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OSCILLATION AND NONOSCILLATION OF FIRST-ORDER DYNAMIC EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

BAŞAK KARPUZ AND ÖZKAN ÖCALAN

Department of Mathematics, Faculty of Science and Arts, ANS Campus
Kocatepe University, 03200 Afyonkarahisar, Turkey

ABSTRACT. In this article, we investigate oscillatory nature of all solutions of a class of delay dynamic equations including positive and negative coefficients. Also we give a nonoscillation criterion for this class of delay dynamic equations. While our results reduce to the well-known oscillation criteria for the particular cases of the time scale, they improve recent results on arbitrary time scales. Further, we give some illustrating examples as applications of our results.

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1. INTRODUCTION

In this paper, we consider first-order delay dynamic equations having the form

$$x^{\Delta} + A(t)x \circ \alpha - B(t)x \circ \beta = 0, \tag{1.1}$$

on a time scale $\mathbb{T}$ (i.e., any nonempty closed subset of reals) with $\infty \in \overline{\mathbb{T}}$, where the coefficients $A, B \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{+})$ and the delay functions $\alpha, \beta : \mathbb{T} \to \mathbb{T}$ are strictly increasing with $\alpha(t) \leq \beta(t) < t$ for all $t \in \mathbb{T}$ with $\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \infty$. Also the delta $^{\Delta}$ derivative is the usual derivative for $\mathbb{T} = \mathbb{R}$, while is the usual forward difference operator for $\mathbb{T} = \mathbb{Z}$. We refer readers to [2, 3] for the fundamental theory of time scales.

On a time scale $\mathbb{T}$, the *forward jump operator* is defined by

$$\sigma(t) := \inf(t, \infty)_{\mathbb{T}}, \text{ where } (t, \infty)_{\mathbb{T}} := (t, \infty) \cap \mathbb{T}$$

and the graininess function by

$$\mu(t) := \sigma(t) - t \geq 0.$$

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The *cylinder transformation* is denoted with

$$\zeta_h(t) := \begin{cases} t, & h = 0 \\ \dfrac{1}{h}\mathrm{Log}(1 + th), & h > 0, \end{cases}$$

for $1 + th \neq 0$ and the *exponential function* with

$$\mathrm{e}_f(t, s) := \exp\left\{ \int_s^t \zeta_{\mu(\eta)}(f(\eta))\Delta\eta \right\}$$

for $t, s \in \mathbb{T}$. As is usual, the set of all *positively regressive and rd-continuous functions* is denoted by

$$\mathcal{R}^+(\mathbb{T}, \mathbb{R}) := \big\{ f \in C_{rd}(\mathbb{T}, \mathbb{R}) : 1 + f(t)\mu(t) > 0 \text{ for all } t \in \mathbb{T} \big\}.$$

It is well known that for any $A \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ there exists a positive function x satisfying the initial value problem

$$x^\Delta = A(t)x \text{ with } x(t_0) = 1,$$

we denote this x by $\mathrm{e}_A(\cdot, t_0)$ and call it exponential function.

(1.1) is called nonoscillatory if it possess a solution of constant sign eventually; otherwise, (1.1) is called oscillatory.

There are some oscillation criteria for the particular case of (1.1) with $B \equiv 0$ having the form

$$x^\Delta + A(t)x \circ \alpha = 0. \tag{1.2}$$

Below, we quote some of these results:

Theorem A ([4, Theorem 1], [9, Theorem 1])**.** Assume that (1.2) has an eventually positive solution on $[t_0, \infty)_\mathbb{T}$, then

$$\limsup_{t\to\infty} \sup_{\substack{\lambda>0 \\ -\lambda A \in \mathcal{R}^+([\alpha(t),t)_\mathbb{T},\mathbb{R})}} \big\{ \lambda \mathrm{e}_{-\lambda A}(t, \alpha(t)) \big\} \geq 1$$

holds.

Corollary A. Assume that

$$\limsup_{t\to\infty} \sup_{\substack{\lambda>0 \\ -\lambda A \in \mathcal{R}^+([\alpha(t),t)_\mathbb{T},\mathbb{R})}} \big\{ \lambda \mathrm{e}_{-\lambda A}(t, \alpha(t)) \big\} < 1,$$

then every solution of (1.2) is oscillatory on $[t_0, \infty)_\mathbb{T}$.

Theorem B ([1, Theorem 3])**.** Assume that there exists a constant $\theta \in (0, 1)$ such that

$$\liminf_{t\to\infty} \int_{\alpha(t)}^t A(\eta)\Delta\eta > \theta \text{ and } \limsup_{t\to\infty} \int_{\alpha(t)}^t A(\eta)\Delta\eta > 1 - \big(1 - \sqrt{1-\theta}\big)^2.$$

Then every solution of (1.2) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

2. MAIN RESULTS

In this section, we present our oscillation results on (1.1). We mention that most of the results on (1.1) are depending on linear delays for the particular selections of $\mathbb{T}$ as $\mathbb{R}$ and $\mathbb{Z}$. Hence our results are new when the delay functions are nonlinear.

2.1. **Oscillation of Delay Dynamic Equation** (1.1)**.** Now, we give the following lemma which plays a major role on oscillation of (1.1).

Lemma 2.1. *Assume that* $\gamma(\mathbb{T}) = \mathbb{T}$ *and*

$$C := A - \gamma^{\Delta} \cdot B \circ \gamma \geq 0, \text{ where } \gamma := \beta^{-1} \circ \alpha, \tag{2.1}$$

and

$$\int_{\gamma(t)}^{t} B(\eta)\Delta\eta \leq 1 \tag{2.2}$$

eventually. If x is an eventually positive solution of (1.1)*, then the companion function of x defined by*

$$z(t) := x(t) - \int_{\gamma(t)}^{t} B(\eta)x(\beta(\eta))\Delta\eta, \text{ for } t \in [\alpha^{-1}(t_0), \infty)_{\mathbb{T}} \tag{2.3}$$

satisfies

$$z^{\Delta} \leq 0 \text{ and } z > 0 \tag{2.4}$$

on $[T, \infty)_{\mathbb{T}}$ for some $T \in \mathbb{T}$.

Proof. Assume that $x(\alpha(t)) \geq 0$, (2.1) and (2.2) hold for all $t \in [t_1, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Using [2, Theorem 1.98], we obtain that

$$\begin{aligned} z(t) &= x(t) - \left(\int_{\gamma(t_1)}^{t} B(\eta)x(\beta(\eta))\Delta\eta + \int_{\gamma(t)}^{\gamma(t_1)} B(\eta)x(\beta(\eta))\Delta\eta \right) \\ &= x(t) - \left(\int_{\gamma(t_1)}^{t} B(\eta)x(\beta(\eta))\Delta\eta + \int_{t}^{t_1} \gamma^{\Delta}(\eta)B(\gamma(\eta))x(\beta(\gamma(\eta)))\Delta\eta \right) \\ &= x(t) - \int_{\gamma(t_1)}^{t} B(\eta)x(\beta(\eta))\Delta\eta + \int_{t_1}^{t} \gamma^{\Delta}(\eta)B(\gamma(\eta))x(\alpha(\eta))\Delta\eta \end{aligned}$$

holds for all $t \in [t_1, \infty)_{\mathbb{T}}$. Applying [2, Theorem 1.117] to the above equation and considering the fact that x is a solution of (1.1), we have

$$\begin{aligned} z^{\Delta}(t) &= x^{\Delta}(t) - B(t)x(\beta(t)) + \gamma^{\Delta}(t)B(\gamma(t))x(\alpha(t)) \\ &= -A(t)x(\alpha(t)) + \gamma^{\Delta}(t)B(\gamma(t))x(\alpha(t)) \\ &= -C(t)x(\alpha(t)) \leq 0 \end{aligned} \tag{2.5}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Therefore z is of constant sign eventually. To prove $z > 0$ eventually, suppose the contrary that there is $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z(t) < 0$ holds for all $t \in [t_2, \infty)_{\mathbb{T}}$. We consider the proof in the following two possible cases.

(i) x is unbounded, that is $\limsup_{t\to\infty} x(t) = \infty$. Thus there exists an increasing divergent sequence of points $\{s_n\}_{n\in\mathbb{N}} \subset [t_2, \infty)_{\mathbb{T}}$ such that $\lim_{n\to\infty} x(s_n) = \infty$ and

$$x(s_n) = \max\{x(\eta) : \eta \in [t_2, s_n]_{\mathbb{T}}\}, \text{ for all } n \in \mathbb{N}.$$

From (2.2) and (2.3), we get

$$\begin{aligned} x(s_n) &= z(s_n) + \int_{\gamma(s_n)}^{s_n} B(\eta)x(\beta(\eta))\Delta\eta \\ &\leq z(t_2) + x(s_n) \int_{\gamma(s_n)}^{s_n} B(\eta)\Delta\eta \leq z(t_2) + x(s_n). \end{aligned}$$

Since $z(t_2) < 0$ the above expression yields to a contradiction.

(ii) x is bounded, that is $\ell \in [0, \infty)$, where $\ell := \limsup_{t\to\infty} x(t)$. Thus there exists an increasing divergent sequence of points $\{s_n\}_{n\in\mathbb{N}} \subset [t_2, \infty)_{\mathbb{T}}$ such that $\lim_{n\to\infty} x(s_n) = \ell$ and define $\{r_n\}_{n\in\mathbb{N}} \subset [t_2, \infty)_{\mathbb{T}}$ as

$$x(r_n) = \max\{x(\eta) : \eta \in [\alpha(s_n), \beta(s_n)]_{\mathbb{T}}\}, \text{ for all } n \in \mathbb{N},$$

which implies $\limsup_{n\to\infty} x(r_n) \leq \ell$. Then we get

$$\begin{aligned} x(s_n) &\leq z(t_2) + \int_{\gamma(s_n)}^{s_n} B(\eta)x(\beta(\eta))\Delta\eta \\ &= z(t_2) + \int_{\gamma(s_n)}^{s_n} \frac{B(\eta)}{\beta^{\Delta}(\eta)} x(\beta(\eta))\beta^{\Delta}(\eta)\Delta\eta \\ &= z(t_2) + \int_{\alpha(s_n)}^{\beta(s_n)} \frac{(B \circ \beta^{-1})(\varsigma)}{(\beta^{\Delta} \circ \beta^{-1})(\varsigma)} x(\varsigma)\Delta\varsigma \\ &\leq z(t_2) + x(r_n) \int_{\alpha(s_n)}^{\beta(s_n)} \frac{(B \circ \beta^{-1})(\varsigma)}{(\beta^{\Delta} \circ \beta^{-1})(\varsigma)} \Delta\varsigma \end{aligned}$$

$$= z(t_2) + x(r_n) \int_{\alpha(s_n)}^{\beta(s_n)} \frac{(B \circ \beta^{-1})(\varsigma)}{(\beta^{\Delta} \circ \beta^{-1})(\varsigma)} \frac{(\beta^{-1})^{\Delta}(\varsigma)}{(\beta^{-1})^{\Delta}(\varsigma)} \Delta\varsigma$$

$$= z(t_2) + x(r_n) \int_{\gamma(s_n)}^{s_n} \frac{(B \circ \beta^{-1} \circ \beta)(\eta)}{\beta^{\Delta}(\eta)((\beta^{-1})^{\Delta} \circ \beta)(\eta)} \Delta\eta$$

$$= z(t_2) + x(r_n) \int_{\gamma(s_n)}^{s_n} B(\eta)\Delta\eta \le z(t_2) + x(r_n),$$

where [2, Theorem 1.98] is used in the third step and [2, Theorem 1.97] is used in the last step. Taking superior limits on both sides of the above inequality, we get $\ell \le z(t_2) + \ell$. This is a contradiction again since $z(t_2) < 0$ holds.

Therefore, contradictions follow in both possible distinct cases, and thus the proof is complete. □

Theorem 2.2. *Assume that the assumptions of Lemma* 2.1 *are satisfied. If* (1.1) *has an eventually positive solution then*

$$\limsup_{t\to\infty} \sup_{\substack{\lambda>0 \\ -\lambda C \in \mathcal{R}^+([\alpha(t),t)_{\mathbb{T}},\mathbb{R})}} \big\{\lambda \mathrm{e}_{-\lambda C}(t,\alpha(t))\big\} \ge 1. \tag{2.6}$$

Proof. By the definition of z in (2.3) and Lemma 2.1, we see that $x \ge z > 0$ holds on $[t_1,\infty)_{\mathbb{T}}$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Considering (2.5), we have

$$z^{\Delta} + C(t) z \circ \alpha \le 0,$$

which implies (2.6) holds (see [4, 9, Theorem 1]). □

Corollary 2.3. *Assume that the assumptions of Lemma* 2.1 *hold. Then every solution of* (1.1) *is oscillating if*

$$\limsup_{t\to\infty} \sup_{\substack{\lambda>0 \\ -\lambda C \in \mathcal{R}^+([\alpha(t),t)_{\mathbb{T}},\mathbb{R})}} \big\{\lambda \mathrm{e}_{-\lambda C}(t,\alpha(t))\big\} < 1.$$

Now we introduce the following recursion to advance our above results:

$$C_n(t) := C(t)\sum_{k=0}^{n} B_k(\alpha(t)) \text{ and } B_n(t) := \begin{cases} 1, & n = 0 \\ \int_{\gamma(t)}^{t} B(\eta)B_{n-1}(\beta(\eta))\Delta\eta, & n \in \mathbb{N} \end{cases}$$

for all $t \in [\alpha^{-n}(t_0),\infty)_{\mathbb{T}}$ and $n \in \mathbb{N}$.

Lemma 2.4. *Assume that all conditions of Lemma* 2.1 *hold. Then z defined by* (2.3) *eventually satisfies*

$$z^{\Delta} + C_n(t) z \circ \alpha \le 0 \tag{2.7}$$

for any $n \in \mathbb{N}_0$.

Proof. Assume that the assumptions of Lemma 2.1 hold for all $t \in [t_0, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Now we claim that

$$x(t) \geq \sum_{k=0}^{n} C_k(t) z(t) \tag{2.8}$$

for all $n \in \mathbb{N}_0$ and all $t \in [t_n, \infty)_{\mathbb{T}}$, where $t_n \in [\alpha^{-n}(t_1), \infty)_{\mathbb{T}}$. Clearly, claim holds for $n = 0$ trivially from (2.3). Suppose that the claim holds for $n \in \mathbb{N}$, then we have

$$\begin{aligned} x(t) = z(t) + \int_{\gamma(t)}^{t} B(\eta) x(\beta(\eta)) \Delta\eta &\geq z(t) + \int_{\gamma(t)}^{t} B(\eta) \sum_{k=0}^{n} B_k(\beta(\eta)) z(\beta(\eta)) \Delta\eta \\ &\geq z(t) \left(1 + \int_{\gamma(t)}^{t} B(\eta) \sum_{k=0}^{n} B_k(\beta(\eta)) \Delta\eta \right). \end{aligned}$$

Since

$$\begin{aligned} 1 + \int_{\gamma(t)}^{t} B(\eta) \sum_{k=0}^{n} B_k(\beta(\eta)) \Delta\eta &= 1 + \sum_{k=0}^{n} \int_{\gamma(t)}^{t} B(\eta) B_k(\beta(\eta)) \Delta\eta \\ &= 1 + \sum_{k=0}^{n} B_{k+1}(t) = \sum_{k=0}^{n+1} B_k(t) \end{aligned}$$

holds, we see that

$$x(t) \geq \sum_{k=0}^{n+1} B_k(t) z(t)$$

for all $t \in [t_{n+1}, \infty)_{\mathbb{T}}$, where $t_{n+1} \in [\alpha^{-1}(t_n), \infty)_{\mathbb{T}}$, which implies that the claim holds. Substituting (2.8) into (2.5), we see that (2.7) holds. □

Proofs of the following results directly follow by [4, Theorem 1], [9, Theorem 1] and [1, Theorem 3] and so are omitted.

Theorem 2.5. *Assume that the assumptions of Lemma* 2.1 *hold. If* (1.1) *has an eventually positive solution* $[t_0, \infty)_{\mathbb{T}}$, *then*

$$\limsup_{t \to \infty} \sup_{\substack{\lambda > 0 \\ -\lambda C_n \in \mathcal{R}^+([\alpha(t), t)_{\mathbb{T}}, \mathbb{R})}} \left\{ \lambda \mathrm{e}_{-\lambda C_n}(t, \alpha(t)) \right\} \geq 1$$

for any $n \in \mathbb{N}$.

Corollary 2.6. *If there exists* $n_0 \in \mathbb{N}$ *such that*

$$\limsup_{t \to \infty} \sup_{\substack{\lambda > 0 \\ -\lambda C_{n_0} \in \mathcal{R}^+([\alpha(t), t)_{\mathbb{T}}, \mathbb{R})}} \left\{ \lambda \mathrm{e}_{-\lambda C_{n_0}}(t, \alpha(t)) \right\} < 1$$

holds, then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$ *provided that the assumptions of Lemma* 2.1 *hold.*

Theorem 2.7. *Assume that the assumptions of Lemma* 2.1 *hold. Furthermore there exists a constant* $\theta \in (0,1)$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\liminf_{t\to\infty} \int_{\alpha(t)}^{t} C_{n_0}(\eta)\Delta\eta > \theta \text{ and } \limsup_{t\to\infty} \int_{\alpha(t)}^{t} C_{n_0}(\eta)\Delta\eta > 1 - \left(1 - \sqrt{1-\theta}\right)^2.$$

Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Corollary 2.8. *All criteria on* (2.7) *preventing positive solutions for any* $n_0 \in \mathbb{N}$, *reveal oscillatory nature of* (1.1) *(i.e., see results in* [8] *and* [10]*).*

2.2. **Nonoscillation of Delay Dynamic Equation** (1.1)**.** In this section, we give a nonoscillation criterion for (1.1). Our result reduces to [9, Theorem 2] (1.2) by letting $q \equiv 0$. On the other hand, our result is not given in the literature thus far even for the well-known time scales $\mathbb{R}$ or $\mathbb{Z}$.

Theorem 2.9. *Assume that* $A \geq B \geq 0$ *on* $[t_0, \infty)_{\mathbb{T}}$ *and there exists* $\lambda_0 > 0$ *such that*

$$-\lambda_0 A + B \in \mathcal{R}^+([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) \text{ and } \lambda_0 \mathrm{e}_{-\lambda_0 A + B}(t, \alpha(t)) \geq 1 \text{ for all } t \geq t_0.$$

Then (1.1) *has a nonincreasing positive solution on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. We employ Schauder's fixed-point theorem (see Theorem 1.7.1 in [7]). Denote Banach space of all real, bounded and rd-continuous functions on $[t_0, \infty)_{\mathbb{T}}$ by Ω with sup norm. Let Ω_0 be the subset of Ω consist of those functions x in Ω which satisfy following properties:

(i) x is nonincreasing on $[t_0, \infty)_{\mathbb{T}}$ and $x \equiv 1$ on $[\alpha(t_0), t_0]_{\mathbb{T}}$.
(ii) $\lambda_0 x(t) \geq x(\alpha(t))$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Clearly, $\Omega_0 \neq \emptyset$, i.e., $x \equiv 1$ on $[\alpha(t_0), \infty)_{\mathbb{T}}$ is such a function in Ω_0.

We claim that all functions in Ω_0 are positive. On the contrary, assume that there is a function $x \in \Omega_0$ and a point $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t_1) \leq 0$ and $x > 0$ on $[t_0, t_1)_{\mathbb{T}}$. By (ii), we see that

$$0 \geq \lambda_0 x(t_1) \geq x(\alpha(t_1)) > 0,$$

which is a contradiction. Thus, the claim holds.

Consider the mapping $\Gamma : \Omega_0 \to \Omega_0$ defined by

$$\Gamma x(t) := \begin{cases} 1, & t \in [\alpha(t_0), t_0]_{\mathbb{T}} \\ \mathrm{e}_{-\omega}(t, t_0), & t \in [t_0, \infty)_{\mathbb{T}}, \end{cases} \tag{2.9}$$

where

$$\omega := \frac{A(t) x \circ \alpha - B(t) x \circ \beta}{x}.$$

Now, we claim that Γ maps Ω_0 into Ω_0.

We show that $-\omega \in \mathcal{R}^+([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and ω is positive. For all $t \in [t_0,\infty)_{\mathbb{T}}$, we have

$$1-\omega(t)\mu(t) \geq 1-\left[A(t)\frac{x(\alpha(t))}{x(t)} - B(t)\frac{x(\beta(t))}{x(t)}\right]\mu(t)$$

$$\geq 1-[\lambda_0 A(t) - B(t)]\mu(t) \geq 0 \tag{2.10}$$

and

$$\omega(t) = \frac{A(t)x(\alpha(t)) - B(t)x(\beta(t))}{x(t)} \geq [A(t)-B(t)]\frac{x(\beta(t))}{x(t)} \geq 0, \tag{2.11}$$

which implies $\xi_{\mu(t)}(-\omega(t)) \leq 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$. So Γx is nonincreasing on $[t_0,\infty)_{\mathbb{T}}$. Also by the definition of Γ, we see that $\Gamma x \equiv 1$ holds on $[\alpha(t_0),t_0)_{\mathbb{T}}$. Thus, (i) holds for Γx.

Now, we show that (ii) is true. From (2.10) and (2.11), we have

$$-\omega \in \mathcal{R}^+([t_0,\infty)_{\mathbb{T}},\mathbb{R}) \text{ and } 0 \leq \omega(t) \leq \lambda_0 A(t) - B(t) \text{ for all } t \in [t_0,\infty)_{\mathbb{T}},$$

then we get

$$\begin{aligned}\lambda_0\Gamma x(t) &= \lambda_0\Gamma x(\alpha(t))\mathrm{e}_{-\omega}(t,\alpha(t)) \geq \lambda_0\mathrm{e}_{-\lambda_0 A+B}(t,\alpha(t))\Gamma x(\alpha(t))\\ &= \Gamma x(\alpha(t)).\end{aligned}$$

Thus, $\Gamma : \Omega_0 \to \Omega_0$. Clearly, Ω_0 is a closed and convex subset of Ω and Γ is continuous. Also $\Gamma\Omega_0$ is a relatively compact subset of Ω because $\Gamma\Omega_0$ is equicontinuous and uniformly bounded. Hence, all hypothesis of Shauder's fixed-point theorem are satisfied, so Γ has a fixed point x. That is, there is a function $x \in \Omega_0$ such that $\Gamma x = x$. It follows from (2.9) that

$$(\Gamma x)^{\Delta}(t) = -\omega(t)\Gamma x(t) \text{ or } x^{\Delta}(t) = -\omega(t)x(t)$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. Considering the definition of ω, we see that

$$x^{\Delta} + A(t)x \circ \alpha - B(t)x \circ \beta = 0$$

holds on $[t_0,\infty)_{\mathbb{T}}$. Hence, x is a nonincreasing positive solution of (1.1) and the proof is done. □

3. SOME APPLICATIONS ON PARTICULAR TIME SCALES

This section is dedicated to some general applications on well-known time scales. The following example illustrates differential equations.

Example 3.1. Let $\mathbb{T} = \mathbb{R}$. Consider the following delay differential equation

$$x'(t) + A(t)x(t-\alpha) + B(t)x(t-\beta) = 0, \tag{3.1}$$

where $A, B \in C([t_0, \infty), \mathbb{R}^+)$, $\alpha > \beta > 0$. Then we get

$$\sup_{\substack{\lambda>0 \\ -\lambda C \in \mathcal{R}^+([t-\alpha,t)_{\mathbb{R}},\mathbb{R})}} \left\{\lambda \mathrm{e}_{-\lambda C}(t, \alpha(t))\right\} = \sup_{\lambda>0} \left\{\lambda \exp\left\{ -\lambda \int_{t-\alpha}^{t} C(\eta)d\eta\right\}\right\}$$

$$= \frac{1}{\mathrm{e} \int_{t-\alpha}^{t} C(\eta)d\eta},$$

where $C(t) = A(t) - B(t - \alpha + \beta) \geq 0 (\not\equiv 0)$. If

$$\liminf_{t\to\infty} \int_{t-\alpha}^{t} C(\eta)d\eta > \frac{1}{\mathrm{e}},$$

then every solution of (3.1) oscillates by Theorem 2.2. Let $a_0 \geq A \geq B \geq b_0 \geq 0$, where a_0, b_0 are constants, then we get

$$\sup_{\substack{\lambda>0 \\ -\lambda A+B \in \mathcal{R}^+([t-\alpha,t)_{\mathbb{R}},\mathbb{R})}} \left\{\lambda \mathrm{e}_{-\lambda A+B}(t, t-\alpha)\right\} = \sup_{\lambda>0} \left\{\lambda \exp\left\{ \int_{t-\alpha}^{t} (-\lambda A(\eta) + B(\eta))\Delta\eta\right\}\right\}$$

$$\geq \sup_{\lambda>0} \left\{\lambda \mathrm{e}^{(-\lambda a_0 + b_0)\alpha}\right\}$$

$$= \frac{\mathrm{e}^{-1+b_0\alpha}}{a_0\alpha}.$$

Therefore, if

$$\frac{\mathrm{e}^{b_0\alpha}}{\mathrm{e}a_0\alpha} \geq 1$$

for all $t \in [t_0, \infty)_{\mathbb{R}}$, then (3.1) has a positive solution by Theorem 2.9 with $\lambda_0 = 1/(a_0\alpha)$.

Note that the result given in [7, Theorem 2.3.1, Theorem 2.3.2 and Theorem 2.6.1] is a particular case of our result given in Section 2.

The following result advances the result of the above example.

Example 3.2. Consider (3.1) with same assumptions. If there is a positive integer n_0 such that

$$\liminf_{t\to\infty} \int_{t-\alpha}^{t} C_{n_0}(\eta)d\eta > \frac{1}{\mathrm{e}},$$

where

$$C_{n_0}(t) = C(t) \sum_{k=0}^{n_0} B_k(t-\alpha) \text{ and } B_n(t) = \begin{cases} 1, & n = 0 \\ \int_{t-\alpha+\beta}^{t} B(\eta)B_{n-1}(\eta-\beta)d\eta, & n \in [1, n_0]_{\mathbb{N}} \end{cases},$$

then every solution of (3.1) is oscillatory by Corollary 2.6.

The following example investigates difference equations.

Example 3.3. Let $\mathbb{T}=\mathbb{Z}$. Consider the following delay difference equation

$$\Delta x(t)+A(t)x(t-\alpha)+B(t)x(t-\beta)=0, \tag{3.2}$$

where A,B are nonnegative sequences of reals and $\alpha,\beta\in\mathbb{Z}^+$ with $\alpha>\beta$. Set $C(t)=A(t)-B(t-\alpha+\beta)\geq 0$(not identically zero) and we get

$$\begin{aligned}\sup_{\substack{\lambda>0\\ -\lambda C\in\mathcal{R}^+([t-\alpha,t)_{\mathbb{Z}},\mathbb{R})}}\left\{\lambda \mathrm{e}_{-\lambda C}(t,t-\alpha)\right\}&=\sup_{\substack{\lambda>0\\ 1-\lambda C(\eta)>0,\\ \eta\in[t-\alpha,t]_{\mathbb{Z}}}}\left\{\lambda\prod_{\eta=t-\alpha}^{t-1}(1-\lambda C(\eta))\right\}\\ &\leq\sup_{\substack{\lambda>0\\ 1-\lambda C(\eta)>0,\\ \eta\in[t-\alpha,t]_{\mathbb{Z}}}}\left\{\lambda(1-\frac{1}{\alpha}\sum_{\eta=t-\alpha}^{t-1}\lambda C(\eta))^{\alpha}\right\}\\ &=\frac{1}{\left(\dfrac{\alpha+1}{\alpha}\right)^{\alpha+1}\sum\limits_{\eta=t-\alpha}^{t-1}C(\eta)}.\end{aligned}$$

Therefore, if

$$\liminf_{t\to\infty}\sum_{\eta=t-\alpha}^{t-1}C(\eta)>\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

holds, then by Theorem 2.2 every solution is oscillating. Further suppose for all $t\in[t_0,\infty)_{\mathbb{Z}}$ that $a_0\geq A\geq B\geq b_0\geq 0$, where a_0,b_0 are constants. Then we get

$$\begin{aligned}\sup_{\substack{\lambda>0\\ -\lambda A+B\in\mathcal{R}^+([t_0,\infty)_{\mathbb{Z}},\mathbb{R})}}\left\{\lambda \mathrm{e}_{-\lambda A+B}(t,\alpha(t))\right\}&=\sup_{\substack{\lambda>0\\ 1-\lambda A+B>0}}\left\{\lambda\prod_{\eta=t-\alpha}^{t-1}(1-\lambda A(\eta)+B(\eta))\right\}\\ &\geq\sup_{\substack{\lambda>0\\ 1-\lambda a_0+b_0>0}}\left\{\lambda\big(1-\lambda a_0+b_0\big)^{\alpha}\right\}\\ &=\frac{\alpha^{\alpha}}{a_0}\left(\frac{b_0+1}{\alpha+1}\right)^{\alpha+1}.\end{aligned}$$

Therefore, if

$$\frac{\alpha^{\alpha}(b_0+1)^{\alpha+1}}{(\alpha+1)^{\alpha+1}a_0}\geq 1$$

holds, then (3.2) has a positive solution by Theorem 2.9 with $\lambda_0=(b_0+1)/[(\alpha+1)a_0]$.

The results obtained in the above example are given in [7, Theorem 7.7.1], which is a particular case of our result.

Now we advance the result of the example given above.

Example 3.4. Consider (3.2) with same assumptions. If there exists a $n_0\in\mathbb{N}$ such that

$$\liminf_{t\to\infty}\sum_{\eta=t-\alpha}^{t-1}C_{n_0}(\eta)>\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

where

$$C_{n_0}(t) = C(t)\sum_{k=0}^{n_0} B_k(t-\alpha) \text{ and } B_n(t) = \begin{cases} 1, & n = 0 \\ \sum\limits_{\eta=t-\alpha+\beta}^{t-1} B(\eta)B_{n-1}(\eta-\beta), & n \in [1, n_0]_{\mathbb{N}} \end{cases},$$

then every solution of (3.2) is oscillatory by Corollary 2.6.

The following examples investigates q-difference equations.

Example 3.5. Let $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ and consider the q-difference equation

$$\Delta_q x(t) + \frac{A(t)}{(q-1)t}x(t/q^{\alpha}) - \frac{B(q^{\alpha-\beta}t)}{(q-1)t}x(t/q^{\beta}) = 0, \tag{3.3}$$

where $A \geq (\not\equiv) B \geq 0$, $\alpha > \beta > 0$ and

$$\Delta_q x(t) = \frac{x(qt) - x(t)}{(q-1)t}.$$

Then letting $C(t) = [A(t) - B(t)]/[(q-1)t]$, we get

$$\begin{aligned} &\sup_{\substack{\lambda>0 \\ -\lambda C \in \mathcal{R}^+([t/q^{\alpha}, t)_{q^{\mathbb{N}_0}}, \mathbb{R})}} \Big\{\lambda \mathrm{e}_{-\lambda C}(t, t/q^{\alpha})\Big\} \\ &= \sup_{\substack{\lambda>0 \\ -\lambda C \in \mathcal{R}^+([t/q^{\alpha}, t)_{q^{\mathbb{N}_0}}, \mathbb{R})}} \bigg\{\lambda \prod_{\eta=1}^{\alpha}[1 - \lambda(A(t/q^{\eta}) - B(t/q^{\eta}))]\bigg\} \\ &\leq \sup_{\substack{\lambda>0 \\ -\lambda C \in \mathcal{R}^+([t/q^{\alpha}, t)_{q^{\mathbb{N}_0}}, \mathbb{R})}} \bigg\{\lambda\bigg(1 - \frac{\lambda}{\alpha}\sum_{\eta=1}^{\alpha}[A(t/q^{\eta}) - B(t/q^{\eta})]\bigg)^{\alpha}\bigg\} \\ &= \frac{1}{\Big(\frac{\alpha+1}{\alpha}\Big)^{\alpha+1}\sum\limits_{\eta=1}^{\alpha}[A(t/q^{\eta}) - B(t/q^{\eta})]}. \end{aligned}$$

Thus, if

$$\liminf_{t\to\infty}\sum_{\eta=1}^{\alpha}[A(t/q^{\eta}) - B(t/q^{\eta})] = \liminf_{t\to\infty}\sum_{\eta=t-\alpha}^{t-1}[A(q^{\eta}) - B(q^{\eta})] > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

then every solution of (3.3) is oscillatory. Let $a_0 \geq A \geq B \geq b_0 \geq 0$ in (3.3), where a_0, b_0 are constants, one can show that if

$$\frac{\alpha^{\alpha}(b_0+1)^{\alpha+1}}{(\alpha+1)^{\alpha+1}a_0} \geq 1$$

holds, then (3.3) a positive solution by Theorem 2.9.

These results for the q-difference equations has not been given in the literature yet.

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REFERENCES

[1] R. P. Agarwal and M. Bohner, An oscillation criterion for first order delay dynamic equations, *Funct. Differ. Equ.*, 2007, In Press.

[2] M. Bohner and A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhäuser, Boston, 2001.

[3] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.

[4] M. Bohner, Some oscillation criteria for first order delay dynamic equations, *Far East J. Appl. Math.*, vol. 18, no. 3, pp. 289–304, (2005).

[5] M. Bohner, B. Karpuz and Ö. Öcalan, Iterated oscillation criteria for delay dynamic equations of first-order, (submitted).

[6] L. Erbe and B. G. Zhang, Oscillation of discrete analogues of delay equations, *Differential and Integral Equations*, vol. 2, no. 3, pp. 30–309, 1989.

[7] I. Gyori and G. Ladas, *Oscillation theory of delay differential equations with applications*, Clarendon Press, Oxford, 1991.

[8] Y. Sahiner and I. P. Stavroulakis, Oscillations of first order delay dynamic equations, *Dynam. Systems and Appl.*, vol. 15, pp. 645–656, (2006).

[9] B. G. Zhang and X. Deng, Oscillation of delay differential equations on time scales, *Math. Comput. Modelling*, vol. 36, no. 11–13, pp. 1307-1318, (2002).

[10] B. G. Zhang, X. Yan and X. Liu, Oscillation criteria of certain delay dynamic equations on time scales, *J. Difference Equ. Appl.*, vol. 11, no. 10, pp. 933–946, (2005).