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OSCILLATION OF SECOND ORDER NEUTRAL EMDEN-FOWLER DELAY DYNAMIC EQUATIONS OF MIXED TYPE

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ABSTRACT. By means of generalized Riccati transformation and averaging technique, we establish some oscillation criteria for the second-order neutral Emden-Fowler delay dynamic equation of mixed type

$$(r(t)x^{\Delta}(t))^{\Delta} + q_1(t)|y(\delta(t))|^{\alpha-1}y(\delta(t)) + q_2(t)|y(\delta(t))|^{\beta-1}y(\delta(t)) = 0,$$

on a time scale $\mathbb{T}$. Our results as a special case when $\mathbb{T} = \mathbb{R}$ improve some well known oscillation criteria for second order neutral Emden-Fowler delay differential equation of mixed type, and when $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = h\mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0}$, i.e., for neutral delay difference equations, neutral delay difference equations with constant step size, and q-neutral difference equations with variable step size. The results obtained here are essentially new and can be applied to different types of time scales. Some applications and examples are considered to illustrate the main results.

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1. INTRODUCTION

Consider the second order neutral Emden-Fowler dynamic equation of mixed type

$$(r(t)x^{\Delta}(t))^{\Delta} + q_1(t)|y(\delta(t))|^{\alpha-1}y(\delta(t)) + q_2(t)|y(\delta(t))|^{\beta-1}y(\delta(t)) = 0, \tag{1.1}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$ where $\mathbb{T}$ is a time scale unbounded above, and $x(t) = y(t)+p(t)y(\tau(t))$. Throughout this paper we assume that:

(A1) α and β are positive constants with $0 < \alpha < 1 < \beta$;
(A2) r, p, q_1, $q_2 \in \mathbf{C}_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ with $0 \le p(t) < 1$ and $\int^{\infty} 1/r(t)\Delta t = \infty$;
(A3) $\tau, \delta \in \mathbf{C}_{rd}([t_0,\infty)_{\mathbb{T}}, [t_0,\infty)_{\mathbb{T}})$ with $\tau(t) \le t$, $\delta(t) \le t$ for $t \in [t_0,\infty)_{\mathbb{T}}$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$.

For completeness, we recall the following concepts related to the notion of time scales, see [1, 7, 15] for more details. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and, since oscillation of solutions is our primary concern, we make the blanket assumption that $\sup \mathbb{T} = \infty$. In this paper we assume that $\mathbb{T}$

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has the topology which inherits from the standard topology on the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$ we define the forward operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\},$$

respectively. In this definition we put $\inf \emptyset := \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset := \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ and $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $\mathbf{C}_{rd}(\mathbb{T})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ which are differentiable and whose derivative is rd-continuous function is denoted by $\mathbf{C}^1_{rd}(\mathbb{T}, \mathbb{R})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

Recall that a solution of (1.1) is a nontrivial real function $y(t)$ such that $y(t) + p(t)y(\tau(t)) \in \mathbf{C}^2_{rd}([t_y, \infty)_\mathbb{T}, \mathbb{R})$ for $t_y \geq t_0$ and satisfies Eq. (1.1) for $t > t_y$. Our attention is restricted to those solutions of Eq. (1.1) which exist on some half-linear $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1\}$ for any $t_1 > t_y$. A solution $y(t)$ of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Equation (1.1) in its general form involves different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = t, \quad \mu(t) = 0, \quad f^{\triangle}(t) = f'(t), \quad \int_a^b f(t) \triangle t = \int_a^b f(t) dt,$$

then Eq. (1.1) becomes the second order neutral delay differential equation

$$(r(t)x'(t))' + q_1(t)|y(\delta(t))|^{\alpha-1}y(\delta(t)) + q_2(t)|y(\delta(t))|^{\beta-1}y(\delta(t)) = 0. \tag{1.2}$$

If $\mathbb{T} = \mathbb{N}$, we have

$$\sigma(t) = t + 1, \; \mu(t) = 1, \; f^{\triangle}(t) = \triangle f(t), \; \int_a^b f(t) \triangle t = \sum_{t=a}^{b-1} f(t),$$

then Eq. (1.1) becomes the neutral difference equation

$$\Delta(r(t)\Delta x(t)) + q_1(t)|y(\delta(t))|^{\alpha-1}y(\delta(t)) + q_2(t)|y(\delta(t))|^{\beta-1}y(\delta(t)) = 0. \tag{1.3}$$

If $\mathbb{T} = h\mathbb{N} := \{hk : k \in \mathbb{N}, h > 0\}$, we have $\sigma(t) = t + h$, $\mu(t) = h$, and

$$x^{\triangle}(t) = \triangle_h x(t) = \frac{x(t+h) - x(t)}{h}, \quad \int_a^b f(t)\triangle t = \sum_{k=0}^{(b-a-h)/h} f(a+kh)h,$$

then Eq. (1.1) becomes the neutral difference equation with constant step size

$$\Delta_h(r(t)\Delta_h x(t)) + q_1(t)|y(\delta(t))|^{\alpha-1}y(\delta(t)) + q_2(t)|y(\delta(t))|^{\beta-1}y(\delta(t)) = 0. \tag{1.4}$$

If $\mathbb{T} = q^{\mathbb{N}_0} := \{t : t = q^n, n \in \mathbb{N}_0, q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, and

$$x^{\triangle}(t) = \triangle_q x(t) = \frac{x(qt) - x(t)}{(q-1)t}, \quad \int_a^b f(t)\triangle t = \sum_{t\in[a,b)} f(t)\mu(t),$$

then Eq. (1.1) becomes the neutral q-difference equation with variable step size

$$\Delta_q(r(t)\Delta_q x(t)) + q_1(t)|y(\delta(t))|^{\alpha-1}y(\delta(t)) + q_2(t)|y(\delta(t))|^{\beta-1}y(\delta(t)) = 0. \tag{1.5}$$

It is well known that the theory of time scales unifies continuous and discrete analysis, and the theory of "dynamic equations" unifies the theories of differential equations and difference equations, and it also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations. Of course, many other interesting time scales exist, and they give rise to plenty of application [3, 7, 8, 15]. The oscillation theory of difference and differential equations has been developed extensively during the past years, we refer the reader to the monographs [1, 2, 4]. In recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solution of the second order Emden-Fowler delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^{\gamma}(\sigma(t)) = 0 \tag{1.6}$$

or

$$(rx^{\Delta})^{\Delta}(t) + q(t)x^{\gamma}(\sigma(t)) = 0, \tag{1.7}$$

see, for example, [5, 6, 9, 10, 13, 14, 17] and the references therein. However, the mixed type Emden-Fowler delay dynamic equation (1.1) has never been the subject of systematic investigations.

In this paper, following the ideas in [11, 16, 18], we establish some oscillation criteria for (1.1) on a time scale $\mathbb{T}$. Our results as a special case when $\mathbb{T} = \mathbb{R}$ improve some well known oscillation criteria [18] for second order neutral mixed type Emden-Fowler delay differential equation (1.2), and when $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = h\mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0}$, i.e., for neutral delay difference equations, neutral delay difference equations with constant step size, and second order q-neutral difference equations with variable step size. The results obtained here are essentially new and can be applied to arbitrary time scales. Finally, some applications and examples are considered to illustrate the main results.

2. MAIN RESULTS

Before we state and prove our main results, we present the following lemma which is important in the proof of our main results. For simplicity of statement, we define the following notation without further mentioning.

$$\Theta(t) := (\beta-\alpha)[1-p(\delta(t))]\left[(\beta-1)^{1-\beta}(1-\alpha)^{\alpha-1}q_1^{\beta-1}(t)q_2^{1-\alpha}(t)\right]^{1/(\beta-\alpha)}.$$

Lemma 2.1. *Assume that*

$$\int_{t_0}^{\infty}\delta(t)\Theta(t)\Delta t=\infty \tag{2.1}$$

holds, and y is an eventually positive solution of Eq. (1.1). *Then there exists a $T\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that:*

(1) *$x(t)>0$, $x^{\Delta}(t)>0$, $x(t)>tx^{\Delta}(t)$ for $t\in[T,\infty)_{\mathbb{T}}$;*
(2) *x is strictly increasing and $x(t)/t$ is strictly decreasing on $[T,\infty)_{\mathbb{T}}$.*

Proof. Let y be an eventually positive solution of Eq. (1.1). Then there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $t_1>0$, and

$$y(t)>0,\quad y(\tau(t))>0,\quad y(\delta(t))>0\quad for\quad t\in[t_1,\infty)_{\mathbb{T}},$$

which follows that $x(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$, since $p(t)>0$.

From (1.1), we have

$$(r(t)x^{\Delta}(t))^{\Delta}=-q_1(t)|y(\delta(t))|^{\alpha-1}y(\delta(t))-q_2(t)|y(\delta(t))|^{\beta-1}y(\delta(t))<0,\quad t\in[t_1,\infty)_{\mathbb{T}},$$

which implies that $r(t)x^{\Delta}(t)$ is an eventually decreasing function. We claim that $r(t)x^{\Delta}(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$. Assume not, then there is a $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $r(t_2)x^{\Delta}(t_2)=:d_1<0$. Then

$$r(t)x^{\Delta}(t)\le r(t_2)x^{\Delta}(t_2)=d_1,\quad t\in[t_2,\infty)_{\mathbb{T}},$$

which follows that

$$x^{\Delta}(t)\le\frac{d_1}{r(t)},\quad t\in[t_2,\infty)_{\mathbb{T}}.$$

Integrating the above from t_2 to t, we get, by (A2),

$$x(t)=x(t_2)+\int_{t_2}^{t}x^{\Delta}(s)\Delta s\le x(t_2)+d_1\int_{t_2}^{t}\frac{1}{r(s)}\Delta s\to-\infty\quad as\quad t\to\infty,$$

which implies $x(t)$ is eventually negative. This is a contradiction. Hence, $r(t)x^{\Delta}(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$ and so $x^{\Delta}(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore, $x(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. Let

$$\chi(t):=x(t)-tx^{\Delta}(t).$$

We claim that there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $\chi(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. If not, then $\chi(t) < 0$ on $[t_2,\infty)_{\mathbb{T}}$. Hence,

$$\Big(\frac{x(t)}{t}\Big)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{\chi(t)}{t\sigma(t)} > 0, \quad t \in [t_2,\infty)_{\mathbb{T}},$$

which implies that $x(t)/t$ is strictly increasing on $[t_2,\infty)_{\mathbb{T}}$. Pick $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $\delta(t) \geq \delta(t_3)$ on $[t_3,\infty)_{\mathbb{T}}$. Then

$$x(\delta(t))/\delta(t) \geq x(\delta(t_3))/\delta(t_3) =: d_2 > 0,$$

so that $x(\delta(t)) \geq d_2\delta(t)$ on $[t_3,\infty)_{\mathbb{T}}$. Since $x(t) > 0$ and $x^{\Delta}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned} y(t) &= x(t) - p(t)y(\tau(t)) \\ &= x(t) - p(t)[x(\tau(t)) - p(\tau(t))y(\tau(\tau(t)))] \\ &\geq x(t) - p(t)x(\tau(t)) \geq [1-p(t)]x(t), \end{aligned}$$

consequently, $y(\delta(t)) \geq [1-p(\delta(t))]x(\delta(t))$. Therefore, (1.1) can be rewritten as

$$(r(t)x^{\Delta}(t))^{\Delta} + q_1(t)[1-p(\delta(t))]^{\alpha}x^{\alpha}(\delta(t)) + q_2(t)[1-p(\delta(t))]^{\beta}x^{\beta}(\delta(t)) \leq 0. \tag{2.2}$$

By the arithmetic and geometric inequality [12, Theorem 9], we have

$$\begin{aligned} &q_1(t)[1-p(\delta(t))]^{\alpha}x^{\alpha}(\delta(t)) + q_2(t)[1-p(\delta(t))]^{\beta}x^{\beta}(\delta(t)) \\ &\quad = x(\delta(t))\{q_1(t)[1-p(\delta(t))]^{\alpha}x^{\alpha-1}(\delta(t)) + q_2(t)[1-p(\delta(t))]^{\beta}x^{\beta-1}(\delta(t))\} \\ &\quad \geq \Theta(t)x(\delta(t)). \end{aligned}$$

Combining the above and (2.2), we get

$$(r(t)x^{\Delta}(t))^{\Delta} + \Theta(t)x(\delta(t)) \leq 0. \tag{2.3}$$

By integrating both sides of (2.3) from t_3 to t, we have

$$r(t_3)x^{\Delta}(t_3) \geq r(t)x^{\Delta}(t) + \int_{t_3}^{t}\Theta(s)x(\delta(s))\Delta s \geq d_2\int_{t_3}^{t}\delta(s)\Theta(s)\Delta s,$$

since $r(t)x^{\Delta}(t) > 0$. This contradicts (2.1). Hence, there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $\chi(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$, which follows that $x(t) > tx^{\Delta}(t)$ on $[t_2,\infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{x^{\Delta}(t)t - x(t)}{t\sigma(t)} = -\frac{\chi(t)}{t\sigma(t)} < 0, \quad t \in [t_2,\infty)_{\mathbb{T}},$$

so, we have that $x(t)/t$ is strictly decreasing on $[t_2,\infty)_{\mathbb{T}}$. □

Now, we employ the generalized Riccati transformation and averaging technique to establish oscillation criteria for Eq. (1.1). In the sequel, we use a class of functions introduced by Philos [16]. Let $D = \{(t,s) \in \mathbb{T}^2 : t_0 \leq s \leq \sigma(t)\}$. We say that a function $H \in \mathbf{C}^1_{rd}(D,[0,\infty))$ belongs to the class $\Im$, defined by $H \in \Im$, if it satisfies the following two conditions:

(H1) $H(t,s) \geq 0$ for $(t,s) \in D$, and $H(\sigma(t),s) = 0$ if and only if $s = t$;

(H2) $H_2^{\Delta}(t,s) \le 0$ for $t_0 \le s < \sigma(t)$, and there exists $h \in \mathbf{C}_{rd}(D,\mathbb{R})$ such that

$$H_2^{\Delta}(\sigma(t),s) = -h(t,s)\sqrt{H(\sigma(t),\sigma(s))},$$

where $H_2^{\Delta}(t,s)$ is delta derivative with respect to s.

Theorem 2.2. *Let* (2.1) *hold. Furthermore, assume that there exist functions* $H \in \Im$, $v \in \mathbf{C}_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *and* $r\eta \in \mathbf{C}_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{t} H(\sigma(t),\sigma(s))\Big[\phi(s) - \frac{1}{4} g(s)\varphi^2(t,s)\Big]\Delta s = \infty, \tag{2.4}$$

where

$$\phi(s) := \big[\delta(s)\Theta(s) + sr(s)\eta^2(s) - \sigma(s)[r(s)\eta(s)]^{\Delta}\big]\frac{v^{\sigma}(s)}{\sigma(s)},$$

$$g(s) := \frac{r(s)\sigma(s)}{sv^{\sigma}(s)}v^2(s), \qquad \varphi(t,s) := \frac{v^{\Delta}(s)}{v(s)} + \frac{2s\eta(s)v^{\sigma}(s)}{\sigma(s)v(s)} - \frac{h(t,s)}{\sqrt{H(\sigma(t),\sigma(s))}}.$$

Then Eq. (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1.1) with $y(\delta(t)) > 0$ and $y(\tau(t)) > 0$ for all $t \ge t_1$ sufficiently large, since the proof in the other case is similar. In view of Lemma 2.1, there is some $t_2 \ge t_1$, such that

$$x^{\Delta}(t) > 0, \quad (r(t)x^{\Delta}(t))^{\Delta} < 0 \quad for \quad t \ge t_2.$$

Define the function $w(t)$ by the generalized Riccati transformation

$$w(t) := v(t)\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)\eta(t)\right], \quad t \ge t_2.$$

Hence,

$$\begin{aligned} w^{\Delta}(t) &= \frac{v^{\Delta}(t)}{v(t)}w(t) + v^{\sigma}(t)\frac{(r(t)x^{\Delta}(t))^{\Delta}}{x^{\sigma}(t)} - v^{\sigma}(t)\frac{r(t)x^{\Delta}(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)} + v^{\sigma}(t)[r(t)\eta(t)]^{\Delta} \\ &\le \frac{v^{\Delta}(t)}{v(t)}w(t) - v^{\sigma}(t)\Theta(t)\frac{x(\delta(t))}{x^{\sigma}(t)} - v^{\sigma}(t)\frac{r(t)x(t)}{x^{\sigma}(t)}\Big(\frac{x^{\Delta}(t)}{x(t)}\Big)^2 + v^{\sigma}(t)[r(t)\eta(t)]^{\Delta}. \end{aligned} \tag{2.5}$$

From the definition of $w(t)$, we get

$$\frac{x^{\Delta}(t)}{x(t)} = \frac{w(t)}{v(t)r(t)} - \eta(t). \tag{2.6}$$

Also from Lemma 2.1, since $x(t)/t$ is strictly decreasing, we have

$$\frac{x(\delta(t))}{x^{\sigma}(t)} \ge \frac{\delta(t)}{\sigma(t)}, \qquad \frac{x(t)}{x^{\sigma}(t)} \ge \frac{t}{\sigma(t)}. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), we obtain

$$\begin{aligned} w^\Delta(t) &\le \frac{v^\Delta(t)}{v(t)}w(t) - v^\sigma(t)\Theta(t)\frac{\delta(t)}{\sigma(t)} \\ &\quad -v^\sigma(t)\frac{tr(t)}{\sigma(t)}\Big[\frac{w(t)}{v(t)r(t)} - \eta(t)\Big]^2 + v^\sigma(t)(r(t)\eta(t))^\Delta \\ &= -\phi(t) + \Big[\frac{v^\Delta(t)}{v(t)} + \frac{2t\eta(t)v^\sigma(t)}{\sigma(t)v(t)}\Big]w(t) - \frac{1}{g(t)}w^2(t). \end{aligned} \tag{2.8}$$

Evaluating both sides of (2.8) at s, multiplying by $H(\sigma(t),\sigma(s))$, integrating from t_2 to t, using integration by parts formula, and rearranging the terms, we get

$$\begin{aligned} &\int_{t_2}^t H(\sigma(t),\sigma(s))\phi(s)\Delta s \le -\int_{t_2}^t H(\sigma(t),\sigma(s))w^\Delta(s)\Delta s \\ &\quad + \int_{t_2}^t H(\sigma(t),\sigma(s))\Big[\frac{v^\Delta(s)}{v(s)} + \frac{2s\eta(s)v^\sigma(s)}{\sigma(s)v(s)}\Big]w(s)\Delta s - \int_{t_2}^t \frac{1}{g(s)}H(\sigma(t),\sigma(s))w^2(s)\Delta s \\ &\quad = H(\sigma(t),t_2)w(t_2) + \int_{t_2}^t H(\sigma(t),\sigma(s))\Big[\varphi(t,s)w(s) - \frac{1}{g(s)}w^2(s)\Big]\Delta s, \end{aligned}$$

which implies that, after completing the square, that

$$\int_{t_2}^t H(\sigma(t),\sigma(s))\phi(s)\Delta s \le H(\sigma(t),t_2)w(t_2) + \frac{1}{4}\int_{t_2}^t H(\sigma(t),\sigma(s))g(s)\varphi^2(t,s)\Delta s,$$

consequently,

$$\int_{t_2}^t H(\sigma(t),\sigma(s))\Big[\phi(s) - \frac{1}{4}g(s)\varphi^2(t,s)\Big]\Delta s \le H(\sigma(t),t_2)w(t_2) \le H(\sigma(t),t_2)|w(t_2)|.$$

Hence, for all $t \ge t_0$,

$$\begin{aligned} &\int_{t_0}^t H(\sigma(t),\sigma(s))\Big[\phi(s) - \frac{1}{4}g(s)\varphi^2(t,s)\Big]\Delta s \\ &\quad = \Big(\int_{t_0}^{t_2} + \int_{t_2}^t\Big)H(\sigma(t),\sigma(s))\Big[\phi(s) - \frac{1}{4}g(s)\varphi^2(t,s)\Big]\Delta s \\ &\quad \le H(\sigma(t),t_0)\Big[\int_{t_0}^{t_2}|\phi(s)|\Delta s + |w(t_2)|\Big]. \end{aligned}$$

Divide the above inequality by $H(\sigma(t),t_0)$ and take lim sup in it as $t \to \infty$, then condition (2.4) gives a desired contradiction. The proof is complete. □

As an immediate conclusion of Theorem 2.2, we obtain

Corollary 2.3. *In Theorem* 2.2, *if* (2.4) *is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)}\int_{t_0}^t H(\sigma(t),\sigma(s))\phi(s)\Delta s = \infty, \tag{2.9}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)}\int_{t_0}^t H(\sigma(t),\sigma(s))g(s)\varphi^2(t,s)\Delta s < \infty, \tag{2.10}$$

then Eq. (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

From Theorem 2.2, we can establish different sufficient conditions for the oscillation of Eq. (1.1) by using different choices of $v(t)$ and $\eta(t)$. For instance, if we consider $v(t) = t$, $\eta(t) = 1/t$, define $H(t, s)$ for $(t, s) \in D$ by $H(\sigma(t), t) = 0$, and $H(t, s) = 1$ otherwise, using Theorem 2.2, we get the following corollary.

Corollary 2.4. *Let* (2.1) *hold, and assume that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[\delta(s)\Theta(s) - \sigma(s)\Big(\frac{r(s)}{s}\Big)^{\Delta} - \frac{5r(s)}{4s}\Big]\Delta s = \infty. \tag{2.11}$$

Then Eq. (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

If in Theorem 2.2, we choose $\eta(t)$ and $v(t)$ such that

$$\eta(t) = -\frac{\sigma(t)v^{\Delta}(t)}{2tv^{\sigma}(t)}, \tag{2.12}$$

then we have

Corollary 2.5. *Let* (2.1) *hold, and assume that there exist functions* $H \in \Im$, $v \in \mathbf{C}^1_{rd}([t_0, \infty), \mathbb{R}^+)$ *such that* (2.12) *holds and*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{t} \Big[H(\sigma(t), \sigma(s))\phi(s) - \frac{\sigma(s)r(s)v^2(s)}{4sv^{\sigma}(s)}h^2(t, s)\Big]\Delta s = \infty, \tag{2.13}$$

where ϕ *is as in Theorem* 2.2. *Then* Eq. (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

If we choose $v(t)$ and $\eta(t)$ such that (2.12) holds, define $H(t, s)$ for $(t, s) \in D$ by $H(\sigma(t), t) = 0$, and $H(t, s) = 1$ otherwise, from Corollary 2.5, we have

Corollary 2.6. *Let* (2.1) *hold, and assume that there exists a function* $v \in \mathbf{C}_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ *such that* (2.12) *holds and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \phi(s)\Delta s = \infty, \tag{2.14}$$

where ϕ *is as in Theorem* 2.2. *Then* Eq. (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

From Corollary 2.6, we can also establish other sufficient conditions for the oscillation of Eq. (1.1) by using different choices of $v(t)$. For example, if $v(t) = t$, then $\eta(t) = -1/(2t)$, and if $v(t) = t^2$, then $\eta(t) = -(t + \sigma(t))/(2t\sigma(t))$, then by Corollary 2.6, we have the following oscillation results, respectively.

Corollary 2.7. *Let* (2.1) *hold, and assume that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[\delta(s)\Theta(s) + \frac{r(s)}{4s} + \frac{\sigma(s)}{2}\Big(\frac{r(s)}{s}\Big)^{\Delta}\Big]\Delta s = \infty. \tag{2.15}$$

Then Eq. (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Corollary 2.8. *Let* (2.1) *hold, and assume that*

$$(2.16)\quad \limsup_{t\to\infty}\int_{t_0}^{t}\sigma^2(s)\Big[\frac{\delta(s)}{\sigma(s)}\Theta(s)+\frac{r(s)(s+\sigma(s))^2}{4s\sigma^3(s)}+\Big(\frac{r(s)(s+\sigma(s))}{2s\sigma(s)}\Big)^{\Delta}\Big]\Delta s=\infty.$$

Then Eq. (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

3. APPLICATIONS

In this section, we apply Theorem 2.2 to different types of time scales and establish some oscillation criteria for Eqs.(1.2)–(1.5). For applications of the main results, we will give five interesting examples which are new and not be studied by any authors mentioned earlier.

We start with the case when $\mathbb{T}=[t_0,\infty)$. By using Theorem 2.2, we have the following oscillation result for Eq. (1.2).

Theorem 3.1. *Assume that*

$$(3.1)\qquad \int_{t_0}^{\infty}\frac{dt}{r(t)}=\infty \quad and \quad \int_{t_0}^{\infty}\delta(t)\Theta(t)dt=\infty.$$

Furthermore, assume that there exist functions $H\in\Im$, $v\in\mathbf{C}^1([t_0,\infty),\mathbb{R}^+)$ *and* $r\eta\in\mathbf{C}^1([t_0,\infty),\mathbb{R})$ *such that*

$$(3.2)\qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\Big[\phi(s)-\frac{1}{4}v(s)r(s)\varphi^2(t,s)\Big]ds=\infty,$$

where

$$\phi(s):=v(s)\Big[\frac{1}{s}\delta(s)\Theta(s)+r(s)\eta^2(s)-[r(s)\eta(s)]'\Big],$$

$$\varphi(t,s):=2\eta(s)+\frac{v'(s)}{v(s)}-\frac{h(t,s)}{\sqrt{H(t,s)}}.$$

Then Eq. (1.2) *is oscillatory on* $[t_0,\infty)$.

Remark 3.1. For Eq. (1.2), Theorem 3.1 improves Theorem 2.1 in Xu [18].

Define $H(t,s)$ by $H(t,t)=0$ and $H(t,s)=1$ for $t_0\le s<t$, by Theorem 3.1, we get

Corollary 3.2. *Let* (3.1) *hold, and assume that there exists a function* $v\in\mathbf{C}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *such that* (2.12) *holds and*

$$(3.3)\qquad \limsup_{t\to\infty}\int_{t_0}^{t}v(s)\Big[\frac{1}{s}\delta(s)\Theta(s)+r(s)\Big(\frac{v'(s)}{2v(s)}\Big)^2+\Big(\frac{r(s)v'(s)}{2v(s)}\Big)'\Big]ds=\infty.$$

Then Eq. (1.2) *is oscillatory on* $[t_0,\infty)$.

We now apply Theorem 2.2 to the time scale $\mathbb{T}=[t_0,\infty)_{\mathbb{N}}$, $t_0\in\mathbb{N}$, and establish some oscillation criteria for the delay difference equation Eq. (1.3).

Theorem 3.3. *Assume that*

$$\sum_{t=t_0}^{\infty}\frac{1}{r(t)}=\infty \quad \text{and} \quad \sum_{t=t_0}^{\infty}\delta(t)\Theta(t)=\infty. \tag{3.4}$$

Furthermore, assume that there exist $H\in\Im$, a sequence $\eta(t)$ and a positive sequence $v(t)$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t+1,t_0)}\sum_{s=t_0}^{t-1}H(t+1,s+1)\Big[\phi(s)-\frac{(s+1)r(s)}{4sv(s+1)}v^2(s)\varphi^2(t,s)\Big]=\infty, \tag{3.5}$$

where

$$\phi(s):=\frac{v(s+1)}{s+1}\Big[\delta(s)\Theta(s)+sr(s)\eta^2(s)-(s+1)\Delta(r(s)\eta(s))\Big],$$
$$\varphi(t,s):=\frac{\Delta v(s)}{v(s)}+\frac{2s\eta(s)v(s+1)}{(s+1)v(s)}-\frac{h(t,s)}{\sqrt{H(t+1,s+1)}}.$$

Then Eq. (1.3) *is oscillatory on* $[t_0,\infty)_{\mathbb{N}}$.

From Theorem 3.3, we can establish different sufficient conditions for the oscillation of (1.3) by using different choices of H, v and η. For instance, let $v(t)=t$, $\eta(t)=1/t$, define $H(t,s)$ by $H(t+1,t)=0$, and $H(t,s)=1$ otherwise, then, by Theorem 3.3, we get

Corollary 3.4. *Assume that* (3.4) *holds, and*

$$\limsup_{t\to\infty}\sum_{s=t_0}^{t-1}\Big[\Theta(s)\delta(s)-(s+1)\Delta\Big(\frac{r(s)}{s}\Big)-\frac{5r(s)}{4s}\Big]=\infty. \tag{3.6}$$

Then Eq. (1.3) *is oscillatory on* $[t_0,\infty)_{\mathbb{N}}$.

When $\mathbb{T}=[n_0h,\infty)_{h\mathbb{N}}$ for $h>0$ and $n_0\in\mathbb{N}$, from Theorem 2.2, we have

Theorem 3.5. *Assume that*

$$\sum_{k=n_0}^{\infty}\frac{1}{r(kh)}=\infty \quad \text{and} \quad \sum_{k=n_0}^{\infty}\delta(kh)\Theta(kh)=\infty. \tag{3.7}$$

Furthermore, assume that there exist $H\in\Im$, a sequence $\eta(t)$ and a positive sequence $v(t)$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{H(nh+h,n_0h)}\sum_{k=n_0}^{n-1}H(nh+h,kh+h)\\ &\qquad\times\Big[\phi(kh)-\frac{(k+1)r(kh)}{4kv(kh+h)}v^2(kh)\varphi^2(nh,kh)\Big]=\infty,\end{aligned} \tag{3.8}$$

where

$$\phi(kh) := \frac{v(kh+h)}{kh+h}[\delta(kh)\Theta(kh) + khr(kh)\eta^2(kh) - (kh+h)\Delta_h(r(kh)\eta(kh))],$$

$$\varphi(nh,kh) := \frac{\Delta_h v(kh)}{v(kh)} + \frac{2k\eta(kh)v(kh+h)}{(k+1)v(kh)} - \frac{h(nh,kh)}{\sqrt{H(nh+h,kh+h)}}.$$

Then Eq. (1.4) *is oscillatory on* $[n_0h,\infty)_{h\mathbb{N}}$.

Define $H(t,s)$ by $H(t+h,t) = 0$, and $H(t,s) = 1$ otherwise, let $v(t) = t$ and $\eta(t) = -1/(2t)$, then, by Theorem 3.5, we have:

Corollary 3.6. *Assume that* (3.7) *holds, and*

$$\limsup_{n\to\infty} \sum_{k=n_0}^{n-1} \left[\delta(kh)\Theta(kh) + \frac{r(kh)}{4kh} + \frac{1}{2}h(k+1)\Delta_h\left(\frac{r(kh)}{kh}\right)\right] = \infty. \tag{3.9}$$

Then Eq. (1.4) *is oscillatory on* $[n_0h,\infty)_{h\mathbb{N}}$.

Next, by applying Theorem 2.2 to Eq. (1.5), we have

Theorem 3.7. *Assume that*

$$\sum_{k=n_0}^{\infty} \frac{q^k}{r(q^k)} = \infty \quad \text{and} \quad \sum_{k=n_0}^{\infty} q^k\Theta(q^k)\delta(q^k) = \infty. \tag{3.10}$$

Furthermore, assume that there exist $H \in \Im$, *a sequence* $\eta(t)$ *and a positive sequence* $v(t)$ *such that*

(3.11)

$$\limsup_{n\to\infty} \frac{1}{H(q^{n+1},q^{n_0})} \sum_{k=n_0}^{n-1} H(q^{n+1},q^{k+1})\left[q^k\phi(q^k) - \frac{q^{k+1}r(q^k)}{4v(q^{k+1})}v^2(q^k)\varphi^2(q^n,q^k)\right] = \infty,$$

where

$$\phi(s) := \frac{v(qs)}{qs}\left[\delta(s)\Theta(s) + sr(s)\eta^2(s) - qs\Delta_q(r(s)\eta(s))\right],$$

$$\varphi(t,s) := \frac{\Delta_q v(s)}{v(s)} + \frac{2\eta(s)v(qs)}{qv(s)} - \frac{h(t,s)}{\sqrt{H(qt,qs)}}.$$

Then Eq. (1.5) *is oscillatory on* $[q^{n_0},\infty)_{q^{\mathbb{N}_0}}$.

If in Theorem 3.7 we choose $v(t) = t$ and $\eta(t) = 1/t$, define $H(t,s)$ by $H(qt,t) = 0$, and $H(t,s) = 1$ otherwise, we have

Corollary 3.8. *Assume that* (3.10) *holds, and*

$$\limsup_{n\to\infty} \sum_{k=n_0}^{n-1} \left[q^k\delta(q^k)\Theta(q^k) - q^{2k+1}\Delta_q\left(\frac{r(q^k)}{q^k}\right) - \frac{5}{4}r(q^k)\right] = \infty. \tag{3.12}$$

Then Eq. (1.5) *is oscillatory on* $[q^{n_0},\infty)_{q^{\mathbb{N}_0}}$.

Finally, we give some examples to show the significance of our main results.

Example 3.1. Consider the second order dynamic equation

$$\Big(\frac{1}{t^2}x^{\Delta}(t)\Big)^{\Delta}+q_1(t)y^{\frac{1}{5}}(t-1)+q_2(t)y^{\frac{9}{5}}(t-1)=0,\quad t\in[2,\infty)_{\mathbb{T}}, \tag{3.13}$$

where $p(t)=1-1/t$, q_1, $q_2\in \mathbf{C}_{rd}([2,\infty)_{\mathbb{T}},\mathbb{R}^+)$ with $q_1(t)=q_2(t)\geq \sigma^2(t)$. Let $v(t)=1$, $\eta(t)=0$, which follows that $\phi(t)\geq 2\sigma(t)$.

In this case (2.14) reads

$$\limsup_{t\to\infty}\int_2^t \phi(s)\Delta s\geq \limsup_{t\to\infty}\int_2^t 2\sigma(s)\Delta s\geq 2\limsup_{t\to\infty}\int_2^t s\Delta s=\infty.$$

Hence, by Corollary 2.6, Eq. (3.13) is oscillatory on $[2,\infty)_{\mathbb{T}}$. In particular,

(1) if $\mathbb{T}=\mathbb{R}$ and $q_1(t)=q_2(t)\geq t^2$, then Eq. (3.13) is oscillatory;
(2) if $\mathbb{T}=\mathbb{N}$ and $q_1(t)=q_2(t)\geq (t+1)^2$, then Eq. (3.13) is oscillatory;
(3) if $\mathbb{T}=h\mathbb{N}$ for $h>0$ and $q_1(t)=q_2(t)\geq (t+h)^2$, then Eq. (3.13) is oscillatory;
(4) if $\mathbb{T}=q^{\mathbb{N}_0}$ for $q>1$ and $q_1(t)=q_2(t)\geq q^2t^2$, then Eq. (3.13) is oscillatory.

Example 3.2. Consider the second-order neutral delay differential equation

$$\Big(\frac{1}{t^5}x'(t)\Big)'+q_1(t)|y(t-1)|^{\alpha-1}y(t-1)+q_2(t)|y(t-1)|^{\beta-1}y(t-1)=0, \tag{3.14}$$

where $t\in[2,\infty)$, $p(t)=1-1/t$, $0<\alpha<1<\beta$ with $\alpha+\beta=2$, and $q_1,q_2\in \mathbf{C}([2,\infty),\mathbb{R}^+)$ with $q_1(t)=q_2(t)\geq \lambda/t^3$, $\lambda>0$. Let

$$v(t)=t^3,\ \eta(t)=-3/(2t),\ H(t,s)=(t-s)^2.$$

A direct computation yields that

$$\phi(t)\geq \frac{2\lambda}{t}-\frac{27}{4t^4},\quad \varphi(t,s)=-\frac{2}{t-s}.$$

Thus, for all $\lambda>0$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,2)}\int_2^t H(t,s)\Big[\phi(s)-\frac{1}{4}v(s)r(s)\varphi^2(t,s)\Big]ds\\&\geq \limsup_{t\to\infty}\frac{1}{(t-2)^2}\int_2^t (t-s)^2\Big[\frac{2\lambda}{s}-\frac{27}{4s^4}-\frac{1}{s^2(t-s)^2}\Big]ds\\&=2\lambda\ln t+\text{constant}=\infty,\end{aligned}$$

i.e., (3.2) holds. Hence, by Theorem 3.1, Eq. (3.14) is oscillatory.

Example 3.3. Consider the following equation

$$\Delta\Big(\frac{1}{t+1}\Delta x(t)\Big)+\frac{t^2}{(t-1)^2}y^{\frac{1}{3}}(t-1)+(t+1)^2y^{\frac{5}{3}}(t-1)=0,\quad t\in[2,\infty)_{\mathbb{N}}, \tag{3.15}$$

where $p(t)=p_0$ with $0\leq p_0<1$. Put $v(t)=1$, $\eta(t)=1+1/t$, define $H(t,s)$ by $H(t+1,t)=0$, and $H(t,s)=1$ otherwise. It is easy to compute that

$$\phi(t)=2(1-p_0)t+\frac{1}{t}+\frac{1}{t(t+1)},\quad \varphi(t,s)=2.$$

Thus, for all $t > 2$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t+1,2)} \sum_{s=2}^{t-1} H(t+1,s+1)\Big[\phi(s) - \frac{(s+1)r(s)}{4sv(s+1)} v^2(s)\varphi^2(t,s)\Big] \\
&= \limsup_{t\to\infty} \sum_{s=2}^{t-1} \Big[2(1-p_0)s + \frac{1}{s(s+1)}\Big] \\
&= \limsup_{t\to\infty} \Big[(1-p_0)(t+1)(t-2) + \frac{1}{2} - \frac{1}{t}\Big] = \infty,
\end{aligned}$$

which follows that (3.5) holds. Therefore, by Theorem 3.3, Eq. (3.15) is oscillatory on $[2,\infty)_{\mathbb{N}}$.

Example 3.4. Consider the following equation

$$\Delta_h(\Delta_h x(t)) + \frac{(t+1)^2}{8(t-1)} y^{\frac{1}{2n+1}}(t-1) + \frac{(t+1)^2}{8(t-1)} y^{\frac{4n+1}{2n+1}}(t-1) = 0, \tag{3.16}$$

where $t \in [3,\infty)_{h\mathbb{N}}$ for $h = 3$, $p(t) = t/(1+t)$, n is a positive integer. Take $v(t) = t$ and $\eta(t) = -1/(2t)$, then

$$\Theta(t) = \frac{(t+1)^2}{4t(t-1)}.$$

Thus, for all $n > 1$,

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{k=1}^{n-1} \Big[\delta(kh)\Theta(kh) + \frac{r(kh)}{4kh} + \frac{kh+h}{2}\Delta_h\Big(\frac{r(kh)}{kh}\Big)\Big] \\
&= \frac{1}{3}\limsup_{n\to\infty} \sum_{k=1}^{n-1} \Big[\frac{(3k+1)^2}{4k} - \frac{1}{4k}\Big] \\
&= \limsup_{n\to\infty} \Big(\frac{3}{8}n^2 + \frac{1}{8}n - \frac{1}{2}\Big) = \infty.
\end{aligned}$$

Hence, (3.9) holds, consequently, by Corollary 3.6, Eq. (3.16) is oscillatory on $[3,\infty)_{h\mathbb{N}}$.

Example 3.5. Consider the 2-difference equation

$$\Delta_q\Big(\frac{1}{t}\Delta_q x(t)\Big) + q_1(t)y^{\frac{1}{3}}(t-1) + q_2(t)y^{\frac{5}{3}}(t-1) = 0, \qquad t \in \mathbb{T}, \tag{3.17}$$

where $\mathbb{T} = [2,\infty)_{2^{\mathbb{N}_0}}$, $p(t) = 1 - 1/t$, and $q_1, q_2 \in \mathbf{C}_{rd}(\mathbb{T}, \mathbb{R}^+)$ with $q_1(t) = q_2(t) \geq 1 - 1/(8t^2)$. Define $v(t) = t$ and $\eta(t) = 1/t$, we get

$$\Theta(t) \geq \frac{2}{t-1}\Big(1 - \frac{1}{8t^2}\Big).$$

Thus, for all $n > 1$,

$$\limsup_{n\to\infty} \sum_{k=1}^{n-1} \Big[q^k \delta(q^k)\Theta(q^k) - q^{2k+1}\Delta_q\Big(\frac{r(q^k)}{q^k}\Big) - \frac{5r(q^k)}{4}\Big]$$
$$\geq \limsup_{n\to\infty} \sum_{k=1}^{n-1} \Big[2^{k+1}\Big(1 - \frac{1}{2^{2k+3}}\Big) + \frac{3}{2^{k+1}} - \frac{5}{2^{k+2}}\Big]$$
$$= \limsup_{n\to\infty} \sum_{k=1}^{n-1} 2^{k+1} = \infty,$$

which follows that (3.12) holds. Hence, by Corollary 3.8, Eq. (3.17) is oscillatory on $[2, \infty)_{2^{\mathbb{N}_0}}$.

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