked out to demonstrate the main results.

At the end of this Section we state the following lemmas, which will be used in Section 3.

Lemma 1.1 ([5, 6]). *Let E be a Banach space and Ω be a bounded open set in E with $\theta \in \Omega$. Suppose that $A: \bar{\Omega} \to E$ is a completely continuous operator. If*

$$Au \neq \mu u, \quad \forall u \in \partial\Omega, \ \mu \geq 1,$$

then the topological degree $deg(I - A, \ \Omega, \ \theta) = 1$.

Lemma 1.2 ([5, 6]). *Let E be a Banach space and Ω be a bounded open set in E. Suppose that $A: \bar{\Omega} \to E$ is a completely continuous operator. If there exists $u_0 \neq \theta$ such that*

$$u - Au \neq \mu u_0, \quad \forall u \in \partial\Omega, \ \mu \geq 0,$$

then the topological degree $deg(I - A, \ \Omega, \ \theta) = 0$.

Lemma 1.3 ([5, 6]). *Let E be a Banach space and $A: E \to E$ is a completely continuous operator. If $A\theta = \theta$ and A'_θ exists, in addition, 1 is not an eigenvalue of A'_θ, then there exists $r > 0$ such that the topological degree*

$$deg(I - A, \ B_r, \ \theta) = deg(I - A'_\theta, \ B_r, \ \theta) = (-1)^\beta,$$

where β is the sum of multiplicities of the eigenvalues, which are less than 1, of A'_θ.

2. PRELIMINARIES

Throughout this paper we assume $f: [0,1] \times R^n \to R$ is continuous.

We first convert BVP(1.1) into another form. Suppose $u(t)$ is a solution of BVP(1.1). Let $v(t) = (-1)^{n-1} u^{(2n-2)}(t)$. Notice that

$$\begin{cases} -[u^{(2n-4)}]''(t) = (-1)^{n-2} v(t), & t \in I; \\ u^{(2n-4)}(0) = u^{(2n-4)}(1) = 0. \end{cases}$$

So $u^{(2n-4)}(t)$ can be written as $u^{(2n-4)}(t) = (-1)^{n-2} A_1 v(t)$, where

$$A_1 v(t) = \int_0^1 G_1(t,s) v(s) ds, \quad t \in I,$$

$$G_1(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1; \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.1}$$

Similarly, notice that

$$(-1)^i u^{(2i)}(t) = \int_0^1 G_1(t,s)(-1)^{i+1} u^{(2i+2)}(s) ds, \quad i = 0, 1, \ldots, n-1.$$

So we can obtain

$$u^{(2i)}(t) = (-1)^i A_{n-i-1} v(t), \quad t \in I, \tag{2.2}$$

here

$$A_j v(t) = \int_0^1 G_j(t,s)v(s)ds, \quad j = 0,1,\ldots,n-1, \tag{2.3}$$

and

$$A_0 v(t) = v(t), \quad G_j(t,s) = \int_0^1 G_1(t,\tau)G_{j-1}(\tau,s)d\tau, \quad i = 2,3,\ldots,n. \tag{2.4}$$

Obviously, $G_j(t,s)$ is the Green's function of homogeneous boundary value problem

$$\begin{cases} u^{(2j)}(t) = 0, \quad t \in (0,1); \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0,1,\ldots,j-1. \end{cases}$$

Therefore, if $u(t)$ is a solution of BVP(1.1), we know by (2.1)–(2.4) that $v(t) = (-1)^{n-1}u^{(2n-2)}(t)$ satisfies the following BVP

$$\begin{cases} -v''(t) = f(t, A_{n-1}v(t), (-1)A_{n-2}v(t), \ldots, (-1)^{n-2}A_1 v(t), (-1)^{n-1}v(t)), \\ \qquad t \in (0,1); \\ v(0) = v(1) = 0. \end{cases} \tag{2.5}$$

Conversely, if $v(t)$ is a solution of BVP(2.5), it is not difficult to see $u(t) = A_{n-1}v(t)$ is a solution of BVP(1.1). So, we need only to study BVP(2.5).

The basic space to be used in this paper is Banach space $C(I)$, which is endowed with the maximum norm $\|\cdot\|$. Let

$$q(t) = \min\{t,\ 1-t\}, \quad t \in I. \tag{2.6}$$

From (2.1) it is easy to see

$$q(t)G_1(\tau,s) \le q(t)G_1(s,s) \le G_1(t,s) \le \min\{t(1-t),\ s(1-s)\}, \quad \forall t,s,\tau \in I, \tag{2.7}$$

where $q(t)$ is defined by (2.6).

Define an operator on $C(I)$ by

$$(Av)(t) =: \int_0^1 G_1(t,s)f(s, A_{n-1}v, (-1)A_{n-2}v, \ldots, (-1)^{n-2}A_1 v, (-1)^{n-1}v)ds. \tag{2.8}$$

Since $f:\ [0,1] \times R^n \to R$ is continuous, the operator A is well defined on $C(I)$.

On the other hand, it is well known that the solution of BVP(2.5) is equivalent to the fixed point of operator A. So, in the following, we need only to investigate the existence of fixed points of A on $C(I)$.

Notice that $f : [0,1] \times R^n \to R$ is continuous. So we have the following conclusion.

Lemma 2.1. *$A : C(I) \to C(I)$ is completely continuous.*

Lemma 2.2. *For each $v \in C(I)$, we have*

$$6^{n-1}\|A_{n-1}v\| \le 6^{n-2}\|A_{n-2}v\| \le \cdots \le 6\|A_1 v\| \le \|v\|. \tag{2.9}$$

Proof. For each $v \in C(I)$, from (2.3) and (2.7), we know

$$\|A_1 v\| \le \int_0^1 s(1-s)\|v\| ds = \frac{1}{6}\|v\|.$$

By induction, one can get (2.9) holds. □

Remark 2.3. Lemma 2.2 indicates the relationship among $v(t)$, $A_1 v(t)$, $\ldots$, $A_{n-1}v(t)$ for each $v \in C(I)$.

Lemma 2.4. *For $d = (d_0, d_1, \ldots, d_{n-1}) \in \mathbf{R}_+^{\mathbf{n}} \setminus \{(0,0,\ldots,0)\}$, define a linear integral operator*

$$L_d v(t) = \int_0^1 k_d(s,t)v(s)ds, \quad \forall t \in I, \ v \in C(I), \tag{2.10}$$

where

$$k_d(s,t) = \sum_{i=0}^{n-2} \int_0^1 G_1(s,\tau) d_i G_{n-i-1}(\tau,t)d\tau + d_{n-1}G_1(s,t). \tag{2.11}$$

Then the generalized eigenvalues of L_d are given by

$$0 < \lambda_1(L_d) < \lambda_2(L_d) < \cdots < \lambda_m(L_d) < \cdots,$$

where

$$\lambda_m(L_d) = \frac{(m\pi)^{2n}}{\sum\limits_{i=0}^{n-1} d_i (m\pi)^{2i}}, \quad m = 1, 2, 3, \ldots. \tag{2.12}$$

The generalized eigenfunction corresponding to $\lambda_m(L_d)$ is

$$\phi_m(t) = \sin(m\pi t). \tag{2.13}$$

Moreover,

$$r(L_d) = \frac{\sum\limits_{i=0}^{n-1} d_i \pi^{2i}}{\pi^{2n}}, \tag{2.14}$$

where $r(L_d)$ denotes the spectral radius of linear operator L_d.

Proof. Suppose there exist λ and $v \neq 0$ such that $v = \lambda L_d v$. Set $u(t) = A_{n-1}v(t)$. Then from (2.1)–(2.4) and (2.10)–(2.11) it is easy to see that

$$\begin{cases} (-1)^n u^{(2n)}(t) = \lambda \sum\limits_{i=0}^{n-1} (-1)^i d_i u^{(2i)}(t), \quad t \in (0,1); \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, \ldots, n-1. \end{cases}$$

This together with [10, Lemma 2] guarantees (2.12)–(2.14) hold. □

3. MAIN RESULTS

We now list the following hypotheses for convenience.

(H1) There exist a positive number r and $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbf{R}_+^{\mathbf{n}} \setminus \{(0, 0, \dots, 0)\}$ such that

$$|f(t, U)| \leq \Big|\sum_{i=0}^{n-1} (-1)^i a_i u_i\Big|, \quad \text{as } |U| \leq r \text{ uniformly in } \ t \in [0, 1],$$

and $r(L_a) < 1$, where L_a is defined as in (2.10) (replacing d with a).

(H2) There exists $b = (b_0, b_1, \dots, b_{n-1}) \in \mathbf{R}_+^{\mathbf{n}} \setminus \{(0, 0, \dots, 0)\}$ such that

$$f(t, U) \geq \sum_{i=0}^{n-1} (-1)^i b_i u_i + o(|U|), \quad \text{as } |U| \to +\infty \text{ uniformly in } \ t \in [0, 1],$$

and $r(L_b) > 1$, where L_b is defined as in (2.10) (replacing d with b).

(H3) There exists a continuous function $g : \mathbf{R}^n \to \mathbf{R}^+$ with $g(U) = o(|U|)$ as $|U| \to +\infty$ satisfying

$$f(t, U) \geq -g(U), \quad \forall U = (u_0, u_1, \dots, u_{n-1}) \in \mathbf{R}^n.$$

Theorem 3.1. *Assume that (H1)–(H3) hold. Then* $BVP(1.1)$ *has at least one non-trivial solution.*

Proof. We first prove that

$$Av \neq \mu v, \quad \forall v \in \partial B_r, \quad \mu \geq 1. \tag{3.1}$$

If this is not true, then there exist $\bar{v} \in \partial B_r$ and $\mu_1 \geq 1$ satisfying $A\bar{v} = \mu_1 \bar{v}$. Without loss of generality, assume $\mu_1 > 1$ (if $\mu_1 = 1$, then $\bar{v}$ is a fixed point of A). So, by (H1), (2.8), and Lemma 2.2, we obtain that

$$\begin{aligned} \mu_1 |\bar{v}(t)| &= |A\bar{v}(t)| \\ &\leq \int_0^1 G_1(t, s) \Big|\sum_{i=0}^{n-1} a_i A_{n-i-1} \bar{v}(s)\Big| ds \\ &\leq \int_0^1 G_1(t, s) \sum_{i=0}^{n-1} a_i A_{n-i-1} |\bar{v}(s)| ds, \quad \forall t \in I. \end{aligned} \tag{3.2}$$

Multiplying (3.2) by $\sin(\pi t)$, then integrating them from 0 to 1 and using Lemma 2.3 we have

$$\begin{aligned} \mu_1 \int_0^1 |\bar{v}(t)| \sin(\pi t) dt &\leq \int_0^1 \sin(\pi t) dt \int_0^1 G_1(t, s) \sum_{i=0}^{n-1} a_i A_{n-i-1} |\bar{v}(s)| ds \\ &\leq \int_0^1 \sin(\pi t) dt \int_0^1 k_a(t, \tau) |\bar{v}(\tau)| d\tau \\ &\leq \int_0^1 |\bar{v}(\tau)| d\tau \int_0^1 k_a(t, \tau) \sin(\pi t) dt \end{aligned}$$

$$\le \int_0^1 |\bar{v}(\tau)| L_a \sin(\pi\tau) d\tau$$
$$\le r(L_a) \int_0^1 |\bar{v}(\tau)| \sin(\pi\tau) d\tau,$$

in contradiction with $\mu_1 > 1$. Therefore, (3.1) holds. This together with Lemma 1.1 guarantees that

$$deg(I - A, B_r, \theta) = 1. \tag{3.3}$$

Next, choose a positive number ε' with $\varepsilon' \le \frac{1}{4}[r(L_b) - 1]$. By condition (H2), there exists a positive number R' with $R' > r$ such that

$$f(t, U) \ge \sum_{i=0}^{n-1} (-1)^i b_i u_i - \varepsilon'|U|, \quad \text{as } |U| \ge R' \text{ uniformly in } \ t \in [0, 1]. \tag{3.4}$$

Let

$$h(t) = \sup\{|f(t, U) - \sum_{i=0}^{n-1} (-1)^i b_i u_i| : \ |U| \le R'\}, \quad t \in [0, 1].$$

This together with (3.4) guarantees that

$$f(t, U) \ge \sum_{i=0}^{n-1} (-1)^i b_i u_i - \varepsilon'|U| - h(t), \quad \forall U \in \mathbf{R}^n, \ t \in [0, 1]. \tag{3.5}$$

Let ε'' be a positive number with $\varepsilon'' \le \frac{1}{8}$. From condition (H3) we know there exists $M_1 > 0$ such that

$$g(U) \le \varepsilon''|U| + M_1, \quad \forall U \in \mathbf{R}^n. \tag{3.6}$$

Choose a positive number R satisfying

$$R > \frac{2}{r(L_b) - 1} \Big[2M_1 r(L_b) + \int_0^1 q(s)h(s)ds \Big]. \tag{3.7}$$

We now prove

$$v - Av \ne \mu \sin(\pi t), \quad \forall v \in C(I), \quad \|v\| = R, \quad \mu \ge 0. \tag{3.8}$$

Suppose, on the contrary, there exist $v_0 \in C(I)$, $\|v_0\| = R$ and $\mu_0 \ge 0$ such that

$$v_0 - Av_0 = \mu_0 \sin(\pi t). \tag{3.9}$$

Let

$$w_0(t) = \int_0^1 G_1(t, s) g(\bar{V}(s)) ds,$$

where

$$\bar{V}(s) = (A_{n-1}v_0(s), (-1)A_{n-2}v_0(s), \dots, (-1)^{n-2}A_1 v_0(s), (-1)^{n-1}v_0(s)), \quad \forall s \in I.$$

By (3.6) and Lemma 2.2 we can get

$$
\begin{aligned}
w_0(t) &= \int_0^1 G_1(t,s)g(\bar{V}(s))ds \\
&\le \varepsilon'' \int_0^1 G_1(t,s)\|\bar{V}(s)\|ds + M_1 \int_0^1 G_1(t,s)ds \\
&\le \frac{M_1 + \varepsilon'' R}{2} t(1-t).
\end{aligned} \tag{3.10}
$$

Therefore, from (2.7) and

$$
\frac{\sin(\pi t)}{\pi^2} = \int_0^1 G_1(t,s)\sin(\pi s)ds
$$

we have

$$
v_0(t) + w_0(t) = \int_0^1 G_1(t,s)\Big[f(s,\bar{V}(s)) + g(\bar{V}(s))\Big]ds + \mu_0 \sin(\pi t) \ge q(t)\|v_0 + w_0\|.
$$

This together with (3.5)–(3.7) and (3.9)–(3.10) guarantees that

$$
\begin{aligned}
&\int_0^1 Av_0(t)\sin(\pi t)dt - \int_0^1 v_0(t)\sin(\pi t)dt \\
&= \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)f(s,\bar{V}(s))ds - \int_0^1 v_0(t)\sin(\pi t)dt \\
&\ge \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)\sum_{i=0}^{n-1} b_i A_{n-i-1}v_0(s)ds \\
&\quad - \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds - \int_0^1 v_0(t)\sin(\pi t)dt \\
&\ge \int_0^1 \sin(\pi t)dt \int_0^1 k_b(t,\tau)v_0(\tau)d\tau - \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds \\
&\quad - \int_0^1 v_0(t)\sin(\pi t)dt \\
&= \int_0^1 v_0(\tau)d\tau \int_0^1 k_b(t,\tau)\sin(\pi t)dt - \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds \\
&\quad - \int_0^1 v_0(t)\sin(\pi t)dt \\
&\ge \int_0^1 v_0(\tau)L_b \sin(\pi\tau)d\tau - \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds \\
&\quad - \int_0^1 v_0(t)\sin(\pi t)dt \\
&= [r(L_b)-1]\int_0^1 v_0(\tau)\sin(\pi\tau)d\tau - \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds \\
&= [r(L_b)-1]\int_0^1 \Big[v_0(\tau)+w_0(\tau)\Big]\sin(\pi\tau)d\tau - [r(L_b)-1]\int_0^1 w_0(\tau)\sin(\pi\tau)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds \\
&\geq [r(L_b)-1]\|v_0+w_0\| \int_0^1 q(\tau)\sin(\pi\tau)d\tau - [r(L_b)-1]\int_0^1 w_0(\tau)\sin(\pi\tau)d\tau \\
&- \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds \\
&\geq [r(L_b)-1][\|v_0\|-\|w_0\|] \int_0^1 q(\tau)\sin(\pi\tau)d\tau - [r(L_b)-1]\int_0^1 w_0(\tau)\sin(\pi\tau)d\tau \\
&- \int_0^1 \sin(\pi t)dt \int_0^1 G_1(t,s)[h(s)+\varepsilon' R]ds \\
&\geq [r(L_b)-1][R-(M_1+\varepsilon'' R)] - [r(L_b)-1](M_1+\varepsilon'' R) - \varepsilon' R - \int_0^1 q(s)h(s)ds \\
&= [r(L_b)-1][R-2(M_1+\varepsilon'' R)] - \varepsilon' R - \int_0^1 q(s)h(s)ds \\
&> 0.
\end{aligned}
$$

On the other hand, from (3.9) we have

$$\int_0^1 Av_0(t)\sin(\pi t)dt - \int_0^1 v_0(t)\sin(\pi t)dt = -\mu_0 \int_0^1 \sin^2(\pi t)dt \leq 0.$$

This is a contradiction. So (3.8) holds. Combining this with Lemma 1.2 we obtain that

$$deg(I-A, B_R, \theta) = 0. \tag{3.11}$$

From the additivity of topological degree, (3.3), and (3.11), it follows that

$$deg(I-A, B_R \setminus \bar{B}_r, \theta) = deg(I_A, B_R, \theta) - deg(I_A, B_r, \theta) = -1.$$

Then by the solution property of topological degree, A has at least one fixed point on $B_R \setminus \bar{B}_r$, which means that BVP(1.1) has at least one nontrivial solution. □

Corollary 3.2. *Suppose conditions (H1)–(H2) hold. In addition, if there exists a constant $l \geq 0$ such that*

$$f(t, u_0, u_1, \ldots, u_{n-1}) \geq -8l, \quad \text{if } (-1)^{n-1}u_{n-1} \geq -l. \tag{3.12}$$

Then BVP(1.1) has at least one nontrivial solution.

Proof. Define

$$\bar{f}(t, u_0, u_1, \ldots, u_{n-1}) = \begin{cases} f(t, u_0, u_1, \ldots, u_{n-1}), & (-1)^{n-1}u_{n-1} \geq -l; \\ f(t, u_0, u_1, \ldots, (-1)^n l), & (-1)^{n-1}u_{n-1} \leq -l, \end{cases} \tag{3.13}$$

and for $\forall v \in C(I)$, let

$$\bar{A}v(t) = \int_0^1 G_1(t,s)\bar{f}(s, A_{n-1}v, (-1)A_{n-2}v, \ldots, (-1)^{n-2}A_1 v, (-1)^{n-1}v)ds.$$

By Theorem 3.1 we know that $\bar{A}$ has at least one nonzero fixed point $\tilde{v}$. Then by (3.12)–(3.13) we have

$$\begin{aligned}\tilde{v}(t) &= \int_0^1 G_1(t,s)\bar{f}(s, A_{n-1}\tilde{v}, (-1)A_{n-2}\tilde{v}, \dots, (-1)^{n-2}A_1\tilde{v}, (-1)^{n-1}\tilde{v})ds \\ &\geq -8l\int_0^1 G_1(t,s)ds \geq -l.\end{aligned}$$

This together with (3.13) guarantees that

$$\begin{aligned}\bar{f}(s, A_{n-1}\tilde{v},&(-1)A_{n-2}\tilde{v}, \dots, (-1)^{n-2}A_1\tilde{v}, (-1)^{n-1}\tilde{v}) \\ &= f(s, A_{n-1}\tilde{v}, (-1)A_{n-2}\tilde{v}, \dots, (-1)^{n-2}A_1\tilde{v}, (-1)^{n-1}\tilde{v}).\end{aligned}$$

Therefore,

$$\tilde{v}(t) = \int_0^1 G_1(t,s)f(s, A_{n-1}\tilde{v}, (-1)A_{n-2}\tilde{v}, \dots, (-1)^{n-2}A_1\tilde{v}, (-1)^{n-1}\tilde{v})ds,$$

which implies that $\tilde{v}(t)$ is a nontrivial solution of BVP(1.1). □

Theorem 3.3. *Suppose (H2)–(H3) are satisfied. In addition, suppose that*

$$f(t,U) = \sum_{i=0}^{n-1}(-1)^i a_i u_i + o(|U|), \quad \text{as } |U| \to 0 \text{ uniformly in } \ t \in [0,1], \tag{3.14}$$

and

$$\lambda_m(L_a) = \frac{(m\pi)^{2n}}{\sum\limits_{i=0}^{n-1} a_i (m\pi)^{2i}} \neq 1, \quad m = 1, 2, 3, \dots. \tag{3.15}$$

Then BVP(1.1) *has at least one nontrivial solution.*

Proof. By (2.8) and (3.14) we know

$$A'_\theta v(t) = \int_0^1 k_d(s,t)v(s)ds = L_d v(t).$$

Then from (3.15) it follows that 1 is not the eigenvalue of A'_θ. By Lemma 1.3, there exists $r_1 > 0$ such that

$$deg(I - A,\ B_{r_1},\ \theta) = deg(I - A'_\theta,\ B_{r_1},\ \theta) = (-1)^\beta = \pm 1, \tag{3.16}$$

where β is the sum of multiplicities of the eigenvalues, which are less than 1, of A'_θ.

Similar to the proof of Theorem 3.1, by (H2)-(H3), there exists $R > 0$ such that (3.11) holds. This together with (3.16) and the additivity of topological degree guarantees that

$$deg(I - A, B_R \setminus \bar{B}_{r_1}, \theta) = deg(I_A, B_R, \theta) - deg(I_A,\ B_{r_1}, \theta) = \mp 1.$$

Then A has at least one fixed point on $B_R \setminus \bar{B}_{r_1}$, which means that BVP(1.1) has at least one nontrivial solution. □

Similar to the Corollary 3.2, we have the following corollary.

Corollary 3.4. *Suppose (H2),* (3.12)*, and* (3.14)–(3.15) *hold. Then* BVP(1.1) *has at least one nontrivial solution.*

Remark 3.5. As immediate consequences of above results, we can give some results on the following BVP when the nonlinear term does not depend on higher derivatives:

$$\begin{cases} (-1)^n u^{(2n)}(t) = f(t, u(t)), & t \in (0,1); \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, i = 0, 1, \ldots, n-1. \end{cases}$$

Example 3.6. Consider the following Lidstone BVP:

$$\begin{cases} -u^{(6)}(t) = \Big| a_0(U(t))u(t) - a_1(U(t))u''(t) + a_2(U(t))u^{(4)}(t) \Big| \\ \qquad\qquad -(1+t^2)a_3(U(t)), \quad t \in (0,1); \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, 2, \end{cases} \tag{3.17}$$

where $a_0(U)$, $a_1(U)$, $a_2(U)$, and $a_3(U)$ are continuous functions given by the following:

$$a_0(U) = \begin{cases} 10, & |U| \le 1; \\ \pi^6, & |U| \ge 2; \end{cases} \qquad a_1(U) = \begin{cases} 5, & |U| \le 1; \\ 10, & |U| \ge 2; \end{cases} \tag{3.18}$$

$$a_2(U) = \begin{cases} 1, & |U| \le 1; \\ 2, & |U| \ge 2; \end{cases} \qquad a_3(U) = \begin{cases} 0, & |U| \le 1; \\ \sqrt{|U|}, & |U| \ge 2; \end{cases} \tag{3.19}$$

where

$$U(t) = \Big(u(t), u''(t), u^{(4)}(t)\Big), \quad |U(t)| = \max\{|u(t)|,\ |u''(t)|,\ |u^{(4)}(t)|\}.$$

Conclusion BVP(3.17) has at least one nontrivial solution.

Proof. In fact, BVP(3.17) can be regarded as the form of BVP(1.1), where $n = 3$, and

$$f(t, u_0, u_1, u_2) = \Big| a_0(U)u_0 - a_1(U)u_1 + a_2(U)u_2 \Big| - (1+t^2)a_3(U), \tag{3.20}$$

$a_0(U), a_1(U), a_2(U)$, and $a_3(U)$ are defined as in (3.18) and (3.19).

Choose $a = (10, 5, 1)$. Then by (2.14) it is easy to see $r(L_a) < 1$. Therefore, condition (H1) holds.

Let $b = (\pi^6, 10, 2)$, $g(U) = 2\sqrt{|U|}$. Then $r(L_b) > 1$. From (3.20) we can get that (H2)–(H3) hold.

Consequently, Theorem 3.1 guarantees that BVP(3.17) has at least one nontrivial solution. □

Remark 3.7. From (3.20), one can find that f is unbounded from below.

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