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OSCILLATIONS OF SCALAR NEUTRAL IMPULSIVE DIFFERENTIAL EQUATIONS OF THE FIRST ORDER WITH VARIABLE COEFFICIENTS

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ABSTRACT. The theory of oscillations of neutral impulsive differential equations is gradually occupying a central place among the theories of oscillations of impulsive differential equations. This could be due to the fact that neutral impulsive differential equations play fundamental roles in the present drive to further develop information technology. Indeed, neutral differential equations appear in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits). In this paper, we generalize and prove the results of oscillations of neutral delay differential equations with constant coefficients obtained by Gyori and Ladas for impulsive differential equations.

Key words and phrases: Oscillatory and non-oscillatory conditions; Neutral delay impulsive differential equations with variable coefficients.

1. PRELIMINARIES

In ordinary differential equations, the solutions are continuously differentiable sometimes at least once, whereas the impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play an important role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differential equations are fundamentally different including the definitions of some of the basic terms. In this section, we examine some of these changes.

In effect, the solution $x(t)$ for $t \in [t_0, T)$ of a given impulsive differential equation or its first derivative $x'(t)$ is a piece-wise continuous function with points of discontinuity $t_k \in [t_0, T)$, $t_k \neq t$, $0 \leq k < \infty$. Consequently, in order to simplify the statements of our assertions later, we introduce the set of functions PC and PC^r which are defined as follows:

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Let $r \in N$, $D := [T, \infty) \subset R$ and let the set $S := \{t_k\}_{k=N}$ be fixed. Except stated otherwise, we will assume that the elements of S are moments of impulse effect and satisfy the property:

C1.1 $0 < t_1 < t_2 < \cdots$ and $\lim_{k \to +\infty} t_k = +\infty$.

We denote by $PC(D, R)$ the set of all functions $\varphi : D \to R$, which are continuous for all $t \in D$, $t \notin S$. They are continuous from the left and have discontinuity of the first kind at the points for which $t \in S$, while by $PC^r(D, R)$, we denote the set of functions $\varphi : D \to R$ having derivative $\frac{d^j \varphi}{dt^j} \in PC(D, R)$, $0 \le j \le r$ ([1]; [4]).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols $PC(D, R; S)$ and $PC^r(D, R; S), r \in N$.

Definition 1.1. A solution x of an impulsive differential equation is said to be

(i) finally positive, if there exists $T \ge 0$ such that $x(t)$ is defined for $t \ge T$ and $x(t) > 0$ for all $t \ge T$;
(ii) finally negative, if there exists $T \ge 0$ such that $x(t)$ is defined for $t \ge T$ and $x(t) < 0$ for all $t \ge T$;
(iii) non-oscillatory, if it is either finally positive or finally negative;
(iv) oscillatory, if it is neither finally positive nor finally negative; and
(v) regular, if it is defined in some half line $[T_x, \infty)$ for some $T_x \in R$ and

$$\sup\{|x(t)| : t \ge T\} > 0, \forall T > T_x \quad ([4]).$$

The following are some basic lemmas and theorems essential in carrying out our investigations. They are extracts from the works by Gyori and Ladas ([3]) and Bainov and Simeonov ([1]).

Lemma 1.1 (Lemma 1.5.1 by Gyori and Ladas [3]). *Let f, g: $[t_0, \infty) \to R$ be such that*

$$f(t) = g(t) + pg(t - \tau), t \ge t_0 + \max\{0, \tau\}, \tag{1.1}$$

where p, $\tau \in R$ and $p \ne -1$. Assume further, that

$$\lim_{t \to \infty} f(t) = L \in R$$

exists. Then the following statements hold:

(i) *If $\lim_{t \to \infty} \inf g(t) \equiv a \in R$, then $L = (1 + p)a$.*
(ii) *If $\lim_{t \to \infty} \sup g(t) \equiv b \in R$, then $L = (1 + p)b$.*
(iii) *If $g(t)$ is bounded and $p \ne -1$, then $\lim t \to \infty g(t) = \frac{L}{1+p}$.*

Lemma 1.2 (Lemma 1.5.2 by Gyori and Ladas [3]). *Let F, G, P: $[t_0, \infty) \to R$ and $c \in R$ be such that*

$$F(t) = G(t) + P(t)G(t - c), \quad t \ge t_0 + \max\{0, c\}. \tag{1.2}$$

Assume that there exist numbers P_1, P_2, P_3, $P_4 \in R$ such that $P(t)$ is in one of the following ranges:

(j) $P_1 \le P(t) \le 0$
(jj) $0 \le P(t) \le P_2 < 1$
(jjj) $1 < P_3 \le P(t) \le P_4$.

Suppose that $G(t) > 0$ for $t \ge t_0$, $\lim_{t\to\infty} \inf G(t) = 0$ and that $\lim_{t\to\infty} F(t) \equiv L \in R$ exists. Then, $L = 0$.

Consider the linear impulsive differential equation with delay ([1])

$$\begin{cases} x'(t) + p(t)x(t-\tau) = 0, & t \notin S \\ \Delta x(t_k) + p_k x(t_k - \tau) = 0, & \forall t_k \in S \end{cases} \tag{1.3}$$

together with the corresponding inequalities

$$\begin{cases} x'(t) + p(t)x(t-\tau) \le 0, & t \notin S \\ \Delta x(t_k) + p_k x(t_k - \tau) \le 0, & \forall t_k \in S \end{cases} \tag{1.4}$$

and

$$\begin{cases} x'(t) + p(t)x(t-\tau) \ge 0, & t \notin S \\ \Delta x(t_k) + p_k x(t_k - \tau) \ge 0, & \forall t_k \in S. \end{cases} \tag{1.5}$$

Let the following condition be fulfilled:
C1.2 $p \in PC(R_+, R)$ and $\tau \ge 0$.

Theorem 1.1 ([1]). *Assume that condition* C1.2 *is satisfied and let there exist a sequence of disjoint intervals $J_n = [\zeta_n, \eta_n)$ with $\eta_n - \zeta_n = 2\tau$, such that:*

(a) *For each $n \in N$, $t \in J_n$ and $t_k \in J_n$*

$$p(t) \ge 0, \quad p_k \ge 0. \tag{1.6}$$

(b) *There exists $v_1 \in N$ such that for $n \ge v_1$*

$$\int_{\eta_n - \tau}^{\eta_n} p(s)ds + \sum_{\eta_n - \tau \le t_k < \eta_n} p_k \ge 1. \tag{1.7}$$

Then

(i) *The inequality* (1.4) *has no finally positive solution.*
(ii) *The inequality* (1.5) *has no finally negative solution.*
(iii) *Each regular solution of equation* (1.3) *is oscillatory.*

Next, consider the linear impulsive differential equation with advanced argument

$$\begin{cases} x'(t) - p(t)x(t+\tau) = 0, & t \notin S \\ \Delta x(t_k) - p_k x(t_k+\tau) = 0, & \forall t_k \in S \end{cases} \tag{1.8}$$

together with the corresponding inequalities

$$\begin{cases} x'(t) - p(t)x(t+\tau) \le 0, & t \notin S \\ \Delta x(t_k) - p_k x(t_k+\tau) \le 0, & \forall t_k \in S \end{cases} \tag{1.9}$$

and

$$\begin{cases} x'(t) - p(t)x(t+\tau) \ge 0, & t \notin S \\ \Delta x(t_k) - p_k x(t_k+\tau) \ge 0, & \forall t_k \in S. \end{cases} \tag{1.10}$$

The following result is valid:

Theorem 1.2 ([1]). *Let condition* C1.2 *be fulfilled and let there exist a sequence of disjoint intervals $J_n = [\zeta_n, \eta_n)$ with $\eta_n - \zeta_n = 2\tau$, such that:*

(a) *For each $n \in N$, $t \in J_n$ and $t_k \in J_n$, condition (1.6) holds.*
(b) *There exists $v_1 \in N$ such that for $n \ge v_1$*

$$\int_{\zeta_n}^{\zeta_n+\tau} p(s)ds + \sum_{\zeta_n < t_k \le \zeta_n+\tau} p_k \ge 1. \tag{1.11}$$

Then

(i) *The inequality* (1.9) *has no finally positive solution.*
(ii) *The inequality* (1.10) *has no finally negative solution.*
(iii) *Each regular solution of equation* (1.8) *is oscillatory.*

Theorem 1.3 ([1]). *Let condition* C1.2 *be fulfilled and let there exist a sequence of disjoint intervals $J_n = [\zeta_n, \eta_n)$ with $\eta_n - \zeta_n \ge 2\tau$, such that:*

(a) *For each $n \in N$, $t \in J_n$ and $t_k \in J_n$, condition* (1.6) *holds.*
(b) *There exist constants $b \ge 0$, $q \ge 0$ and $v_1 > 0$ such that for $n \ge v_1$ and $t \in [\zeta_n, \eta_n - \tau)$ the following inequalities hold:*

$$\int_{t}^{t+\tau} p(s)ds \ge b, \qquad \sum_{t < t_k < t+\tau} p_k \ge q, \tag{1.12}$$

$$be + q > 1. \tag{1.13}$$

(c) *There exist constants $\delta > 0$ and $v_2 \in N$ such that for each $n \ge v_2$ there exists $t_n^* \in [\zeta_n, \zeta_n + \tau)$ such that*

$$\tilde{B}_n(t_n^*)\tilde{C}_n(t_n^*) \ge \delta, \tag{1.14}$$

where

$$\tilde{B}_n(t_n^*) = \int_{\zeta_n}^{t_n^*} p(s)ds + \sum_{\zeta_n < t_k \leq t_n^*} p_k$$

and

$$\tilde{C}_n(t_n^*) = \int_{t_n^*}^{\eta_n+\tau} p(s)ds + \sum_{t_n^* < t_k \leq \eta_n+\tau} p_k.$$

(d) $\lim_{n\to+\infty} = +\infty.$

Then

(i) *The inequality* (1.9) *has no finally positive solution.*

(ii) *The inequality* (1.10) *has no finally negative solution.*

(iii) *Each regular solution of equation* (1.8) *is oscillatory.*

2. AUXILIARY RESULTS

Now we consider the neutral delay impulsive differential system with variable coefficients

$$\begin{cases} [x(t) + P(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0, & t \notin S \\ \Delta[x(t_k) + P(t_k)x(t_k-\tau)] + Q_k x(t_k-\sigma) = 0, & \forall t_k \in S, \end{cases} \tag{2.1}$$

where

$$P \in PC^1(R_+, R), \quad Q \in PC(R_+, R_+), \quad Q_k \geq 0, \quad \tau \geq 0, \quad \sigma \geq 0 \tag{2.2}$$

that is, a differential equation together with its impulsive conditions in which the first-order derivative of the unknown function appears in the equation both with and without delay. The oscillations in neutral delay and neutral delay impulsive equations with constant coefficients have been studied by many researchers ([5]; [6]; [7]; [2]).

Before proceeding, we establish the following lemmas, which are not only interesting, but will also be useful in discussing the main results.

Lemma 2.1. *Assume that*

C2.1 $$\int_{t_0>0}^{\infty} x(s)ds \to \infty \Rightarrow \int_{t_0>0}^{\infty} Q(s)x(s-\sigma)ds = \infty$$

for any $x \in PC(R_+, R_+)$ *and* $\forall \sigma \geq 0$.

Let $x(t)$ *be a finally positive solution of equation* (2.1) *and set*

$$z(t) = x(t) + P(t)x(t-\tau). \tag{2.3}$$

Then the following statements are true:

(a) $z(t)$ is a finally non-increasing function;
(b) If $P(t) < -1$, then $z(t)$ is finally negative;
(c) If $-1 \leq P(t) \leq 0$, then $z(t) > 0$ and $\lim_{t\to+\infty} z(t) = 0$.

Definition 2.1. The solution $x(t)$ is said to be

(i) finally non-increasing if $t_1 < t_2$ implies $x(t_1) \geq x(t_2)$ for $t_1, t_2 > T$ and $T > 0$ and
(ii) finally non-decreasing if $t_1 < t_2$ implies $x(t_1) \leq x(t_2)$ for $t_1, t_2 > T$ and $T > 0$.

Proof of Lemma 2.1

(a) We have

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma) \leq 0, & t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma) \leq 0, & \forall t_k \in S \end{cases} \tag{2.4}$$

and so $z(t)$ is a finally non-increasing function.

(b)(i) Assume, on the contrary, that $z(t) > 0 \; \forall t \geq T_0$. But then,

$$\neg\text{“}z(t) < 0\text{”} \Rightarrow z(t) \geq 0.$$

If however, $z(t) = 0$, then by condition C2.1, equation (2.4) and the fact that $x(t) > 0, \forall t \geq T_0 \Rightarrow z(t) < 0 \; \forall t \geq T_1$. Hence,

$$\neg\text{“}z(t) < 0\text{”} \Leftrightarrow z(t) > 0, \forall t \geq T_0.$$

(ii) Let us start with the statement

$$x(t) \geq -P(t)x(t-\tau) \geq x(t-\tau). \tag{2.5}$$

We show that $x(t) \geq \beta > 0$ for $[t_k - \tau,\ t_k]$. Also, we show that the statement holds for $[t_\ell,\ t_{\ell+1}]$. Since $x(t) > 0$ for all continuity points $(t_\ell,\ t_{\ell+1}]$, only $\lim_{t\to t_\ell+0} x(t) = 0$ can contradict our statement. Actually, if $\lim_{t\to t_\ell+0} x(t) = 0$, then by (2.5), $\lim_{t\to t_\ell+0} x(t-\tau) = 0$ also. Then, $\lim_{t\to t_\ell+0} z(t) = 0$ follows and from equation (2.4), z fulfils the initial condition in $(t_\ell,\ t_{\ell+1}]$, that is,

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma) \\ z(t_\ell) = 0 \end{cases}$$

hence

$$0 = z(t_\ell) \geq z(s), \quad s \in (t_\ell,\ t_{\ell+1}]$$

which contradicts the hypothesis that $z(t) > 0$, $t_0 \leq t < \infty$. Therefore $\lim_{t\to t_\ell+0} x(t) > 0$. Consequently,

$$\min_{t_\ell < t \leq t_{\ell+1}} x(t) > \beta_\ell > 0.$$

Hence

$$\min_{t_{k-\tau}\le t\le t_k} x(t) \min_{t_{k-\tau}\le t_\ell\le t_k} \min_{t_\ell\le t\le t_{\ell+1}} x(t) = \min_{t_{k-\tau}\le t_\ell\le t_k} \beta_\ell = \beta > 0.$$

Thus, $x(t)$ is bounded from below by a positive constant on the sequence $t + k\tau$, $0 \le k < \infty$. Therefore from equation (2.4), we see that

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma) \le -Q(t)\beta, & t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma) \le -Q_k\beta, & \forall t_k \in S \end{cases} \tag{2.6}$$

which, in view of condition C2.1, implies

$$\lim_{t\to+\infty} z(t) = -\beta\left(\int_{t_0}^{+\infty} Q(t)dt + \sum_{k=1}^{\infty} Q_k\right) = -\infty.$$

This is a contradiction and so completes the proof of Lemma 2.1(b). Notice, in the last equation, that the condition

$$\int_{t_0}^{+\infty} Q(t)dt + \sum_{k=1}^{\infty} Q_k = \infty$$

constitutes a special case of C2.1.

(c) Let us claim inversely that $z(t) < 0$. We recall that

$$\neg\text{“}z(t) > 0\text{”} \Leftrightarrow z(t) \le 0.$$

Hence, reasoning like in b(i) above, $z(t) < 0$ for $t > T_0$. Thus, $x(t) \le x(t-\tau)$, hence $x(t)$ is a bounded function and so also is $z(t)$. Since $z(t) < 0$,

$$\lim_{t\to+\infty} z(t) = L < 0.$$

Hence

$$\int_{t_0}^{\infty} z(s)ds = -\infty.$$

On the other hand,

(2.7)
$$\begin{aligned}
&\int_{t_0}^{\infty} (x(s) + P(s)x(s-\tau))\, ds = \lim_{T\to\infty} \int_{t_0}^{T} (x(s) + P(s)x(s-\tau))\, ds \\
&= \lim_{T\to\infty} \left(\int_{t_0}^{T} x(s)ds + \int_{t_0-\tau}^{T-\tau} P(s+\tau)x(s)ds \right) \\
&= \lim_{T\to\infty} \left(\int_{t_0}^{T-\tau} x(s)ds + \int_{t_0}^{T-\tau} P(s+\tau)x(s)ds + \int_{T-\tau}^{T} x(s)ds + \int_{t_0-\tau}^{t_0} P(s+\tau)x(s)ds \right) \\
&= \int_{t_0}^{\infty} x(s)\left(1 + P(s+\tau)\right) ds + \lim_{T\to\infty} \int_{T-\tau}^{T} x(s)ds + \int_{t_0-\tau}^{t_0} P(s+\tau)x(s)ds.
\end{aligned}$$

But the component

$$\int_{t_0}^{\infty} x(s)\left(1 + P(s+\tau)\right) ds + \lim_{T\to\infty} \int_{T-\tau}^{T} x(s)ds \geq 0$$

and

$$\int_{t_0-\tau}^{t_0} P(s+\tau)x(s)ds \leq 0,$$

meaning that (2.7) cannot tend to -∞. This is a contradiction, therefore $z(t) \nrightarrow L < 0$ which implies that $z(t) \nless 0$.

Hence we have established that $z(t) > 0$, $t \geq T_0$ and that $z(t) \to L \geq 0$. Clearly, if $L > 0$, then $\int_{t_0}^{\infty} z(s)ds = \infty$, hence $\int_{t_0}^{\infty} x(s)ds = \infty$. Thus, by condition C2.1, $z(t) \to 0$ and this completes the proof of Lemma 2.1.

Now consider the neutral equation

$$\begin{cases} [x(t) + px(t-\tau)]' + Q(t)x(t-\sigma) = 0, & t \notin S \\ \Delta[x(t_k) + px(t_k-\tau)] + Q_k x(t_k-\sigma) = 0, & \forall t_k \in S, \end{cases} \tag{2.8}$$

where $t_k \in R$ and $1 \leq k < \infty$. We introduce the following conditions:
C2.2 There exist nonnegative integers m_1 and m_2 such that

$$t_{k+m_1} = t_k + \tau,\ t_{k+m_2} = t_k + \sigma, \quad k \in N.$$

Lemma 2.2. *Let us assume that conditions* C2.1 *and* C2.2 *hold. We further assume that $p \neq -1$ in equation* (2.8) *and that*

$$Q \in PC(R_+, R_+), \quad Q_k \geq 0, \quad \tau \geq 0, \quad \sigma \geq 0. \tag{2.9}$$

Let $x(t)$ be a finally positive solution of equation (2.8) and set

$$z(t) = x(t) + px(t-\tau).$$

Then

(a) *$z(t)$ is a finally non-increasing function and either*

$$\lim_{t\to+\infty} z(t) = -\infty \tag{2.10}$$

or

$$\lim_{t\to+\infty} z(t) = 0^+. \tag{2.11}$$

(b) *The following statements are equivalent:*
   (i) (2.10) *holds;*
   (ii) $p < -1$*;*
   (iii) $\lim\limits_{t\to+\infty} x(t) = +\infty$.

(c) *The following statements are equivalent:*
   (j) (2.11) *holds;*
   (ii) $p > -1$*;*
   (iii) $\lim\limits_{t\to+\infty} x(t) = 0^+$.

Proof

(a) From equation (2.8), bearing inequalities (2.9) in mind, we obtain.

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma), & t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma), & \forall t_k \in S \end{cases} \tag{2.12}$$

which implies z is a non-increasing function. It converges therefore, either to $-\infty$ or to a number L, where $-\infty < L < +\infty$, for $t \to \infty$.

If z converges to $-\infty$, then the proof of (a) is complete. Otherwise, if $z(t)$ converges to L as $t \to \infty$, then

$$\int_{t_0}^{\infty} z(t)dt = \pm\infty \text{ (depending on whether } L > 0 \text{ or } L \le 0).$$

Again, if $L = 0$, the proof of (a) is complete. Otherwise, we integrate both sides of equation (2.12) from t to ∞ for sufficiently large t, to obtain

$$L - z(t_0) = -\int_{t_0}^{+\infty} Q(t)x(t-\sigma)dt - \sum_{t_k > t_0} Q_k x(t_k - \sigma). \tag{2.13}$$

Observe that by selection in Lemma 2.2, $Q \geq 0$; x is finally positive and what is more, $Q_k \geq 0$ by the condition of Lemma 2.2. Therefore,

$$\int_{t_0}^{+\infty} Q(t)x(t-\sigma)dt + \sum_{t_k > t_0} Q_k x(t_k - \sigma) < \infty$$
$$\Rightarrow \int_{t_0}^{+\infty} Q(t)x(t-\sigma)dt < \infty.$$

Hence, by contraposition of C2.1,

$$\int_{t_0}^{\infty} z(s)ds < \infty.$$

Notice the modification

$$\int_{t_0}^{+\infty} x(s)ds + \sum_{t_k \geq t_0} x(t_k) = \infty \Rightarrow \int_{t_0}^{+\infty} Q(s)x(s-\sigma)dt + \sum_{t_k \geq t_0} Q_k x(t_k - \sigma) = \infty, \quad \forall \sigma \geq 0$$

of condition C2.1 or equivalently,

$$\int_{t_0}^{+\infty} Q(t)x(t-\sigma)dt + \sum_{t_k \geq t_0} Q_k x(t_k - \sigma) < \infty \Rightarrow \int_{t_0}^{+\infty} x(s)ds + \sum_{t_k \geq t_0} x(t_k) < \infty,$$

where in this case, $\sigma \geq 0$ is assumed to exist. Statement (2.13) contradicts the hypothesis that $\int_{t_0}^{+\infty} z(t)dt = \infty$. Hence $L = 0$ and this completes the proof of (a).

(b) Let (i) hold, that is, condition (2.10) is fulfilled. We are to prove that

(i) $\Rightarrow$ (ii) By definition,

$$z(t) = x(t) + px(t-\tau).$$

Both $x(t)$ and $x(t-\tau)$ are positive functions, meaning that the above expression can be negative only if $p < 0$. Consequently, $z(t) \to -\infty$ only if $x(t)$ is unbounded.

We show that there exists $T_0 \in R$ such that

$$z(T_0^+) < 0 \quad \text{and} \quad x(T_0^+) \geq \sup_{t \leq T_0} x(t).$$

Let us assume conversely that such T_0 does not exist. Then

$$x(T_0^+) < \sup_{t \leq T_0} x(t) \quad \forall T_0 \in R.$$

Consequently, $\exists \varepsilon > 0$ such that $\forall s$, $T_0 < s < T_0 + \varepsilon$, $x(s) \geq \sup_{t \leq T_0} x(t)$. Hence

$$\sup \left\{ s : x(s) < \sup_{t \leq T_0} x(t) \right\} = T \in R$$

must exist, otherwise $x(s)$ is bounded contrary to our earlier assertion.

But then for T,

$$\sup_{t\le T_0} x(t) \le x(T)$$

holds. With this $T_0 := T$, we obtain the inequality

$$0 > z(T_0^+) = x(T_0^+) + p\,x(T_0 - \tau^+) \ge x(T_0^+)(1+p).$$

This is only possible if $p < -1$, since $x(T_0^+) > 0$.

(ii) $\Rightarrow$ (iii) Let $p < -1$. Also, let us assume that z is finally positive. Then z is decreasing and $z \to 0$, by Lemma 2.1. If

$$0 < z(t) = x(t) + p\,x(t-\tau)$$

then

$$x(t) > (-p)\,x(t-\tau). \tag{2.14}$$

On the other hand, by $z(t) \to 0$ and

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma), & t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma), & \forall t_k \in S, \end{cases}$$

$$0 - z(t) = -\int_t^{+\infty} Q(s)x(s-\sigma)ds - \sum_{t\le t_k} Q_k x(t_k - \sigma) > -\infty.$$

Hence, by condition C2.1,

$$\int_{t-\sigma}^{\infty} x(s)ds < \infty, \quad \sum_{t\le t_k} x(t_k - \sigma) < \infty. \tag{2.15}$$

Consequently, inequality (2.14) brings contradiction since

$$x(t_k + i\sigma) > (-p)^i x(t_k - \sigma), \quad 1 \le i < \infty$$

would have led to infinity in (2.15). Hence z cannot be finally positive. Thus, by Lemma 2.1, $z \to -\infty$ if $t \to \infty$. Therefore, there exists T_0 such that $z(s) < 0$ if $s > T_0$.

Since

$$z(t) = x(t) + p\,x(t-\tau)$$

and $z(t) \to -\infty$,

$$0 > z(t) > p\,x(t-\tau)$$

which implies

$$0 < \frac{z(t)}{p} < x(t-\tau) \to +\infty.$$

(iii) $\Rightarrow$ (i) Assume that $x(t) \to \infty$ for $t \to \infty$. We show that if $z(t) \to 0$, it implies that $x(t) \nrightarrow \infty$. Really,

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma), & t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma), & \forall t_k \in S. \end{cases}$$

Hence

$$0 - z(t) = -\int_t^{\infty} Q(s)x(s-\sigma)ds - \sum_{t \le t_k} Q_k x(t_k - \sigma) < \infty,$$

which, by condition C2.1, implies

$$\int_t^{\infty} x(s-\sigma)ds < \infty \quad \text{and} \quad \sum_{t \le t_k} x(t_k - \sigma) < \infty.$$

This contradicts the statement that $x(t) \to \infty$. Hence, $z(t) \to 0 \Rightarrow x(t) \nrightarrow \infty$. Therefore, $x(t) \to \infty \Rightarrow z(t) \to -\infty$ by Lemma 2.1. This completes the proof (b).

(c) Applying contraposition to the statements of Lemma 2.2(a), we obtain

$$\neg(j) \Rightarrow \neg(jj) \Rightarrow \neg(jjj).$$

Thus,

$$\neg(j) \Rightarrow \neg(jj) \text{ means } z(t) \to 0 \Rightarrow p \ge -1;$$
$$\neg(j) \Rightarrow \neg(jjj) \text{ means } z(t) \to 0 \Rightarrow x(t) \nrightarrow \infty.$$

(j) $\Rightarrow$ (jj) We know that $z(t) \to 0 \Rightarrow p \ge -1$. Let us assume that $p = -1$. If z, being a decreasing function, has negative values, then $z(t)$ finally tends to $-\infty$ by Lemma 2.1. Hence, $z(t) \to 0$ implies that z is finally positive. Thus,

$$0 < z(t) = x(t) - x(t-\tau), \quad \forall t > T_0.$$

Hence

$$x(t-\tau) < x(t), \quad \forall t > T_0.$$

Iterating the above inequality, we obtain

$$x(t + i\tau) > x(t-\tau) > 0. \tag{2.16}$$

On the other hand,

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma), & t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma), & \forall t_k \in S, \end{cases}$$

where t_k belongs to the set of points of impulse effect. Hence

$$0 - z(t) = -\int_t^{+\infty} Q(s)x(s-\sigma)ds - \sum_{t \le t_k} Q_k x(t_k - \sigma) < \infty.$$

This follows from condition C2.1, that $\sum_{t\le t_k} x(t_k-\sigma) < \infty$, which contradicts condition (2.16). Hence the assumption that, $z(t) \to 0$ when $p = -1$ leads to a contradiction. Therefore, $p > -1$ is admissible only.

(jj) $\Rightarrow$ (jjj) Now we are familiar with the fact when $p > -1$, $x(t) \nrightarrow \infty$. Let us check what happens when $p \le 0$. Since, whenever $x(t) \nrightarrow \infty$ implies $z(t) \nrightarrow -\infty$, it follows by Lemma 2.1 that $z(t) \to 0$. Therefore

$$z(t) = x(t) + px(t-\tau) > x(t) > 0, \quad \forall t > T_0.$$

Hence $x(t) \to 0$.

Let $-1 < p < 0$. Then, the fact that z is a strictly decreasing function and $t \in [T_0, T_0+\tau]$, it implies

$$x(t) = (-p)x(t-\tau) + z(t) < (-p)x(t-\tau) + z(T_0-\tau).$$

We rewrite the above inequality in the form:

$$x(t) < (-p)x(t-\tau) + z(T_0-\tau)$$

and replace the function $x(t-\tau)$ with its supremum

$$x(t-\tau) \le \sup_{s\in[T_0-\tau,\, T_0]} x(s).$$

Then

$$x(t) < (-p) \sup_{s\in[T_0-\tau,\, T_0]} x(s) + z(T_0-\tau),$$

hence

$$\sup_{s\in[T_0,\, T_0+\tau]} x(s) < (-p) \sup_{s\in[T_0-\tau,\, T_0]} x(s) + z(T_0-\tau). \tag{2.17}$$

Let

$$\theta_k := T_0 + k\tau, \quad M_k := \sup_{s\in[\theta_k-\tau,\, \theta_k]} x(s) \quad \forall -1 \le k < \infty.$$

Then we get

$$M_{k+1} < (-p)M_k + z(\theta_k - \tau).$$

Applying this iteratively, we obtain, for $\ell > k$:

$$M_\ell \le (-p)^{\ell-k} M_k + z(\theta_{k-1}) \sum_{j=k}^{\ell} (-p)^i < (-p)^{\ell-k} M_k + z(\theta_{k-1}) \frac{1}{1+p}.$$

Hence for $\ell \to \infty$,

$$\limsup M_\ell \le z(\theta_{k-1}) \frac{1}{1+p} \to 0,$$

therefore

$$M_\ell \to 0 \Rightarrow x(t) \to 0.$$

This proves (jj) $\Rightarrow$ (jjj) and thus completes the proof of Lemma 2.2.

Lemma 2.3. *Let us assume that conditions* C2.1 *and* (2.9) *hold,* $p \neq -1$ *and there exists* $m \in N$ *such that*

$$Q(t+\tau) = Q(t),\ t_{k+m} = t_k + \tau, \quad Q_{k+m} = Q_k, \quad t \in R, \quad k \in Z. \tag{2.18}$$

Let $x(t)$ *be a solution of equation* (2.8) *and set*

$$z(t) = x(t) + p\,x(t-\tau) \text{ and } w(t) = z(t) + pz(t-\tau).$$

Then $z(t)$ *and* $w(t)$ *are solutions of equation* (2.8), $z(t) \in PC^1$, $w(t) \in PC^2$ *and*

$$w(t) > 0, \quad w'(t) \geq w'(t-\tau), \quad \Delta w(t_k) \geq \Delta w(t_k - \tau), \quad t \in R, \quad k \in Z. \tag{2.19}$$

Proof. If $x(t)$ is a solution of equation (2.8), it can be proved by direct substitution that the functions $z(t)$ and $w(t)$ are also solutions of the same equation. What is more, $z(t)$ is differentiable while $w(t)$ is twice differentiable.

Recall that by Lemma 2.2, $z(t)$ is a decreasing function, hence either (2.10) or (2.11) holds. In either case, the inequality

$$w'(t) \geq w'(t-\tau), \quad \Delta w(t_k) \geq \Delta w(t_k - \tau) \tag{2.20}$$

is applicable provided $p \neq -1$. Consequently, the condition $w(t) > 0$ holds. This completes the proof of Lemma 2.3.

3. MAIN RESULTS

The following theorems represent the basic results of this paper.

Theorem 3.1. *Assume that conditions* C2.1, (2.2) *and* (2.9) *are satisfied. Then every solution of the equation*

$$\begin{cases} [x(t) - x(t-\tau)]' + Q(t)x(t-\sigma) = 0, & t \notin S \\ \Delta[x(t_k) - x(t_k - \tau)] + Q_k x(t_k - \sigma) = 0, & \forall t_k \in S \end{cases} \tag{3.1}$$

is oscillatory.

Proof. By the definition of $z(t)$, the expression

$$z(t) = x(t) - x(t-\tau),$$

immediately implies $p = -1$. Hence by the implication $(j) \Rightarrow (jj)$ of Lemma 2.2(c), $x(t)$ is neither finally positive nor finally negative. Consequently, the solution of equation (3.1) oscillates. This completes the proof of Theorem 3.1.

Theorem 3.2. *Assume that condition* (2.2) *and equation* (2.3) *hold,* $P(t) \leq -1$, $\tau > \sigma$ *and every solution of the equation*

$$\begin{cases} z'(t) + \frac{Q(t)}{P(t+\tau-\sigma)} z(t + (\tau - \sigma)) = 0, & t \notin S \\ \Delta z(t_k) + \frac{Q_k}{P(t_k+\tau-\sigma)} z(t_k + (\tau-\sigma)) = 0, & \forall t_k \in S \end{cases} \tag{3.2}$$

is oscillatory. Then every solution of equation (2.1) *is oscillatory.*

Proof. Let us assume that equation (2.1) has a finally positive solution $x(t)$. Set $z(t) = x(t) + P(t)x(t-\tau)$. Then by Lemma 2.1(b),

$$z(t) < 0, \quad t \geq t_0. \tag{3.3}$$

Observe that

$$z(t) > P(t)x(t-\tau)$$

and so

$$\begin{cases} \frac{Q(t)}{P(t+\tau-\sigma)} z(t+(\tau-\sigma)) < Q(t)x(t-\sigma) = -z'(t), & t \notin S \\ \frac{Q_k}{P(t_k+\tau-\sigma)} z(t_k+(\tau-\sigma)) < Q_k x(t_k-\sigma) = -\Delta z(t_k), & \forall t_k \in S, \end{cases}$$

or

$$\begin{cases} z'(t) + \frac{Q(t)}{P(t+\tau-\sigma)} z(t+(\tau-\sigma)) < 0, & t \notin S \\ \Delta z(t_k) + \frac{Q_k}{P(t_k+\tau-\sigma)} z(t_k+(\tau-\sigma)) < 0, & \forall t_k \in S. \end{cases}$$

We rewrite the above inequality in the following equivalent form:

$$\begin{cases} z'(t) - \left[\frac{Q(t)}{-P(t+\tau-\sigma)}\right] z(t+(\tau-\sigma)) < 0, & t \notin S \\ \Delta z(t_k) - \left[\frac{Q_k}{-P(t_k+\tau-\sigma)}\right] z(t_k+(\tau-\sigma)) < 0, & \forall t_k \in S. \end{cases} \tag{3.4}$$

In view of condition C2.1 and Theorem 1.2(ii), the advanced impulsive differential inequality (3.4) cannot have a finally negative solution. This contradicts condition (3.3), hence all the solutions of equation (2.1) must be oscillatory. This proves Theorem 3.2.

Remark 3.1. Theorem 3.2 is still valid if we assume that some sufficient conditions for the oscillation of equation (3.2) hold. For example, this is true by Theorem 1.3 if

$$\begin{cases} \lim_{t\to+\infty} \inf \int\limits_{t}^{t+\tau-\sigma} \frac{Q(s)}{-P(s+\tau-\sigma)} ds \geq \varpi > 0, \\ \lim_{t\to+\infty} \sum \inf_{t \leq t_k < t+\tau-\sigma} \frac{Q_k}{-P(t_k+\tau-\sigma)} \geq q > 0, \end{cases}$$

where

$$\varpi e + q > 1.$$

Theorem 3.3. *Assume that conditions* C2.1 *and* (2.2) *hold,* $-1 < P(t) \leq 0$ *and every solution of the equation*

$$\begin{cases} z'(t) + Q(t)z(t-\sigma) = 0, & t \geq t_0, t \notin S \\ \Delta z(t_k) + Q_k z(t_k-\sigma) = 0, & \forall t_k \in S, \forall t_k \in S \end{cases}$$

is oscillatory. Then every solution of equation (2.1) *is oscillatory.*

Proof. Assume conversely, that equation (2.1) has a finally positive solution $x(t)$. Like in Theorem 3.2, we set

$$z(t) = x(t) + P(t)x(t-\tau).$$

Then by Lemma 2.1 (c),

$$z(t) > 0, \quad t \geq t_0. \tag{3.5}$$

Since $z(t) \leq x(t)(t \geq t_0)$, it follows from the equation

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma), & t \notin S \\ \Delta z(t_k) = -Q_k z(t_k - \sigma), & \forall t_k \in S \end{cases}$$

that

$$\begin{cases} z'(t) + Q(t)z(t-\sigma) \leq 0, & t \geq t_0 \\ \Delta z(t_k) + Q_k z(t_k - \sigma) \leq 0. \end{cases} \tag{3.6}$$

In view of condition C2.1 and Theorem 1.2(i), the delay impulsive differential inequality (3.6) cannot have a finally positive solution and this contradicts condition (3.5). Thus, the proof of Theorem 3.3 is complete.

Theorem 3.4. *Assume that conditions* (2.9) *and* (2.18) *hold,* $p \neq -1$, $(1+p)(\sigma-\tau) > 0$ *and every solution of the equation*

$$\begin{cases} w'(t) + \frac{1}{1+q}Q(t)w(t+\tau-\sigma) = 0, & t \geq t_0, t \notin S \\ \Delta w(t_k) + \frac{1}{1+p}Q_k w(t_k + \tau - \sigma) = 0, & \forall t_k \in S \end{cases} \tag{3.7}$$

is oscillatory. Then every solution of equation (2.8) *is oscillatory.*

Proof. Assume conversely that equation (2.8) has a finally positive solution $x(t)$. We set $z(t) = x(t) + px(t-\tau)$ and $w(t) = z(t) + pz(t-\tau)$. By direct substitution, it is possible to show that $z(t)$ and $w(t)$ are differentiable solutions of equation (2.8). That is,

$$\begin{cases} z'(t) + p\, z'(t-\tau) + Q(t)z(t-\sigma) = 0, & t \geq t_0, t \notin S \\ \Delta z(t_k) + p\, \Delta z(t_k - \tau) + Q_k z(t_k - \sigma) = 0, & \forall t_k \in S, \end{cases} \tag{3.8}$$

and

$$\begin{cases} w'(t) + p\, w'(t-\tau) + Q(t)w(t-\sigma) = 0, & t \geq t_0, t \notin S \\ \Delta w(t_k) + p\, \Delta w(t_k - \tau) + Q_k w(t_k - \sigma) = 0, & \forall t_k \in S. \end{cases} \tag{3.9}$$

By Lemma 2.2(a), $z(t)$ is a decreasing function and either condition (2.10) or (2.11) holds. In either case, we shall claim that

$$\begin{cases} w'(t-\tau) \leq w'(t), & t \geq t_0, t \notin S \\ \Delta w(t_k - \tau) \leq \Delta w(t_k), & \forall t_k \in S. \end{cases} \tag{3.10}$$

Indeed,

$$\begin{cases} w'(t) = -Q(t)z(t-\sigma) \geq -Q(t)z(t-\sigma-\tau), & t \notin S \\ \Delta w(t_k) = -Q_k z(t_k-\sigma) \geq -Q_k z(t_k-\sigma-\tau), & \forall t_k \in S, \end{cases}$$

therefore

$$\begin{cases} w'(t) \geq -Q(t-\tau)z(t-\sigma-\tau) = w'(t-\tau), & t \notin S \\ \Delta w(t_k) \geq -Q(t_k-\tau)z(t_k-\sigma-\tau) = \Delta w(t_k-\tau), & \forall t_k \in S. \end{cases}$$

Furthermore, it follows from Lemma 2.2 that as long as $p \neq -1$,

$$w(t) > 0, \quad t \geq t_0. \tag{3.11}$$

Taking advantage of the inequalities in (3.10), we transform equation (3.9) to obtain

$$\begin{cases} (1+p)w'(t-\tau) + Q(t)w(t-\sigma) \leq 0, & t \notin S \\ (1+p)\Delta w(t_k-\tau) + Q_k w(t_k-\sigma) \leq 0, & \forall t_k \in S. \end{cases}$$

From equation (2.18), that is, the τ-periodicity of Q , it follows that

$$\begin{cases} w'(t) + \frac{Q(t)}{1+q} w(t-(\sigma-\tau)) \leq 0, & t \geq t_0, t \notin S \\ \Delta w(t_k) + \frac{Q_k}{1+p} w(t_k-(\sigma-\tau)) \leq 0, & \forall t_k \in S \end{cases} \tag{3.12}$$

if $1+p > 0$ and

$$\begin{cases} w'(t) - \left[\frac{Q(t)}{-(1+q)}\right] w(t+(\tau-\sigma)) \geq 0, & t \geq t_0, t \notin S \\ \Delta w(t_k) - \left[\frac{Q_k}{-(1+p)}\right] w(t_k+(\tau-\sigma)) \geq 0, & \forall t_k \in S \end{cases} \tag{3.13}$$

if $1+p < 0$.

From equation (3.8) and Theorem 1.1(i), the delay impulsive differential inequalities (3.12) and (3.13) cannot have finally positive solutions. This contradicts condition (3.11), and hence completes the proof of Theorem 3.4.

Remark 3.2. Similar to Theorem 3.2, it is also observed that if some sufficient conditions for the oscillation of the solutions of equation (3.7) are given as

$$\frac{1}{1+p}\left[\lim_{t\to+\infty}\inf\left\{\int_{t-\sigma}^{t-\tau} Q(s)ds + \sum_{t-\sigma\leq t+k<t-\tau} Qk\right\}\right] \geq K > e^{-1},$$

then the validity of Theorem 3.4 can still be established.

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