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FUNCTIONAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH STATE-DEPENDENT DELAY

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ABSTRACT. In this paper we study the existence of solutions for the initial value problem for functional differential equations, as well as, for neutral functional differential equations of fractional order with state-dependent delay. The nonlinear alternative of Leray-Schauder type is the main tool in our analysis.

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1. INTRODUCTION

The purpose of this paper is to study the existence of solutions for initial value problems (IVP for short) for both functional differential equations of fractional order with state-dependent delay

$$D^\beta y(t) = f(t, y_{\rho(t,y_t)}), \quad \text{for each } \ t \in J = [0, b], \quad 0 < \beta < 1, \tag{1.1}$$

$$y(t) = \varphi(t), \quad t \in (-\infty, 0] \tag{1.2}$$

as well as for neutral functional differential equations of fractional order with state-dependent delay

$$D^\beta[y(t) - g(t, y_{\rho(t,y_t)})] = f(t, y_{\rho(t,y_t)}), \quad \text{for each } \ t \in J, \tag{1.3}$$

$$y(t) = \varphi(t), \ t \in (-\infty, 0], \tag{1.4}$$

where D^β is the standard Riemman-Liouville fractional derivative.

Here, $f : J \times \mathcal{B} \to \mathbb{R}$, $g : J \times \mathcal{B} \to \mathbb{R}$ and $\rho : J \times \mathcal{B} \to (-\infty, b]$ are appropriate given functions, $\varphi \in \mathcal{B}$, $\varphi(0) = 0$, $g(0, \varphi) = 0$ and $\mathcal{B}$ is called a *phase space* that will be defined later (see Section 2).

For any function y defined on $(-\infty, b]$ and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

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The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [12] (see also Kappel and Schappacher [19] and Schumacher [30]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [18].

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equation has received a significant amount of attention in the last years, we refer to [2, 3, 5, 7, 13, 14, 15] and the references therein. On the other hand, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [9, 17, 20, 22, 24, 25, 26, 27] and references therein.

Differential equations of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics and other fields can be described with the help of fractional differential equations, see [8, 10, 17, 23, 27, 28, 29] and references therein. The theory of differential equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to fractional differential equations, for example see [1, 20, 21, 26, 31].

Our approach is based on the nonlinear alternative of Leray-Schauder type [11]. These results can be considered as a contribution to this emerging field.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, and let $C^0(\mathbb{R}^+)$ be the space of all continuous functions on $\mathbb{R}^+$. Consider also the space $C^0(\mathbb{R}_0^+)$ of all continuous real functions on

$$\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\},$$

which later identify by abuse of notation, with the class of all $f \in C^0(\mathbb{R}^+)$ such that $\lim_{t\to 0^+} f(t) = f(0^+) \in \mathbb{R}$.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\},$$

where $|\cdot|$ denotes a suitable complete norm on $\mathbb{R}$.

Now, we recall some definitions and facts about fractional derivatives and fractional integrals of arbitrary orders, see [20, 25, 26, 27].

Definition 2.1. The fractional primitive of order $\beta > 0$ of a function $h : \mathbb{R}^+ \to \mathbb{R}$ of order $\beta \in \mathbb{R}^+$ is defined by

$$I_0^\beta h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s)ds,$$

provided the right hand side exists pointwise on $\mathbb{R}^+$. Γ is the gamma function. For instance, $I^\beta h$ exists for all $\beta > 0$, when $h \in C^0(\mathbb{R}^+) \cap L^1_{loc}(\mathbb{R}^+)$; note also that when $h \in C^0(\mathbb{R}_0^+)$ then $I^\beta h \in C^0(\mathbb{R}_0^+)$ and moreover $I^\beta h(0) = 0$.

Definition 2.2. The fractional derivative of order $\beta > 0$ of a continuous function $h : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\begin{aligned} \frac{d^\beta h(t)}{dt^\beta} &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t (t-s)^{-\beta} h(s)ds \\ &= \frac{d}{dt} I_a^{1-\beta} h(t). \end{aligned}$$

For our existence results we will need the following lemma.

Lemma 2.3. [6] *Let $0 < \beta < 1$ and let $h : (0, b] \to \mathbb{R}$ be continuous and $\lim_{t \to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(y(s))ds,$$

if and only if, y is a solution of the initial value problem for the fractional differential equation

$$D^\beta y(t) = h(y(t)), \ \ t \in (0, b],$$

$$y(0) = 0.$$

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [18]. More precisely, $\mathcal{B}$ will be a linear space of all functions from $(-\infty, 0]$ to $\mathbb{R}$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ satisfying the following axioms:

(A) If $y : (-\infty, b] \to \mathbb{R}$, $b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:
  (i) $y_t \in \mathcal{B}$,
  (ii) $\|y_t\|_{\mathcal{B}} \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}$,
  (iii) $|y(t)| \le H\|y_t\|_{\mathcal{B}}$,
  where $H > 0$ is a constant, $K : [0, \infty) \to [1, \infty)$ is continuous, $M : [0, \infty) \to [1, \infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in (A), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A-2) The space $\mathcal{B}$ is complete.

The next lemma is a consequence of the phase space axioms and is proved in [13].

Lemma 2.4. *Let $\varphi \in \mathcal{B}$ and $I = (\gamma, 0]$ be such that $\varphi_t \in \mathcal{B}$ for every $t \in I$. Assume that there exists a locally bounded function $J^\varphi : I \to [0, \infty)$ such that $\|\varphi_t\|_\mathcal{B} \le J^\varphi(t)\|\varphi\|_\mathcal{B}$ for every $t \in I$. If $y : (\infty, b] \to \mathbb{R}$ is continuous on J and $y_0 = \varphi$, then*

$$\|y_t\|_\mathcal{B} \le (M_b + J^\varphi(\max\{\gamma, -|s|\})\|\varphi\|_\mathcal{B} + K_b \sup\{|y(\theta)| : \theta \in [0, \max\{0, s\}]\}, \quad s \in (\gamma, b],$$

where, we denoted $K_b = \sup_{t\in J} K(t)$ and $M_b = \sup_{t\in J} M(t)$.

Finally, we state the following generalization of Gronwall's lemma for singular kernels, whose proof can be found in [16], Lemma 7.1.1, will be essential for our main results.

Lemma 2.5. *Let $v : [0, b] \to [0, \infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, b]$ and there are constants $a > 0$ and $0 < \beta < 1$ such that*

$$v(t) \le w(t) + a \int_0^t \frac{v(s)}{(t-s)^\beta} ds,$$

then, there exists a constant $K = K(\beta)$ such that

$$v(t) \le w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\beta} ds,$$

for every $t \in [0, b]$.

3. FDEs OF FRACTIONAL ORDER

In this section, the nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions of problem (1.1)–(1.2).

Let us start by defining what we mean by a solution of problem (1.1)–(1.2).

Definition 3.1. A function $y : (-\infty, b] \to \mathbb{R}$ is said to be a solution of (1.1)–(1.2) if $y_0 = \varphi, y_{\rho(s,y_s)} \in \mathcal{B}$ for every $s \in J$ and

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s, y_{\rho(s,y_s)})}{(t-s)^{1-\beta}} ds, \quad t \in J.$$

In what follows we assume that $\rho : J \times B \to (-\infty, b]$ is continuous and $\varphi \in \mathcal{B}$ and f satisfies the following hypotheses:

(H1) f is a continuous function;

(H2) There exist $p, q \in C(J, \mathbb{R}^+)$ such that

$$|f(t, u)| \le p(t) + q(t)\|u\|_\mathcal{B}$$

for $t \in J$ and each $u \in B$, and $\|I^\beta p\|_\infty < +\infty$;

(H3) The function $t \to \varphi_t$ is well defined and continuous from the set $\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in J \times B, \rho(s,\psi) \le 0\}$ into $\mathcal{B}$. Moreover, there exists a continuous and bounded function $J^{\varphi} : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \le J^{\varphi}(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

Remark 3.2. The hypothesis (H3) is adapted from [13], where we refer for remarks concerning this hypothesis.

Theorem 3.3. *Assume that the hypotheses (H1)–(H3) hold. If $\rho(t,\psi) \le t$ for every $(t,\psi) \in J \times \mathcal{B}$, then the IVP* (1.1)–(1.2) *has at least one solution on* $(-\infty, b]$.

Proof. Let $Y = \{u \in C(J,\mathbb{R}) : u(0) = \varphi(0) = 0\}$ endowed with the uniform convergence topology and $N : Y \to Y$ be the operator defined by

$$Ny(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s, \bar{y}_{\rho(s,\bar{y}_s)})}{(t-s)^{1-\beta}}\, ds, \quad t \in J,$$

where $\bar{y} : (-\infty, b] \to \mathbb{R}$ is such that $\bar{y}_0 = \varphi$ and $\bar{y} = y$ on J. From axiom (A) and our assumption on φ, we infer that $Ny(\cdot)$ is well defined and continuous.

Let $\bar{\varphi} : (-\infty, b] \to \mathbb{R}$ be the extension of φ to $(-\infty, b]$ such that $\bar{\varphi}(\theta) = \varphi(0) = 0$ on J and $\tilde{J}^{\varphi} = \sup\{J^{\varphi} : s \in \mathcal{R}(\rho^-)\}$.

We will prove that $N(\cdot)$ is completely continuous from $B_r(\bar{\varphi}|_J, Y)$ into $B_r(\bar{\varphi}|_J, Y)$.

Step 1: *N is continuous on $B_r(\bar{\varphi}|_J, Y)$.*

This was proved in [13, p. 515, Step 3].

Step 2: *The set $N(B_r(\bar{\varphi}|_J, Y))(t) = \{Ny(t) : y \in B_r(\bar{\varphi}|_J, Y)\}$ is relatively compact in $\mathbb{R}$ for every $t \in J$.*

The case $t = 0$ is obvious. Let $0 < \epsilon < t \le b$. If $y \in B_r(\bar{\varphi}|_J, Y)$, from Lemma 2.4 follows that

$$\|\bar{y}_{\rho(t,\bar{y}_t)}\|_{\mathcal{B}} \le r^* = (M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b r$$

and so

$$\begin{aligned} |(Ny)(t)| &= \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s, \bar{y}_{\rho(s,\bar{y}_s)})}{(t-s)^{1-\beta}}\, ds \\ &\le \frac{1}{\Gamma(\beta)} \int_0^t \frac{p(s) + q(s)\|\bar{y}_{\rho(s,\bar{y}_s)}\|_{\mathcal{B}}}{(t-s)^{1-\beta}}\, ds \\ &= \frac{b^{\beta}\|p\|_{\infty}}{\Gamma(\beta+1)} + \frac{b^{\beta}\|q\|_{\infty}}{\Gamma(\beta+1)} r^* := \ell \end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of Y.*

Let $t_1, t_2 \in [0,b]$, $t_1 < t_2$ and let B_r as in Step 2. Let $y \in B_r$. Then for each $t \in [0,b]$, we have

$$\begin{aligned}
|(Ny)(t_2)-(Ny)(t_1)| &= \frac{1}{\Gamma(\beta)}\Big|\int_0^{t_1}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]f(s,\bar{y}_{\rho(s,\bar{y}_s)})\,ds \\
&\quad +\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}\frac{f(s,\bar{y}_{\rho(s,\bar{y}_s)})}{(t_2-s)^{1-\beta}}\,ds\Big| \\
&\le \frac{\|p\|_\infty + r^*\|q\|_\infty}{\Gamma(\beta)}\int_0^{t_1}[(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}]ds \\
&\quad +\frac{\|p\|_\infty + r^*\|q\|_\infty}{\Gamma(\beta)}\int_{t_1}^{t_2}\frac{ds}{(t_2-s)^{1-\beta}} \\
&\le \frac{\|p\|_\infty + r^*\|q\|_\infty}{\Gamma(\beta+1)}[(t_2-t_1)^\beta + t_1^\beta - t_2^\beta] \\
&\quad +\frac{\|p\|_\infty + r^*\|q\|_\infty}{\Gamma(\beta+1)}(t_2-t_1)^\beta \\
&\le \frac{2(\|p\|_\infty + r^*\|q\|_\infty)}{\Gamma(\beta+1)}(t_2-t_1)^\beta.
\end{aligned}$$

As $t_1 \longrightarrow t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \le 0$ and $t_1 \le 0 \le t_2$ is obvious.

As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that N is continuous and completely continuous.

Step 4: (A priori bounds). *We now show there exists an open set $U \subseteq Y$ with $y \ne \lambda N(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$.*

Let $y \in Y$ and $y = \lambda N(y)$ for some $0 < \lambda < 1$. Then for each $t \in [0,b]$ we have

$$y(t) = \lambda\Big[\frac{1}{\Gamma(\beta)}\int_0^t \frac{f(s,\bar{y}_{\rho(s,\bar{y}_s)})}{(t-s)^{1-\beta}}\,ds\Big].$$

This implies by (H2)

$$\begin{aligned}
|y(t)| &= \frac{1}{\Gamma(\beta)}\int_0^t \frac{f(s,\bar{y}_{\rho(s,\bar{y}_s)})}{(t-s)^{1-\beta}}\,ds \\
&\le \frac{1}{\Gamma(\beta)}\int_0^t \frac{p(s)+q(s)[(M_b+\tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b\sup\{|\bar{y}(s)| : s\in[0,t]\}]}{(t-s)^{1-\beta}}\,ds \\
&\le \frac{b^\beta\|p\|_\infty}{\Gamma(\beta+1)} + \frac{\|q\|_\infty}{\Gamma(\beta)}\int_0^t \frac{(M_b+\tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b\sup\{|\bar{y}(s)| : s\in[0,t]\}}{(t-s)^{1-\beta}}\,ds,
\end{aligned}$$

since $\rho(s,\bar{y}_s) \le s$ for every $s \in J$. Here $\bar{J}^\phi = \sup\{J^\phi(s) : s \in \mathcal{R}(\rho^-)\}$.

If $\mu(t) = (M_b+\tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b\sup\{|\bar{y}(s)| : s\in[0,\max\{0,\rho(s,\bar{y}_t)\}]$ then we obtain

$$\mu(t) \le \frac{b^\beta\|p\|_\infty}{\Gamma(\beta+1)} + \frac{\|q\|_\infty}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mu(s)ds.$$

Then from Lemma 2.5, there exists $K = K(\beta)$ such that we have

$$\begin{aligned}
|\mu(t)| &\leq \frac{b^\beta \|p\|_\infty}{\Gamma(\beta+1)} + \frac{\|q\|_\infty}{\Gamma(\beta)} K(\beta) \int_0^t (t-s)^{\beta-1} \frac{b^\beta \|p\|_\infty}{\Gamma(\beta+1)} ds \\
&= \frac{b^\beta \|p\|_\infty}{\Gamma(\beta+1)} \left\{ 1 + \frac{\|q\|_\infty}{\Gamma(\beta)} K(\beta) \int_0^t (t-s)^{\beta-1} ds \right\} \\
&\leq \frac{b^\beta \|p\|_\infty}{\Gamma(\beta+1)} \left\{ 1 + \frac{\|q\|_\infty}{\Gamma(\beta+1)} K(\beta) b^\beta \right\}.
\end{aligned}$$

Then

$$\|\mu\|_\infty \leq \frac{b^\beta \|p\|_\infty}{\Gamma(\beta+1)} \left\{ 1 + \frac{\|q\|_\infty}{\Gamma(\beta+1)} K(\beta) b^\beta \right\} := M^*.$$

Set

$$U = \{ y \in Y : \|y\|_\infty < M^* + 1 \}.$$

$N : \overline{U} \to Y$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N(y)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [11], we deduce that N has a fixed point y in U. □

4. NFDEs OF FRACTIONAL ORDER

In this section we give existence results for the IVP (1.3)–(1.4).

Definition 4.1. A function $y : (-\infty, b] \to \mathbb{R}$ is said to be a solution of (1.3)–(1.4) if $y_0 = \varphi, y_{\rho(s,y_s)} \in \mathcal{B}$ for every $s \in J$ and

$$y(t) = g(s, y_{\rho(s,y_s)}) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y_{\rho(s,y_s)}) ds, \quad t \in J.$$

Theorem 4.2. *Assume (H1)–(H2) and the following condition:*

(H4) *the function g is continuous and completely continuous, and for any bounded set Q in $\mathcal{B} \cap C([0,b],\mathbb{R})$, the set $\{t \to g(t, y_t) : y \in Q\}$ is equicontinuous in $C([0,b],\mathbb{R})$, and there exist constants $0 \leq K_b d_1 < 1$, $d_2 \geq 0$ such that*

$$|g(t,u)| \leq d_1 \|u\|_{\mathcal{B}} + d_2, \quad t \in [0,b], \quad u \in B.$$

If $\rho(t,\psi) \leq t$ for every $(t,\psi) \in J \times \mathcal{B}$, then the IVP (1.3)*–*(1.4) *has at least one solution on $(-\infty, b]$.*

Proof. Consider the operator $N_0 : C((-\infty,b],\mathbb{R}) \to C((-\infty,b],\mathbb{R})$ defined by,

$$N_0(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, 0], \\ \varphi(0) - g(0,\varphi) + g(t, y_{\rho(t,y_t)}) & \\ +\dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} f(s, y_{\rho(s,y_s)}) ds, & \text{if } t \in [0,b]. \end{cases}$$

In analogy to Theorem 3.3, we consider the operator $N_1 : Y \to Y$ defined by

$$(N_1y)(t) = \begin{cases} 0, & t \le 0, \\ g(t, \bar{y}_{\rho(s,\bar{y}_s)}) + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} f(s, \bar{y}_{\rho(s,\bar{y}_s)}) ds, & t \in [0,b]. \end{cases}$$

We shall show that the operator N_1 is continuous and completely continuous. Using (H4) it suffices to show that the operator $N_2 : Y \to Y$ defined by,

$$N_2(y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds, \quad t \in [0,b],$$

is continuous and completely continuous. This was proved in Theorem 3.3.

We now show there exists an open set $U \subseteq Y$ with $y \neq \lambda N_1(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$.

Let $y \in Y$ and $y = \lambda N_1(y)$ for some $0 < \lambda < 1$. Then

$$y(t) = \lambda \left[g(s, \bar{y}_{\rho(s,\bar{y}_s)}) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds \right], \quad t \in [0,b],$$

and

$$\begin{aligned} |y(t)| &\le d_1((M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{|y(s)| : s \in [0,t]\}) + d_2 \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [p(s) + q(s)((M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{|y(s)| : s \in [0,t]\})ds \\ &\le d_1((M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{|y(s)| : s \in [0,t]\}) + d_2 \\ &\quad + \frac{b^{\beta}\|p\|_{\infty}}{\Gamma(\beta+1)} + \frac{\|q\|_{\infty}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}((M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \sup\{|y(s)| : s \in [0,t]\})ds, \end{aligned}$$

for $t \in (0,b]$. If $\mu(t) = \sup\{|y(s)| : s \in [0,t]\}$ then

$$\begin{aligned} \mu(t) &\le d_1(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + d_1K_b\mu(t) + d_2 \\ &\quad + \frac{b^{\beta}\|p\|_{\infty}}{\Gamma(\beta+1)} + \frac{b^{\beta}\|q\|_{\infty}}{\Gamma(\beta+1)} \int_0^t (t-s)^{\beta-1}(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} \\ &\quad + \frac{\|q\|_{\infty}}{\Gamma(\beta)} K_b \int_0^t (t-s)^{\beta-1}\mu(s)\, ds \\ &\le d_1(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + d_1K_b\mu(t) + d_2 \\ &\quad + \frac{b^{\beta}\|p\|_{\infty}}{\Gamma(\beta+1)} + \frac{b^{\beta}\|q\|_{\infty}}{\Gamma(\beta+1)} b^{\beta-1}(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} \\ &\quad + K_b \frac{\|q\|_{\infty}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\mu(s)\, ds, \quad t \in (0,b], \end{aligned}$$

or

$$\begin{aligned}
&\mu(t)\\
&\leq \frac{1}{1-K_b d_1}\left[d_1(M_b+\tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}}+d_2+\frac{b^{\beta}\|p\|_{\infty}}{\Gamma(\beta+1)}+\frac{b^{\beta}\|q\|_{\infty}}{\Gamma(\beta+1)}(M_b+\tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}}\right]\\
&\quad+\frac{K_b}{1-K_b d_1}\frac{\|q\|_{\infty}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mu(s)\,ds, \quad t\in(0,b].
\end{aligned}$$

Consequently

$$\begin{aligned}
&\|\mu\|_{\infty}\\
&\leq \frac{1}{1-K_b d_1}\left[d_1(M_b+\tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}}+d_2+\frac{b^{\beta}\|p\|_{\infty}}{\Gamma(\beta+1)}+\frac{b^{\beta}\|q\|_{\infty}}{\Gamma(\beta+1)}(M_b+\tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}}\right]\\
&\quad+\frac{K_b}{1-K_b d_1}\frac{\|q\|_{\infty}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mu(s)\,ds
\end{aligned}$$

and by Lemma 2.5, there exists $K=K(\beta)$ such that

$$\|\mu\|_{\infty}\leq \Lambda_1+\Lambda_2 K(\beta)\int_0^t (t-s)^{\beta-1}\,\Lambda_1\,ds\leq \Lambda_1+\Lambda_2 K(\beta)\Lambda_1 b^{\beta}:=L^*,$$

where

$$\Lambda_1=\frac{1}{1-K_b d_1}\left[d_1(M_b+\tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}}+d_2+\frac{b^{\beta}\|p\|_{\infty}}{\Gamma(\beta+1)}+\frac{b^{\beta}\|q\|_{\infty}}{\Gamma(\beta+1)}(M_b+\tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}}\right],$$

$$\Lambda_1=\frac{K_b}{1-K_b d_1}\frac{\|q\|_{\infty}}{\Gamma(\beta)}.$$

Set

$$U_1=\{y\in Y:\|y\|_{\infty}<L^*+1\}.$$

From the choice of U there is no $y\in\partial U_1$ such that $y=\lambda N_1(y)$ for $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [11], we deduce that N_1 has a fixed point y in U_1. Then N_1 has a fixed point, which is a solution of the IVP (1.3)–(1.4). □

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