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CONVERGENCE CRITERIA FOR A CLASS OF SECOND-ORDER DIFFERENCE EQUATIONS

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ABSTRACT. This paper studies a class of nonlinear second order difference equations of the type

$$x_{n+1} = \frac{f(x_n, x_{n-1})}{x_n},$$

where f is symmetric and monotonic with initial conditions x_{-1}, x_0 being positive real numbers. Some sufficient conditions under which every positive solution of such equation converges to a period two solution or to the cycle $\{0, \infty\}$ are established.

Keywords: Difference equation, equilibrium, period two solution, convergence, symmetric functions, coordinate-wise monotonicity.

1. INTRODUCTION

Nonlinear difference equations appear as discrete analogues and as numerical solutions of differential and delay differential equations. Such equations also have direct applications in the modeling of diverse phenomena in various sciences such as, biology (see [3, 4]), ecology (see [9]), economics (see [8, 11]), medical sciences (see [13]), military sciences (see [6, 14]). Our objective in the study of difference equations is to understand as much as possible about the asymptotic behavior (such as stability, boundedness, convergence, etc.) of solutions without the knowledge of an explicit formula for the solutions.

In this paper, we study the global behavior of the nonlinear second order difference equation

$$x_{n+1} = \frac{f(x_n, x_{n-1})}{x_n}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where f is symmetric and monotonic and the initial values x_{-1}, x_0 are positive real numbers. Thus f belongs to the class of coordinate-wise monotonic functions, i.e., functions which are monotonic in each coordinate. For more about coordinate-wise

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monotonicity, see [14]. Most of the results about Eq. (1) treat the case when f is a rational function. For second order rational difference equations, see monograph [10]. For higher order rational difference equations, see [2, 12]. For nonlinear difference equations see [15, 16].

A solution $\bar{x}$ of the equation $f(x,x) = x^2$ is a fixed point (or equilibrium) of Eq. (1). Also the sequence

$$\alpha, \beta, \alpha, \beta, \alpha, \beta, \ldots$$

where $\alpha\beta = f(\alpha, \beta)$ is a period two solution of Eq. (1). In this note, we obtain sufficient conditions under which every positive solution of Eq. (1) converges to a period two solution and study the boundedness and convergence of this equation.

2. CONVERGENCE TO A PERIOD TWO SOLUTION AND DIVERGENCE TO INFINITY

Lemma 1. *Assume that the function* $f : (0,\infty)^2 \to (0,\infty)$ *is differentiable. Then*

(i): f *satisfies the following functional equation*

$$f(x,y) - f(y,z) = (x - z)g(x,y,z) \tag{2}$$

where $g : (0,\infty)^3 \to [0,\infty)$ *and* g *is continuous if and only if*

(a): f *is symmetric, i.e.,* $f(x,y) = f(y,x)$,

(b): f *is increasing in both arguments.*

(ii): f *satisfies Eq.* (2) *where* $g : (0,\infty)^3 \to (-\infty, 0]$ *and* g *is continuous if and only if* (a) *together with the following hypothesis hold*

(c): f *is decreasing in both arguments*

Proof. We only give the proof of (i). The argument for (ii) is similar and will be omitted. Suppose that Eq. (2) holds and g is a continuous and nonnegative-valued function. In Eq. (2) let $x = z$, which yields $f(x,y) = f(y,x)$, so f is symmetric and (a) holds. Next Assume that $(a,b) \in (0,\infty)^2$. By the continuity of g we have

$$\frac{\partial f}{\partial x}(a,b) = \lim_{x\to a} \frac{f(x,b) - f(a,b)}{x - a} = \lim_{x\to a} g(x,b,a) = g(a,b,a) \geq 0$$

so f is increasing relative to the first argument and since f is symmetric it is increasing relative to the second argument too.

Now assume that conditions (a) and (b) hold. Define the function

$$g(x,y,z) = \begin{cases} \dfrac{f(x,y) - f(z,y)}{x - z} & x \neq z \\ \dfrac{\partial f}{\partial x}(x,y) & x = z \end{cases}$$

so by (a) Eq. (2) holds. Also by the differentiability of f, g is continuous and by (b), g is a nonnegative-valued function. The proof is complete.

Lemma 2. *Assume that $f \in C^2$ and condition (a) in Lemma 1 holds. Then*

(i): g is increasing (decreasing) in the first and third arguments in $(0,\infty)^3$ if and only if for every $(x,y) \in (0,\infty)^2$, $\frac{\partial^2 f}{\partial x^2}(x,y) \geq 0$ ($\frac{\partial^2 f}{\partial x^2}(x,y) \leq 0$).

(ii): g is increasing (decreasing) in the second argument in $(0,\infty)^3$ if and only if for every $(x,y) \in (0,\infty)^2$, $\frac{\partial^2 f}{\partial x \partial y}(x,y) \geq 0$ ($\frac{\partial^2 f}{\partial x \partial y}(x,y) \leq 0$).

Proof. Using Lemma 1 we have

$$g(x,y,z) = \begin{cases} \frac{f(x,y)-f(z,y)}{x-z} & x \neq z \\ \frac{\partial f}{\partial x}(x,y) & x = z \end{cases}$$

Now suppose $x \neq z$. Then using mean value theorem there exist η_{xyz} between x and z and ξ_{xyz} between η_{xyz} and x so that

$$\begin{aligned} g_x(x,y,z) &= \frac{f_x(x,y)(x-z) - f(x,y) + f(y,z)}{(x-z)^2} = \frac{f_x(x,y)(x-z) + (z-x)f_x(\eta_{xyz},y)}{(x-z)^2} \\ &= \frac{f_x(x,y) - f_x(\eta_{xyz},y)}{x-z} = \frac{(x-\eta_{xyz})f_{xx}(\xi_{xyz},y)}{(x-z)} \end{aligned}$$

and for $x = z$, $g_x(x,y,z) = f_{xx}(x,y)$. Therefore, g is increasing (decreasing) in the first argument if and only if $f_{xx}(x,y) \geq 0$ ($f_{xx}(x,y) \leq 0$) for every $(x,y) \in (0,\infty)^2$. Since g is symmetric relative to the first and third arguments similar argument holds for the third argument too.

Again assume $x \neq z$. Then using mean value theorem there exists η_{xyz} between x and z so that

$$g_y(x,y,z) = \frac{f_x(y,x) - f_x(y,z)}{x-z} = f_{xy}(y,\eta_{xyz})$$

and for $x = z$, $g_y(x,y,z) = f_{xy}(x,y)$. Therefore, g is increasing (decreasing) in the second argument if and only if $f_{xy}(x,y) \geq 0$ ($f_{xy}(x,y) \leq 0$) for every $(x,y) \in (0,\infty)^2$. The proof is complete.

Theorem 1. *Assume that $f : (0,\infty)^2 \to (0,\infty)$ is differentiable and hypothesis (a) and (b) in lemma 1 hold and the sequence $\{x_n\}$ of positive values satisfies Eq. (1). Let $\rho = f(x_0, x_{-1}) - x_0 x_{-1}$, $l = \min\{x_{-1}, x_0\}$ and define*

$$S = \{(x,y,z,w) |\ l \leq z \leq x, l \leq w \leq y\}, \quad A = \{(x,y,z) |\ l \leq x,y,z, \quad z \leq x\}$$

Then

(i): If $\rho \leq 0$ then the sequence $\{x_n\}$ converges to a period 2 solution.

(ii): If $\rho > 0$ and there exists $0 < U < 1$ such that for $(x,y,z,w) \in S$

$$\limsup \frac{g(x,y,z)g(y,z,w)}{xy} < U$$

as $(x,y,z,w) \to (\mu,\infty,\mu,\infty)$ for all $\mu > 0$ and as $(x,y,z,w) \to (\infty,\infty,\infty,\infty)$, then the sequence $\{x_n\}$ converges to a period 2 solution.

(iii): If $\rho > 0$ and there exists $L > 1$ such that for $(x, y, z, w) \in S$

$$\liminf \frac{g(x,y,z)g(y,z,w)}{xy} > L \qquad as\ (x,y,z,w) \to (\mu,\infty,\mu,\infty) \quad \forall \mu > 0$$

and

$$\frac{f_x(\mu,\eta)f_x(\eta,\mu)}{\mu\eta} > L, \quad \forall \mu, \eta > 0$$

then the sequence $\{x_n\}$ diverges to infinity.

(iv): If $\rho > 0$ and there exists $L > 1$ such that for $(x, y, z) \in A$

$$\liminf \frac{g(x,y,z)}{x} > 0 \qquad as\ (x,y,z) \to (\infty,\mu,\infty) \quad \forall \mu > 0$$

and for all $x, y > 0$

$$\frac{f_x(x,y)}{x} > L$$

then the sequence $\{x_n\}$ diverges to infinity.

Proof. Subtracting x_{n-1} from the left and right hand sides of Eq. (1), we obtain

$$x_{n+1} - x_{n-1} = \frac{f(x_n, x_{n-1}) - x_n x_{n-1}}{x_n}.$$

From the fact that

$$x_n x_{n-1} = f(x_{n-1}, x_{n-2}),$$

and by (2), we have for $n \geq 1$ that

$$x_{n+1} - x_{n-1} = \frac{f(x_n, x_{n-1}) - f(x_{n-1}, x_{n-2})}{x_n} = \frac{(x_n - x_{n-2})g(x_n, x_{n-1}, x_{n-2})}{x_n}. \tag{3}$$

So the signum of $x_n - x_{n-2}$ is invariant for all $n \geq 1$ since g is positive-valued (by Lemma 1).

Now assume that $\rho \leq 0$. Then both of subsequences of even and odd terms are decreasing. Hence, the sequence $\{x_n\}$ converges to a period 2 solution and (i) is verified.

Next, assume that $\rho > 0$. thus, both of subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are strictly increasing by Lemma 1(note that by Lemma 1 $g(x, y, z) > 0$ if $x \neq z$). Also note that by (3) and the fact that $\rho > 0$, $x_n \geq l$ for all $n \geq -1$. Therefore $(x_{2n}, x_{2n-1}, x_{2n-2}, x_{2n-3}) \in S$ for all $n \geq 1$.

Suppose that the hypothesis in (ii) hold. We claim that the sequence $\{x_n\}$ converges to a period 2 solution. Assume for the sake of contradiction that this is not true. So, either both of subsequences of even and odd terms diverges to infinity or, one of them is convergent and the other one is divergent. Without loss of generality one can assume that either $x_{2n+1} \to \infty$ and $x_{2n} \to \infty$ or, $x_{2n+1} \to \infty$ and $x_{2n} \to \mu > 0$. Thus, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\frac{g(x_{2n}, x_{2n-1}, x_{2n-2})g(x_{2n-1}, x_{2n-2}, x_{2n-3})}{x_{2n}x_{2n-1}} < (U + \epsilon) \tag{4}$$

Therefore by (4) and (3) one can write for $n \geq N$ that

$$\begin{aligned} x_{2n+1} - x_{2n-1} &= (x_{2n-1} - x_{2n-3})\frac{g(x_{2n}, x_{2n-1}, x_{2n-2})g(x_{2n-1}, x_{2n-2}, x_{2n-3})}{x_{2n}x_{2n-1}} \\ &\leq (x_{2n-1} - x_{2n-3})(\epsilon + U), \end{aligned}$$

and therefore by induction one can write for $n \geq N$ that

$$x_{2n+1} - x_{2n-1} < (U + \epsilon)^{n-N+1}(x_{2N-1} - x_{2N-3})$$

Choose $0 < \epsilon < 1 - U$. Thus for all $n \geq N$

$$\begin{aligned} x_{2n+1} &< (x_{2n+1} - x_{2n-1}) + (x_{2n-1} - x_{2n-3}) + \ldots + (x_{2N-1} - x_{2N-3}) + x_{2N-3} \\ &\leq (x_{2N-1} - x_{2N-3}) \sum_{i=0}^{n-N+1} (U + \epsilon)^i + x_{2N-3} \\ &< (x_{2N-1} - x_{2N-3}) \sum_{i=0}^{\infty} (U + \epsilon)^i + x_{2N-3} \\ &= (x_{2N-1} - x_{2N-3}).\frac{1}{1 - (U + \epsilon)} + x_{2N-3} \end{aligned}$$

which contradicts the fact that the subsequence $\{x_{2n+1}\}$ is unbounded. One can prove (iii) with an analysis similar to that of (ii). Just note that by the continuity of g

$$\liminf_{(x,y,z,w)\to(\mu,\eta,\mu,\eta)} \frac{g(x, y, z)g(y, z, w)}{xy} = \frac{f_x(\mu, \eta)f_x(\eta, \mu)}{\mu\eta}$$

Now assume that the hypothesis in (iv) are held. For the sake of contradiction assume that the sequence $\{x_n\}$ does not diverge to infinity. then there are two possible cases to consider:

Case I: One of subsequences of even and odd terms is convergent and the other one is divergent. Without loss of generality assume that $x_{2n+1} \to \infty, x_{2n} \to \mu > 0$. Now, we claim that

$$\liminf_{n\to\infty}(x_{2n+1} - x_{2n-1}) = 0$$

otherwise

$$\liminf_{n\to\infty}(x_{2n+2} - x_{2n}) \geq \liminf_{n\to\infty}(x_{2n+1} - x_{2n-1}). \liminf_{n\to\infty} \frac{g(x_{2n+1}, x_{2n}, x_{2n-1})}{x_{2n+1}} > 0$$

which simply is a contradiction. Thus there exists a subsequence $\{x_{2n_k+1} - x_{2n_k-1}\}$ of the sequence $\{x_{2n+1} - x_{2n-1}\}$ such that $\lim_{k\to\infty} x_{2n_k+1} - x_{2n_k-1} = 0$. Now, using mean value theorem we have

$$\frac{g(x_{2n_k+1}, x_{2n_k}, x_{2n_k-1})}{x_{2n_k+1}} = \frac{f_x(\xi_k, x_{2n_k})}{x_{2n_k+1}}$$

where $x_{2n_k-1} < \xi_k < x_{2n_k+1}$. Since $\lim_{k\to\infty}(x_{2n_k+1} - \xi_k) = 0$ then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $x_{2n_k+1} - \xi_k < \epsilon$. Thus for $k \geq N$

$$\frac{g(x_{2n_k+1}, x_{2n_k}, x_{2n_k-1})}{x_{2n_k+1}} > \frac{f_x(\xi_k, x_{2n_k})}{\xi_k + \epsilon} = \frac{f_x(\xi_k, x_{2n_k})}{\xi_k}.\frac{\xi_k}{\xi_k+\epsilon} > L.\frac{\xi_k}{\xi_k+\epsilon}.$$

Similarly, one can write

$$\frac{g(x_{2n_k}, x_{2n_k-1}, x_{2n_k-2})}{x_{2n_k}} = \frac{f_x(\delta_k, x_{2n_k-1})}{x_{2n_k}} = \frac{f_x(\delta_k, x_{2n_k-1})}{\delta_k}.\frac{\delta_k}{x_{2n_k}} > L.\frac{\delta_k}{x_{2n_k}},$$

where $x_{2_k-2} < \delta_k < x_{2n_k}$. So for $k \geq N$

$$x_{2n_k+2} - x_{2n_k} = (x_{2n_k} - x_{2n_k-2}).\frac{g(x_{2n_k+1}, x_{2n_k}, x_{2n_k-1})}{x_{2n_k+1}}.\frac{g(x_{2n_k}, x_{2n_k-1}, x_{2n_k-2})}{x_{2n_k}} \tag{5}$$
$$> (x_{2n_k} - x_{2n_k-2}).L^2.\frac{\xi_k}{\xi_k+\epsilon}.\frac{\delta_k}{x_{2n_k}}.$$

Note that the sequence $\{\xi_k\}$ diverges increasingly to infinity and $\lim_{k\to\infty}(x_{2n_k} - \delta_k) = 0$. Therefore,

$$\lim_{k\to\infty} \frac{\xi_k}{\xi_k+\epsilon}.\frac{\delta_k}{x_{2n_k}} = 1$$

This together with the fact that $L > 1$ and (5) implies that the subsequence $\{x_{2n_k}\}$ is unbounded which is a contradiction.

Case II: Both of subsequences of even and odd terms are convergent. Assume that $x_{2n} \to \mu, x_{2n+1} \to \eta$. Then by the continuity of g

$$\lim_{n\to\infty} \frac{g(x_{n+1}, x_n, x_{n-1})g(x_n, x_{n-1}, x_{n-2})}{x_{n+1}x_n} = \frac{f_x(\mu,\eta).f_x(\eta,\mu)}{\mu\eta} > L^2$$

which implies that the sequence $\{x_n\}$ is unbounded, a contradiction. The proof is complete.

Corollary 1. *Assume that hypothesis in Theorem* 1 *are held. Assume also that* $f \in C^2$ *and for every* $(x, y) \in (0,\infty)^2$ *both of partial derivatives* $\frac{\partial^2 f}{\partial x^2}$ *and* $\frac{\partial^2 f}{\partial x \partial y}$ *are positive or negative. Then section (ii) in Theorem* 1 *could be replaced by the following more simpler section: If* $\rho > 0$ *and there exists* $0 < U < 1$ *such that*

$$\limsup \frac{f_x(x,y).f_x(y,x)}{xy} < U \quad \text{as } (x,y) \to (\mu,\infty) \quad \forall \mu > 0$$

then the sequence $\{x_n\}$ *converges to a period* 2 *solution.*

Proof. Let $(x, y, z, w) \in S$. If both of partial derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are positive then by Lemma 2 g is increasing in all of it's arguments and we have

$$\frac{g(x,y,z)g(y,z,w)}{xy} \leq \frac{f(x,y,x)g(y,x,y)}{xy} = \frac{f_x(x,y)f_x(y,x)}{xy}$$

Next, if both of partial derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are negative then by Lemma 2 g is decreasing in all of it's arguments and we have

$$\frac{g(x,y,z)g(y,z,w)}{xy} \leq \frac{g(z,w,z)g(w,z,w)}{zw} = \frac{f_x(z,w)f_x(w,z)}{zw}$$

which completes our proof.

Example 1. Let $f(x,y) = \frac{axy}{x+y}$ which defines the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{x_n + x_{n-1}} \tag{6}$$

Eq. (6) was investigated in [10] where it was shown that every solution of Eq. (6) converges to a period two solution.

Clearly conditions (a) and (b) in lemma 1 are held and

$$g(x,y,z) = \frac{ay^2}{(x+y)(y+z)}.$$

Therefore

(i). If $\rho = \frac{ax_0x_{-1}}{x_0+x_{-1}} - x_0x_{-1} \leq 0$, i.e., $x_0 + x_{-1} \leq a$ then by Theorem 1(i) both of subsequences of even and odd terms are convergent decreasingly. So Eq. (6) converges to a period two solution.

(ii). If $\rho > 0$, i.e., $x_0 + x_{-1} > a$ then since $g(x,y,z)g(y,x,w)/xy \to 0$ as $(x,y,z,w) \to (\mu,\infty,\mu,\infty)$ or $(\infty,\infty,\infty,\infty)$ for every $\mu > 0$, by Theorem 1(ii) both of subsequences of even and odd terms are convergent increasingly. Therefore, Eq. (6) converges to a period two solution again.

Example 2. Consider the difference equation

$$x_{n+1} = a\frac{x_n^2 + x_{n-1}^2}{x_n}, \quad a > 0, \quad a \neq \frac{1}{2}. \tag{7}$$

which is obtained from Eq. (1) by setting $f(x,y) = a(x^2+y^2)$. Conditions (a) and (b) in lemma 1 are simply held and

$$g(x,y,z) = a(x+z).$$

Thus

(i). If $\rho = a(x_{-1}^2 + x_0^2) - x_{-1}x_0 \leq 0$ then using Theorem 1(i) both of subsequences of even and odd terms are convergent decreasingly and therefore Eq. (7) converges to a period two solution.

(ii). If $\rho > 0$ and $a < \frac{1}{2}$ then for points $(x,y,z) \in S$

$$\frac{g(x,y,z)g(y,z,w)}{xy} = a^2\frac{x+z}{x}.\frac{y+w}{y} < 4a^2 < 1$$

thus by Theorem 1(ii) both of subsequences of even and odd terms are convergent increasingly. Hence, Eq. (7) again converges to a period two solution. This result could be obtained by Corollary 1 since $f_{xx}(x,y) = 2a \geq 0, f_{xy}(x,y) = 0$. Note that

$$\frac{f_x(x,y)f_x(y,x)}{xy} = \frac{(2ax)(2ay)}{xy} = 4a^2 < 1$$

So if $a < \frac{1}{2}$ then by Corollary 1 similar result obtains.

(iii). $\rho > 0$ and $a > \frac{1}{2}$. Note that $\liminf g(x,y,z)/x = a(x+z)/x = a + a \liminf z/x > 0$ as $(x,y,z) \to (\infty,\mu,\infty)$ for every $\mu > 0$ and for points $(x,y,z) \in A$

$$\frac{f_x(x,y)}{x} = \frac{2ax}{x} = 2a > 1$$

thus by Theorem 1(iv) Eq. (7) diverges to infinity.

Example 3. Consider the difference equation

$$x_{n+1} = \frac{a(x_n + x_{n-1}) + b(\sin x_n + \sin x_{n-1}) + c}{x_n}, \quad a,b,c > 0, \quad a \geq b \tag{8}$$

which is obtained from Eq. (2) by setting $f(x,y) = a(x+y) + b(\sin x + \sin y + c)$. Conditions (a) and (b) in lemma 1 are simply held and

$$g(x,y,z) = \begin{cases} a + b\frac{\sin x - \sin z}{x-z} & x \neq z \\ a + b\cos x & x = z \end{cases}$$

Thus

(i). If $\rho = a(x_{-1}+x_0)+b(\sin x_{-1}+\sin x_0)+c-x_{-1}x_0 \leq 0$ then by part (i) in Theorem 1 both of subsequences of even and odd terms are convergent decreasingly. Hence, Eq. (8) converges to a period two solution.

(ii). If $\rho > 0$ since $g(x,y,z)g(y,x,w)/xy \to 0$ as $(x,y,z,w) \to (\mu,\infty,\mu,\infty)$ or $(\infty,\infty,\infty,\infty)$ for every $\mu > 0$ then by part (ii) in Theorem 1 both of subsequences of even and odd terms are convergent increasingly. Hence, Eq. (8) again converges to a period two solution.

Example 4. Set $f(x,y) = \sqrt{ax^2 + bxy + ay^2}$ which satisfies conditions (a) and (b) in Lemma 1. So we have the following difference equation

$$x_{n+1} = \frac{\sqrt{ax_n^2 + bx_nx_{n-1} + ax_{n-1}^2}}{x_n}, \quad a,b > 0 \tag{9}$$

with

$$g(x,y,z) = \frac{ax + by + az}{\sqrt{ax^2 + bxy + ay^2} + \sqrt{ay^2 + byz + az^2}}.$$

Therefore

(i). If $\rho = \sqrt{ax_{-1}^2 + bx_{-1}x_0 + ax_0^2} - x_{-1}x_0 \leq 0$ then by Theorem 1(i) Eq. (9) converges to a period two solution with decreasing subsequences of even and odd terms.

(*ii*) If $\rho > 0$ then $g(x,y,z)g(y,z,w)/xy \to 0$ as $(x,y,z,w) \to (\mu,\infty,\mu,\infty)$ or $(\infty,\infty,\infty,\infty)$ for every $\mu > 0$. So, by Theorem 1(*ii*) Eq. (9) converges to a period two solution with increasing subsequences of even and odd terms.

Example 5. Let $f(x,y) = \exp x + \exp y$ and note that conditions (a) and (b) in Lemma 1 are simply satisfied. Thus the following difference equation is obtained by Eq. (1)

$$x_{n+1} = \frac{\exp x_n + \exp x_{n-1}}{x_n} \tag{10}$$

with

$$g(x,y,z) = \begin{cases} \frac{\exp x - \exp z}{x-z} & x \neq z \\ \exp x. & x = z \end{cases}$$

Note that $\rho = \exp x_{-1} + \exp x_0 - x_{-1}x_0 > 0$ for every $x_{-1}, x_0 > 0$. Also for points $(x,y,z) \in A$

$$\frac{g(x,y,z)}{x} = \frac{\exp x - \exp z}{x(x-z)} = \frac{\sum_{i=1}^{\infty} \frac{(x^i - z^i)}{i!}}{x(x-z)} > \frac{(x^3 - z^3)}{3!x(x-z)} = \frac{x^2 + xz + z^2}{3!x}$$

thus $\frac{g(x,y,z)}{x} \to \infty$ as $(x,y,z) \to (\infty,\mu,\infty)$. Also for all $x, y > 0$

$$\frac{f_x(x,y)}{x} = \frac{\exp x}{x} \geq \exp 1 > 1$$

Therefore, by Theorem 1(*iv*) every positive solution of Eq. (10) diverges to infinity.

Lemma 3. *Assume that the function* $f : (0,\infty)^2 \to (0,\infty)$ *is differentiable and conditions* (a) *and* (c) *in Lemma 1 hold. Assume also that the sequence* $\{x_n\}$ *of positive numbers satisfies Eq. (1). Let* $\rho = f(x_0, x_{-1}) - x_0 x_{-1}$. *Then*

- *(i): If* $\rho > 0$, *then the subsequence of odd terms is increasing and another one is decreasing.*
- *(ii): If* $\rho < 0$, *then the subsequence of even terms is increasing and another one is decreasing.*
- *(iii): The solution* $\{x_n\}$ *either converges to a period two solution or converges to* $\{0,\infty\}$.

Proof. By Lemma 1, g is a negative-valued function. Using this and Eq. (3) parts (i) and (ii) are easily proved. It remains to verify (iii). If $\rho = 0$ then using Eq. (3), Eq. (1)converges to the period two solution (x_{-1}, x_0). Now, assume that $\rho \neq 0$. Thus one of cases (i) and (ii) occurs and in both of them one of subsequences of even and odd terms is increasing and another one is decreasing. Without loss of generality assume that $\rho > 0$. Thus, four cases are possible, i.e., $(x_{2n+1}, x_{2n}) \to (p,q)$, $p < \infty, q > 0$ or, $(x_{2n+1}, x_{2n}) \to (\infty, 0)$ or, $(x_{2n+1}, x_{2n}) \to (p, 0)$, $p < \infty$ or, $(x_{2n+1}, x_{2n}) \to (\infty, q)$, $q > 0$. We show that only the first two cases occurs and this completes our proof.

Now if $(x_{2n+1}, x_{2n}) \to (p, 0)$, $p < \infty$ then

$$\lim_{n\to\infty} f(x_{2n}, x_{2n-1}) = \lim_{n\to\infty} x_{2n+1}x_{2n} = 0,$$

using the fact that $x_{2n-1} < p$ and $x_{2n} > 0$ for all $n \in \mathbb{N}$ and by the fact that f is decreasing we have for all $x > p, y > 0$ that

$$f(x, y) < \lim_{n\to\infty} f(x_{2n}, x_{2n-1}) = 0$$

which simply is a contradiction. Next suppose $(x_{2n+1}, x_{2n}) \to (\infty, q)$, $q > 0$. Thus

$$\lim_{n\to\infty} f(x_{2n}, x_{2n-1}) = \lim_{n\to\infty} x_{2n+1}x_{2n} = \infty$$

again using the fact that $q < x_{2n}$ for all $n \in \mathbb{N}$ and by the fact that f is decreasing we have for all $x < q, y > 0$ that

$$f(x, y) > \lim_{n\to\infty} f(x_{2n}, x_{2n-1}) = \lim_{n\to\infty} x_{2n+1}x_{2n} = \infty$$

which again is a contradiction. The proof is complete.

Theorem 2. *Suppose $f : (0,\infty)^2 \to (0,\infty)$ is differentiable and conditions (a) and (c) in Lemma 1 hold. Assume also that the sequence $\{x_n\}$ of positive numbers satisfies Eq. (1). Let $\rho = f(x_0, x_{-1}) - x_0x_{-1}$. Then*

(i): If $\rho = 0$ then the sequence $\{x_n\}$ converges to the period two point (x_{-1}, x_0).

(ii): If there exists $L > 1$ such that for every period two points (p, q) of Eq. (1)

$$\frac{f_x(p, q)f_x(q, p)}{pq} > L$$

then the subsequence of odd terms diverges increasingly to ∞ and the other subsequence converges decreasingly to 0 when $\rho > 0$ and the subsequence of even terms diverges increasingly to ∞ and the other subsequence converges decreasingly to 0 when $\rho < 0$.

(iii): If there exists $0 < U < 1$ such that for every $(x, y, z, w) \in \mathcal{C}$

$$\limsup \frac{g(x, y, z)g(y, z, w)}{xy} < U \qquad \text{as } (x, y, z, w) \to (0, \infty, 0, \infty)$$

then the sequence $\{x_n\}$ converges to a period two solution with increasing odd terms and decreasing even terms when $\rho > 0$ and increasing even terms and decreasing odd terms when $\rho < 0$ where

$$\mathcal{C} = \{(x, y, z, w) \mid xy = f(y, z),\ yz = f(z, w), \quad 0 < x \le z, \quad w \le y\}$$

(iv): Assume that $f \in C^2$. If for every $(x, y) \in (0,\infty)^2$, $\frac{\partial^2 f}{\partial x^2}(x, y) \ge 0, \frac{\partial^2 f}{\partial x \partial y}(x, y) \ge 0$ and also there exists $0 < U < 1$ such that for period two points (p, q)

$$\limsup \frac{f_x(p, q)f_x(q, p)}{pq} < U \qquad \text{as } (p, q) \to (0, \infty)$$

then the sequence $\{x_n\}$ converges to a period two solution with increasing odd terms and decreasing even terms when $\rho > 0$ and increasing even terms and decreasing odd terms when $\rho < 0$.

Proof. (i) is easily proved using Eq. (3). So assume that $\rho \neq 0$. Then using Lemma 2 there are only two possible scenarios: either the sequence $\{x_n\}$ converges to a period two solution or, it converges to $\{0, \infty\}$. Now if hypothesis in (ii) hold then the first scenario is not possible. Otherwise, suppose that $(x_{2n}, x_{2n-1}) \to (p, q)$. Then using Eq. (3)

$$x_{n+1} - x_{n-1} = (x_{n-1} - x_{n-3}) \frac{g(x_n, x_{n-1}, x_{n-2}) g(x_{n-1}, x_{n-2}, x_{n-3})}{x_n x_{n-1}} \tag{11}$$

also by the continuity of g

$$\lim_{n\to\infty} \frac{g(x_n, x_{n-1}, x_{n-2}) g(x_{n-1}, x_{n-2}, x_{n-3})}{x_n x_{n-1}} = \frac{f_x(p, q) f_x(q, p)}{pq} > L$$

this together the fact that $L > 1$ and Eq. (11) implies that $\{x_n\}$ is unbounded which is a contradiction. Therefore, $\{x_n\}$ converges to $\{0, \infty\}$.

Next assume that hypothesis in (iii) hold. We claim that the second scenario does'nt occur. For the sake of contradiction, and without loss of generality suppose that $\{x_{2n}\}$ converges decreasingly to 0 and $\{x_{2n+1}\}$ diverges increasingly to ∞(Note that this occurs when $\rho > 0$. The proof for the case $\rho < 0$ is similar and will be omitted). Therefore, $(x_{2n}, x_{2n-1}, x_{2n-2}, x_{2n-3}) \in \mathcal{C}$ for all $n \geq 1$ and Eq. (3) yields

$$x_{2n+1} - x_{2n-1} = (x_{2n-1} - x_{2n-3}) \frac{g(x_{2n}, x_{2n-1}, x_{2n-2}) g(x_{2n-1}, x_{2n-2}, x_{2n-3})}{x_{2n} x_{2n-1}}$$

this together with the fact that $0 < U < 1$ and an analysis precisely similar to that of Theorem 1(ii) implies that $\{x_{2n+1}\}$ is bounded, a contradiction.

Finally, assume that hypothesis in (iv) hold. Define

$$M = \left\{ \frac{g(x, y, z) g(y, z, w)}{xy} \mid (x, y, z, w) \in \mathcal{C} \right\}.$$

Now assume that $(x, y, z, w) \in \mathcal{C}$. Since $\frac{\partial^2 f}{\partial x^2}(x, y) \geq 0, \frac{\partial^2 f}{\partial x \partial y}(x, y) \geq 0$ in $(0, \infty)^2$ then by Lemma 2 g is increasing in all of it's arguments in $(0, \infty)^3$. This fact and also the fact that g is a negative-valued function yield

$$\frac{g(x, y, z) g(y, z, w)}{xy} \leq \frac{g(x, w, x) g(w, x, w)}{xw} = \frac{f_x(x, w) f_x(w, x)}{xw}$$

this means that the maximum point for the set M occurs when $x = z$ and $w = y$. Also for such a point we have $xw = f(w, x)$, i.e., (x, w) is a period two point. Therefore

$$\sup M \leq \sup T \tag{12}$$

where

$$T = \left\{ \frac{f_x(x, w) f_x(w, x)}{xw} \mid x, w > 0, (x, w) \text{ is a period 2 point } \right\}$$

Now we claim that the sequence $\{x_n\}$ converges to a period two point. For the sake of contradiction, and without loss of generality assume that $\{x_{2n}\}$ converges decreasingly to 0 and $\{x_{2n+1}\}$ diverges increasingly to ∞. (Again note that this occurs when $\rho > 0$. The proof for the case $\rho < 0$ is similar and will be omitted). Thus $(x_{2n}, x_{2n-1}, x_{2n-2}, x_{2n-3}) \in \mathcal{C}$ for all $n \geq 2$ and by (11) one can write

$$\limsup_{n\to\infty} \frac{g(x_{2n}, x_{2n-1}, x_{2n-2})g(x_{2n-1}, x_{2n-2}, x_{2n-3})}{x_{2n}x_{2n-1}} \leq \limsup_{(x,w)\to(0,\infty)} T \leq U$$

this together with the fact that $0 < U < 1$ and an analysis precisely similar to that of Theorem 1(ii) implies that the subsequence $\{x_{2n+1}\}$ is bounded which simply is a contradiction. The proof is complete.

Example 6. Consider the following difference equation

$$x_{n+1} = \frac{1}{x_n^{s+1}} + \frac{1}{x_n x_{n-1}^s} \tag{13}$$

which is obtained from Eq. (1) with

$$f(x,y) = \frac{1}{x^s} + \frac{1}{y^s}, \quad s \geq 0$$

conditions (a) and (c) in Lemma 1 are simply held. Also $f_{xx}(x,y) = \frac{s(s+1)}{x^{s+2}} \geq 0, f_{xy}(x,y) = 0$. Now assume that (p,q) is a period two point for Eq. (13). Then $pq = \frac{1}{p^s} + \frac{1}{q^s}$ and we have

$$\frac{f_x(p,q)f_x(q,p)}{pq} = \frac{s^2}{(pq)^{s+2}} = \frac{s^2}{(\frac{1}{p^s} + \frac{1}{q^s})^{s+2}} \to 0 \quad \text{as } (p,q) \to (0,\infty)$$

So all the conditions in Theorem 2(iv) are held. Therefore, Eq. (13) converges to a period two solution.

Example 7. Let $f(x,y) = \frac{1}{(xy)^s}$, $s \geq 0$, $s \neq 1$. Note that this function satisfies conditions (a) and (c) in Lemma 1 and defines the following difference equation

$$x_{n+1} = \frac{1}{x_n^{s+1}x_{n-1}^s} \tag{14}$$

assume that (p,q) is a period 2 solution of Eq. (13). Then $pq = 1$ and we can write

$$\frac{f_x(p,q)f_x(q,p)}{pq} = \frac{s^2}{(pq)^{s+2}} = s^2,$$

Now if $\rho = \frac{1-(x_0x_{-1})^{s+1}}{x_0^{s+1}x_{-1}^s} = 0$, i.e., $x_0x_{-1} = 1$ then by Theorem 2(i) Eq. (14) converges to the period 2 point (x_0, x_{-1}). Next if $\rho \neq 0$, $s > 1$ then by Theorem 2(ii) Eq. (14) converges to $\{0,\infty\}$, i.e., one of subsequences of even and odd terms converges to 0 and the other one diverges to ∞, based on ρ is positive or negative.

Finally if $\rho \neq 0$, $s < 1$ then since $f_{xx}(x,y) = \frac{s(s+1)}{x^{s+2}y^s} \geq 0, f_{xy}(x,y) = \frac{s^2}{(xy)^{s+1}} \geq 0$ by Theorem 2(iv) Eq. (14) converges to a period 2 solution.

3. CONCLUSION

In this paper, we have studied a class of nonlinear second-order difference equations which cover a very wide class of difference equations. We have obtained sufficient conditions under which every positive solution of this equation converges to a period two solution. Also we have discussed the boundedness, permanence, and convergence properties of the solutions. In the end, we have verified the obtained results through some comprehensive examples. The authors truly believe that these results can be conveniently extended to the following higher order difference equation

$$x_{n+1} = \frac{f(x_n, \ldots, x_{n-m})}{x_n}, \quad m > 1.$$

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