Dynamic Systems and Applications 18 (2009) 667-676

SOLVABILITY OF SOME FUNCTIONAL INTEGRAL EQUATION

LESZEK OLSZOWY

Department of Mathematics, Rzeszów University of Technology
W. Pola 2, 35-959 Rzeszów, Poland

ABSTRACT. In this paper we present results on the existence and asymptotic behaviour of solutions of a functional integral equation. Considering the equation in Banach space and proving a new fixed point theorem we establish existence theorems which generalize several ones obtained earlier by other authors. The applicability of the results is illustrated by an example.

AMS (MOS) Subject Classification. Primary 45N05; Secondary 47B38, 47H09

1. INTRODUCTION

In the paper [4] the authors proved an interesting theorem on the existence of solutions of the following functional integral equation

$$x(t) = f\Big(t, \int_0^t x(s)ds,\ \int_0^t x(h(s,x(s)))ds\Big), \quad t \in \mathbb{R}_+ = [0,\infty). \tag{1.1}$$

Their investigations were situated in the Banach space $C_p(\mathbb{R}_+)$ consisting of all real functions defined and continuous on $\mathbb{R}_+$ and tempered by some function $p(t)$. This method can be improved. The aim of the present paper is to study the problem of the existence of solutions of Eq. (1.1) in the appropriate Fréchet space. Such method allows us to obtain results which generalize those obtained in [4].

In this paper we first prove a new fixed point theorem formulated in terms of the Kuratowski measure. And then by making use of it, we generalize [4] by considering the equation (1.1) in the Fréchet space $C(\mathbb{R}_+, E)$ where E is a real Banach space. Next, as a simple consequence of the established theorem, we get result for equation (1.1) in $C(\mathbb{R}_+)$ which improves Theorem 3.1 [4].

The equation of the type (1.1) or its particular cases were investigated in [1, 2, 3, 4, 6, 7, 8, 9, 10, 13]. The result obtained in this paper generalizes those obtained in the above mentioned papers.

Received November 14, 2008

Let us mention that the theory of functional integral equations has many useful applications in describing numerous events and problems of the real world. For example, integral equations are often applicable in engineering, mathematical physics, economics and biology (cf. [1, 2, 3, 6, 8, 11]).

2. NOTATION AND AUXILIARY FACTS

In this section, we list a few auxiliary results which will be applied further on. Assume that E is a real Banach space with the norm $||\cdot||$ and the zero element θ. Denote by $B(r)$ the closed ball centered at θ and with radius r.

Consider

$$C(\mathbb{R}_+, E) = \{x : \mathbb{R}_+ \to E,\ x \ \text{ is continuous}\},$$

equipped with the family of seminorms $|x|_n = \sup\{||x(t)|| : t \in [0, n]\}$, $n \in \mathbb{N}$.

It is known that $C(\mathbb{R}_+, E)$ furnished with the standard distance is a locally convex Fréchet space. Let us recall two facts [12]:

(A) a sequence $\{x_n\}$ is convergent to x in $C(\mathbb{R}_+, E)$ if and only if $\{x_n\}$ is uniformly convergent to x on compact subsets of $\mathbb{R}_+$,

(B) a family $A \subset C(\mathbb{R}_+, E)$ is relatively compact if and only if for each $T > 0$, the restrictions to $[0, T]$ of all functions from A form an equicontinuous set and $A(t)$ is relatively compact in E for each $t \in \mathbb{R}_+$.

If X is a subset of E (or $C(\mathbb{R}_+, E)$), then $\overline{X}$ and $\mathrm{Conv}X$ denote the closure and convex closure of X, respectively.

Let α denote the Kuratowski measure of noncompactness in E, the properties of which may be found in [5].

For any $X \subset C(\mathbb{R}_+, E)$ and $t \geq 0$ let $X(t) = \{x(t) :\ x \in X\}$.
Moreover, the symbol $\mathrm{Conv}X(t)$ stands for $(\mathrm{Conv}X)(t)$.

To prove the main result in this paper, we need the following Lemma.

Lemma 2.1 (Fixed Point Theorem). *Let Ω be equicontinuous on compact intervals of $\mathbb{R}_+$, closed and convex subset of $C(\mathbb{R}_+, E)$ such that $\Omega(t)$ is bounded set in E for each $t \geq 0$, operator $F : \Omega \to \Omega$ be continuous. For any $X \subset \Omega$, set*

$$\widehat{F}^1(X) = F(X), \quad \widehat{F}^n(X) = F(\mathrm{Conv}\widehat{F}^{n-1}(X)), \quad n = 2, 3, \ldots. \tag{2.1}$$

If for any $X \subset \Omega$ and $T > 0$

$$\lim_{n\to\infty} \sup_{t \leq T} \alpha\Big(\widehat{F}^n(X)(t)\Big) = 0$$

then F has a fixed point in Ω.

Proof. Let us put $\Omega_0 = \Omega$. Applying repeatedly (2.1) we obtain a sequence of naturals $\{n_k\}$ and a sequence of sets $\{\Omega_k\}$, such that

$$\Omega_k = \mathrm{Conv}\widehat{F}^{n_k}(\Omega_{k-1}) \quad \text{for } k = 1, 2, \ldots$$
$$\sup_{t \le k} \alpha(\Omega_k(t)) \le 1/k \quad \text{for } k = 1, 2, \ldots$$

i.e.

$$\lim_{k\to\infty} \sup_{t\le k} \alpha(\Omega_k(t)) = 0. \tag{2.2}$$

The sets $\{\Omega_k\}$ are obviously closed and convex subsets in $C(\mathbb{R}_+, E)$. Next let us observe that

$$F(\Omega_k) \subset \Omega_k \quad \text{for } k = 0, 1, 2, \ldots. \tag{2.3}$$

Indeed

$$\widehat{F}^1(\Omega_0) = F(\Omega_0) \subset \Omega_0 \quad \text{hence } \mathrm{Conv}\widehat{F}^1(\Omega_0) \subset \Omega_0,$$
$$\widehat{F}^2(\Omega_0) = F(\mathrm{Conv}\widehat{F}^1(\Omega_0)) \subset F(\Omega_0) = \widehat{F}^1(\Omega_0),$$
$$\widehat{F}^3(\Omega_0) = F(\mathrm{Conv}\widehat{F}^2(\Omega_0)) \subset F(\mathrm{Conv}\widehat{F}^1(\Omega_0)) = \widehat{F}^2(\Omega_0),$$
$$\cdots$$
$$\widehat{F}^{n_1}(\Omega_0) = F(\mathrm{Conv}\widehat{F}^{n_1-1}(\Omega_0)) \subset F(\mathrm{Conv}\widehat{F}^{n_1-2}(\Omega_0)) = \widehat{F}^{n_1-1}(\Omega_0).$$

This implies that

$$\Omega_1 = \mathrm{Conv}\widehat{F}^{n_1}(\Omega_0) \subset \mathrm{Conv}\widehat{F}^{n_1-1}(\Omega_0),$$
$$F(\Omega_1) \subset F(\mathrm{Conv}\widehat{F}^{n_1-1}(\Omega_0)) = \widehat{F}^{n_1}(\Omega_0) \subset \mathrm{Conv}\widehat{F}^{n_1}(\Omega_0) = \Omega_1.$$

Employing the same method, we can prove that $F(\Omega_k) \subset \Omega_k$ for $k = 2, 3, \ldots$.

Now we show that $\{\Omega_k\}$ is decreasing sequence of sets. Indeed,

$$\widehat{F}^1(\Omega_0) = F(\Omega_0) \subset \Omega_0,$$
$$\widehat{F}^2(\Omega_0) = F(\mathrm{Conv}\widehat{F}^1(\Omega_0)) \subset F(\mathrm{Conv}\Omega_0) \subset \Omega_0,$$
$$\cdots$$
$$\widehat{F}^{n_1}(\Omega_0) = F(\mathrm{Conv}\widehat{F}^{n_1-1}(\Omega_0)) \subset F(\mathrm{Conv}\Omega_0) \subset \Omega_0,$$

and therefore $\Omega_1 \subset \Omega_0$. Applying (2.3) and repeating this argumentation we obtain that $\Omega_k \subset \Omega_{k-1}$ for $k = 1, 2, \ldots$.

Now we prove that

$$\bigcap_{k=0}^{\infty} \Omega_k \ne \emptyset.$$

Suppose $\{t_i\}, i = 1, 2, \ldots$ is a sequence of reals dense in $\mathbb{R}_+$. We take the sequence of functions $\{x_k\}$ such that $x_k \in \Omega_k, k = 0, 1, \ldots$. Keeping in mind (2.2) we can choose x_k to be pointwise convergent at each point $t_i, i = 1, 2, \ldots$. It is always possible by taking consecutive subsequences converging successively at $t_1, t_2, \ldots$ and by applying the diagonal procedure. Put $x_\infty(t_i) = \lim_{k\to\infty} x_k(t_i)$ for $i = 1, 2, \ldots$. Repeating reasoning from [5, Th. 11.2] we obtain that x_∞ can be extended to the $\mathbb{R}_+$ and that x_∞ is uniform limit of x_k on every compact interval. Linking (A) and the closedness of Ω_k we get that $x_\infty \in \Omega_k$ and $\bigcap_{k=0}^{\infty} \Omega_k \neq \emptyset$.

Now let us denote

$$\widetilde{\Omega} = \bigcap_{k=0}^{\infty} \Omega_k.$$

Applying (2.3) we get

$$F(\widetilde{\Omega}) = F(\bigcap_{k=0}^{\infty} \Omega_k) = \bigcap_{k=0}^{\infty} F(\Omega_k) \subset \bigcap_{k=0}^{\infty} \Omega_k = \widetilde{\Omega} \quad .\text{e. } F(\widetilde{\Omega}) \subset \widetilde{\Omega}.$$

Next, in view of (2.2) we get $\alpha(\widetilde{\Omega}(t)) = 0$ for each $t \geq 0$. Linking this fact together with closedness of $\widetilde{\Omega}$, (B) and previously established properties we obtain that $\widetilde{\Omega}$ is nonempty, convex and compact subset of $C(\mathbb{R}_+, E)$. Finally, applying the Tikhonov fixed point theorem for $F : \widetilde{\Omega} \to \widetilde{\Omega}$ we infer that F has at least one fixed point in $\widetilde{\Omega} \subset C(\mathbb{R}_+, E)$.

This completes the proof. □

Lemma 2.2 ([11]). *Assume $X \subset C(\mathbb{R}_+, E)$ is equicontinuous on compact intervals of $\mathbb{R}_+$ and $X(t)$ is bounded for all $t \geq 0$. Then*

$$\alpha\Big(\int_0^t X(s)ds\Big) \leq \int_0^t \alpha(X(s))ds, \quad \textit{for all } t \geq 0.$$

Lemma 2.3 ([11]). *Let $X \subset C(\mathbb{R}_+, E)$ be equicontinuous on compact intervals of $\mathbb{R}_+$, then* Conv$X$ *is also equicontinuous on ones.*

3. MAIN RESULT

In this section we will study the functional integral equation (1.1). We will consider this equation under the following assumptions:

(H_1) the function $f : \mathbb{R}_+ \times E \times E \to E$ is uniformly continuous on $[0, T] \times B(R) \times B(R)$ for any $T, R > 0$,

(H_2) the function $h : \mathbb{R}_+ \times E \to \mathbb{R}_+$ is uniformly continuous on $[0, T] \times B(R)$ for any $T, R > 0$ and $h(s, x) \leq s$ for all $s \in \mathbb{R}_+$ and $x \in E$,

(H_3) there exist three continuous functions $L_0, L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following inequality

$$\|f(t, x, y)\| \leq L_0(t) + L_1(t)\|x\| + L_2(t)\|y\|$$

holds for each $t \in \mathbb{R}_+$ and $x, y \in E$,

(H_4) there are two continuous functions $K_1, K_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\alpha\Big(f(t,U,V)\Big) \le K_1(t)\alpha(U) + K_2(t)\alpha(V)$$

for any bounded sets $U, V \subset E$.

Now, we formulate the following theorem which is the main result of the paper.

Theorem 3.1. *Under the assumptions* (H_1)–(H_4) *the equation* (1.1) *has at least one solution* $x \in C(\mathbb{R}_+, E)$.

Proof. Consider the operator F defined on the space $C(\mathbb{R}_+, E)$ by the formula

$$(Fx)(t) = f\Big(t, \int_0^t x(s)ds, \int_0^t x(h(s,x(s)))ds\Big), \quad t \ge 0.$$

Observe that the operator F is well-defined and transforms the space $C(\mathbb{R}_+, E)$ into itself.

Let us denote: $\overline{L}_i(t) = \sup_{s\le t} L_i(s)$ where $i = 0,1,2$ and consider the Cauchy problem for the linear equation

$$\overline{L}_0(t) + \Big(\overline{L}_1(t) + \overline{L}_2(t)\Big)\phi(t) = \phi'(t), \quad \phi(0) = 0.$$

The solution $\phi(t)$ is defined for $t \ge 0$ and we put $\psi(t) = \phi'(t)$. Obviously $\psi(t) \ge 0$, $\psi(t)$ is increasing and moreover $\psi(t)$ satisfies the equation

$$\overline{L}_0(t) + \Big(\overline{L}_1(t) + \overline{L}_2(t)\Big)\int_0^t \psi(s)ds = \psi(t).$$

Let us observe that $\psi(t)$ fulfils the inequality

$$L_0(t) + \Big(L_1(t) + L_2(t)\Big)\int_0^t \psi(s)ds \le \psi(t). \tag{3.1}$$

Indeed

$$L_0(t) + \Big(L_1(t) + L_2(t)\Big)\int_0^t \psi(s)ds \le \overline{L}_0(t) + \Big(\overline{L}_1(t) + \overline{L}_2(t)\Big)\int_0^t \psi(s)ds = \psi(t).$$

Next we put $\Delta = \{x \in C(\mathbb{R}_+, E) : \|x(t)\| \le \psi(t) \text{ for } t \ge 0\}$. Obviously the set Δ is convex and closed. Now we prove $F(\Delta) \subset \Delta$. Taking $x \in \Delta$ and keeping in mind (H_3), increasing of ψ and (3.1) we get

$$\begin{aligned}\|(Fx)(t)\| &\le L_0(t) + L_1(t)\int_0^t \|x(s)\|ds + L_2(t)\int_0^t \|x(h(s,x(s)))\|ds \\ &\le L_0(t) + L_1(t)\int_0^t \psi(s)ds + L_2(t)\int_0^t \psi(s)ds \le \psi(t).\end{aligned}$$

Now we show that the set $F(\Delta)$ is equicontinuous on compact intervals of $\mathbb{R}_+$. To prove this fact let us fix $x \in \Delta$, $T > 0$, $\varepsilon > 0$ and take $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| \leq \varepsilon$.

$$\begin{aligned}\|(Fx)(t_1) - (Fx)(t_2)\| &= \|f\Big(t_1, \int_0^{t_1} x(s)ds, \int_0^{t_1} x(h(s, x(s)))ds\Big) \\ &\quad - f\Big(t_2, \int_0^{t_2} x(s)ds, \int_0^{t_2} x(h(s, x(s)))ds\Big)\| \leq \nu^T(f, \varepsilon)\end{aligned}$$

where

$$\begin{aligned}\nu^T(f, \varepsilon) = \sup\{&\|f(t_1, x_1, y_1) - f(t_2, x_2, y_2)\| : t_1, t_2 \in [0, T], \\ &|t_1 - t_2| \leq \varepsilon, \quad x_1, x_2 \in B(T\psi(T)), \quad \|x_1 - x_2\| \leq \varepsilon\psi(T), \\ &y_1, y_2 \in B(T\psi(T)), \quad \|y_1 - y_2\| \leq \varepsilon\psi(T)\}.\end{aligned}$$

Taking into account the uniform continuity of the function $f = f(t, x, y)$ on the set $[0, T] \times B(T\psi(T)) \times B(T\psi(T))$ we conclude that $\nu^T(f, \varepsilon) \to 0$ as $\varepsilon \to 0$ what confirms that $F(\Delta)$ is equicontinuous on compact intervals $[0, T]$.

In what follows we show that F is continuous on Δ. To do this let us take $x, x_n \in \Delta$ such that $x_n \to x$ in $C(\mathbb{R}_+, E)$ i.e. according to (A), $x_n \to x$ uniformly on $[0, T]$ for any $T > 0$. We will prove that $Fx_n \to Fx$ uniformly on $[0, T]$. Let us fix $T > 0$ and take $t \in [0, T]$.

$$\begin{aligned}\|(Fx)(t) - (Fx_n)(t)\| &= \|f\Big(t, \int_0^{t} x(s)ds, \int_0^{t} x(h(s, x(s)))ds\Big) \\ &\qquad - f\Big(t, \int_0^{t} x_n(s)ds, \int_0^{t} x_n(h(s, x_n(s)))ds\Big)\| \\ &\leq \overline{\nu}^T\Big(f, T \cdot \sup_{s \leq T} \|x(s) - x_n(s)\| \\ &\qquad + T \cdot \overline{\nu}_1^T(h, x, T \cdot \sup_{s \leq T} \|x(s) - x_n(s)\|)\Big)\end{aligned}$$

where

$$\begin{aligned}\overline{\nu}^T(f, \delta) = \sup\{&\|f(t, x_1, y_1) - f(t, x_2, y_2)\| : t \in [0, T],\ x_1, x_2 \in B(T\psi(T)), \\ &\|x_1 - x_2\| \leq \delta,\ y_1, y_2 \in B(T\psi(T)),\ \|y_1 - y_2\| \leq \delta\}\end{aligned}$$

and

$$\begin{aligned}\overline{\nu}_1^T(h, x, \delta) = \sup\{&\|x(h(s, y_1)) - x(h(s, y_2))\| : s \in [0, T],\ y_1, y_2 \in B(T\psi(T)), \\ &\|y_1 - y_2\| \leq \delta\}.\end{aligned}$$

In view of uniform continuity of f and h on bounded sets we get that $Fx_n \to Fx$ uniformly on $[0,T]$. Let us put $\Omega = \mathrm{Conv}F(\Delta)$. In view of Lemma 2.3 we obtain that Ω is equicontinuous on compact intervals of $\mathbb{R}_+$.

In what follows we show that the continuous mapping $F : \Omega \to \Omega$ fulfils the assumptions of Lemma 2.1. To prove this fact fix $T > 0$ and denote $M = \sup\{K_1(t) + K_2(t) : t \leq T\}$. Moreover, we define a mapping $H : C(\mathbb{R}_+, E) \to C(\mathbb{R}_+, E)$ by

$$(Hx)(t) = x(h(t, x(t))).$$

Let fix $X \subset \Omega$ and let us observe that

$$\mathrm{Conv}X(t) \subset \mathrm{Conv}(X(t)). \tag{3.2}$$

The proof is standard and will be omitted.

In further investigations we need the following inequality

$$\alpha\Big(H(X)(t)\Big) \leq \sup\{\alpha(X(s)) : s \leq t\}. \tag{3.3}$$

To prove this let us fix $\varepsilon > 0$. Since X is equicontinuous we can choose enough dense a sequence $\{t_i\}_{i=0}^{i=n}$, $0 = t_0 < t_1 < \cdots < t_n = T$ such that $\|x(s_1) - x(s_2)\| \leq \varepsilon$ for $s_1, s_2 \in [t_{i-1}, t_i]$ and any $x \in X$. Hence, in view of (H_2) and properties of measure α, we infer that

$$\begin{aligned}
\alpha\Big(H(X)(t)\Big) &\leq \alpha\Big(\bigcup_{s\leq t} X(s)\Big) = \alpha\Big(\bigcup_{i=1}^{n} \bigcup_{s\in[t_{i-1},t_i]} X(s)\Big) \\
&= \max\left\{\alpha(\bigcup_{s\in[t_{i-1},t_i]} X(s)) : i = 1, \ldots, n\right\} \\
&\leq \sup\{\alpha(X(s)) : s \leq t\} + 2\varepsilon.
\end{aligned}$$

Free choice of $\varepsilon > 0$ implies (3.3).

Next let us put $A = \sup\{\alpha(X(s)) : s \leq T\}$. Obviously $A < \infty$. Bearing in mind the inclusion $F(X) \subset f\Big(t, \int_0^t X(s)ds, \int_0^t H(X)(s)ds\Big)$, (H_4), Lemma 2.2, (3.3) and above denotations we get for any $t \leq T$

$$\begin{aligned}
\alpha\Big(\widehat{F}^1(X)(t)\Big) &\leq \alpha\Big(f\Big(t, \int_0^t X(s)ds, \int_0^t H(X)(s)ds\Big)\Big) \\
&\leq K_1(t) \int_0^t \alpha(X(s))ds + K_2(t) \int_0^t \alpha\Big(H(X)(s)\Big)ds \\
&\leq K_1(t)tA + K_2(t)tA = AMt.
\end{aligned}$$

Hence, in view of (3.2) we get

$$\alpha\Big(\mathrm{Conv}\widehat{F}^1(X)(t)\Big) \leq AMt.$$

Applying above inequality and arguing in the same way as above we have

$$\alpha\Big(\widehat{F}^2(X)(t)\Big) \le \alpha\bigg(f\Big(t, \int_0^t \mathrm{Conv}\widehat{F}^1(X)(s)ds, \int_0^t H\Big(\mathrm{Conv}\widehat{F}^1(X)(s)\Big)ds\Big)\bigg)$$

$$\le K_1(t)\int_0^t \alpha\Big(\mathrm{Conv}\widehat{F}^1(X)(s)\Big)ds + K_2(t)\int_0^t \alpha\Big(H\Big(\mathrm{Conv}\widehat{F}^1(X)(s)\Big)\Big)ds$$

$$\le K_1(t)\int_0^t AMsds + K_2(t)\int_0^t AMsds \le AM^2t^2/2!,$$

$$\alpha\Big(\mathrm{Conv}\widehat{F}^2(X)(t)\Big) \le AM^2t^2/2!.$$

By the method of mathematical induction, we can prove

$$\alpha\Big(\widehat{F}^n(X)(t)\Big) \le AM^nt^n/n!.$$

Hence

$$\sup_{t\le T}\alpha\Big(\widehat{F}^n(X)(t)\Big) \le AM^nT^n/n! \to 0 \quad \text{as } n\to\infty.$$

Thus, joining the properties of F and Ω, and taking into account Lemma 2.1 we infer that the operator F has at least one fixed point in Ω.

The proof of our theorem is complete. □

Remark 3.2. Let us observe that if $E = \mathbb{R}$ then the assumption H_4 is obviously fulfilled and the uniform continuity of f and h can be replaced by continuity, so we can formulate $H_1 - H_4$ in the next, simpler form:

(i) the function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist three continuous functions $L_0, L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,x,y)| \le L_0(t) + L_1(t)|x| + L_2(t)|y|$$

for $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$,

(ii) the function $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous and $h(t,x) \le t$ for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

As an immediately consequence of Theorem 3.1 we get.

Theorem 3.3. *Under the assumptions* (i)–(ii) *the equation* (1.1) *has at least one solution* $x \in C(\mathbb{R}_+)$.

Remark 3.4. Keeping in mind that the solutions x of Eq. (1.1) belong to Δ and applying exact formula for solution of linear equation we get following estimation

$$||x(t)|| \le \exp\Big(\int_0^t (\overline{L}_1(s) + \overline{L}_2(s))ds\Big)\int_0^t \overline{L}_0(s)\exp\Big(\int_0^s (\overline{L}_1(\tau) + \overline{L}_2(\tau))d\tau\Big)ds.$$

Remark 3.5. Let us observe that above Theorem 3.3 improves Theorem 3.1 [4] in which authors assumed additionally that

$$\lim_{t\to\infty} t(L_1(t) + L_2(t)) = 0, \tag{3.4}$$

$$\lim_{t\to\infty} L_0(t)\exp(-\int_0^t L_0(s)ds) = 0, \tag{3.5}$$

and there exists a continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g(t) \to \infty \text{ as } t \to \infty \text{ and } g(t) \le h(t,x) \text{ for } t \in \mathbb{R}_+ \text{ and } x \in \mathbb{R}. \tag{3.6}$$

Example 3.6. Consider the following functional equation

$$x(t) = \sqrt{t}|\sin t|^{(t+1)^4} + t\int_0^t x(s)ds + t^2\int_0^t x\Big(\frac{2s^2}{1+s^2+x^2(s)}\Big)ds, \tag{3.7}$$

for $t \ge 0$.

Observe that this equation is a special case of (1.1), where $f(t,x,y) = \sqrt{t}|\sin t|^{(t+1)^4} + tx + t^2y$ and $h(t,x) = 2t^2/(1+t^2+x^2)$.

We show that equation (3.7) satisfies the assumptions of Theorem 3.3 with $L_0(t) = \sqrt{t}|\sin t|^{(t+1)^4}$, $L_1(t) = t$, $L_2(t) = t^2$. In fact, let us observe that

$$|f(t,x,y)| \le \sqrt{t}|\sin t|^{(t+1)^4} + t|x| + t^2|y| = L_0(t) + L_1(t)|x| + L_2(t)|y|.$$

This shows that assumptions (i) is satisfied. Moreover, we have

$$h(t,x) = 2t^2/(1+t^2+x^2) \le 2t^2/(1+t^2) \le t \quad \text{for all } t \in \mathbb{R}_+$$

and (ii) is also fulfilled. Hence, on the basis of Theorem 3.3 we deduce that equation (3.7) has at least one solution $x = x(t)$ in the space $C(\mathbb{R}_+)$.

Let us observe that functions $L_1(t)$, $L_2(t)$ and $h(t,x)$ do not satisfy conditions (3.4) and (3.6). We show that (3.5) also is not satisfied. We apply easy to prove inequality (simple proof is omitted)

$$\int_0^\infty \sin^n t dt \le \sqrt{2\pi/n} \quad \text{for } n \in \mathbb{N}.$$

$$\begin{aligned}\int_0^\infty L_0(t)dt &= \sum_{k=1}^\infty \int_{(k-1)\pi}^{k\pi} \sqrt{t}|\sin t|^{(t+1)^4} \le \sum_{k=1}^\infty \int_{(k-1)\pi}^{k\pi} \sqrt{k\pi}|\sin t|^{((k-1)\pi+1)^4}\\ &= \sum_{k=1}^\infty \int_0^\pi \sqrt{k\pi}(\sin t)^{k^4} \le \pi\sum_{k=1}^\infty \sqrt{2k}/k^2 < \infty\end{aligned}$$

what, together with the equality $\limsup_{t\to\infty} L_0(t) = \infty$ implies that (3.5) is not satisfied.

REFERENCES

[1] R. P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations,* Kluwer Academic Publishers, Dordrecht, 2001.

[2] J. Appell, P. P. Zabrejko, *Nonlinear Superposition Operators*, in: Cambridge Tracts in Mathematics, vol. 95, Cambridge Univ. Press, 1990.

[3] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, *Introduction to Theory of Functional - Differential Equations,* Nauka Moscow, 1990.

[4] J. Banaś , I. J. Cabrera, On existence and asymptotic behaviour of solutions of a functional integral equation, *Nonlinear Anal.*, 66:2246–2254, 2007.

[5] J. Banaś , K. Goebel, *Measures of Noncompactness in Banach Spaces,* Lecture Notes in Pure and Applied Math., 60, Dekker, New York and Basel, 1980.

[6] C. Corduneanu, *Integral Equations and Applications,* Cambridge Univ. Press, Cambridge, 1991.

[7] S. Czerwik, The existence of global solutions of a functional - differential equation, *Collog. Math.*, 36:121–125, 1976.

[8] K. Deimling, *Nonlinear Functional Analysis,* Springer Verlag, Berlin, 1985.

[9] B. C. Dhage, S. K. Ntouyas, Existence results for nonlinear functional integral equations via a fixed point theorem of Krasnoselskii - Schaefer type, *Nonlin. Studies*, 9:307–317, 2002.

[10] L. J. Grimm, Existence and uniqueness for nonlinear neutral - differential equations, *Bull. Amer. Math. Soc.*, 77:374–376, 1971.

[11] D. J. Guo, V. Lakshmikantham, X. Z. Liu, *Nonlinear Integral Eguations in Abstract Spaces,* Kluwer Academic , Dordrecht, 1996.

[12] L. Olszowy, On existence of solutions of a quadratic Urysohn integral equation on an unbounded interval, *Comment. Math.*, Vol.48 No.1:103–112, 2008.

[13] M. Zima, A certain fixed point theorem and its applications to integral-functional equations, *Bull. Austal. Math. Soc.*, 46:179–186, 1992.